# Banks' Liquidity Management and Systemic Risk*

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**Abstract**

We study a novel mechanism to explain the interaction between banks' liquidity management and the emergence of systemic financial crises, in the form of self-fulfilling runs. To this end, we develop an environment where banks offer insurance to their depositors against both idiosyncratic and aggregate shocks, by holding a portfolio of liquidity and illiquid productive assets. Moreover, banks' asset portfolios and the probability of a depositors' self-fulfilling run are jointly determined via a "global game". In equilibrium, an endogenous pecking order emerges: there exists a unique threshold recovery rate, associated with the early liquidation of the productive assets, such that, for any value lower than that, the banks first employ liquidity and then liquidate, in order to finance depositors' early withdrawals. Ex ante, the banks hold more liquidity than in a full-information economy, where there are no self-fulfilling runs and risk is only due to idiosyncratic and aggregate shocks.

Keywords: systemic risk, excess liquidity, bank runs, global games

JEL Classification: G01, G21

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# 1 Introduction

Why do banks hold excess liquidity in the face of systemic financial crises? Many explanations have been proposed for this observation, from precautionary motivations (Ashcraft et al., 2011; Acharya and Merrouche, 2013) to counterparts risk (Heider et al., 2015). However, there is some extensive evidence that the emergence of these events is connected not only to extreme fluctuations of the fundamentals of the economy, but also to investors' self-fulfilling expectations of crises themselves. This is particularly true in the banking system: in fact, liquidity and maturity transformation, i.e. the issuance of short-term liquid liabilities (most often, deposits) to finance the purchase of long-term illiquid assets, creates a mismatch in banks' balance sheets that makes them vulnerable to self-fulfilling runs by their depositors. Self-fulfilling runs are not a phenomenon of the past: in fact, Argentina in 2001 and Greece in 2015 are only the two most recent examples of such systemic events. Moreover, there exists a wide consensus that many U.S. money market funds and life insurance funds (that, by offering liquidity and maturity transformation, can be likened to traditional banks) have experienced self-fulfilling runs after the bankruptcy of Lehman Brothers in 2008 (Foley-Fisher et al., 2015) and, more generally, that the U.S. financial crisis of 2007-2009 can be interpreted as a systemic self-fulfilling run of financial intermediaries on other financial intermediaries (Gorton and Metrick, 2012). Therefore, there is the paramount need to understand how banks' liquidity management influences the formation of investors' self-fulfilling expectations, and how these in turns might lead to systemic runs. This is the aim of the present paper.

To this end, we develop a positive theory of banking with three main ingredients. First, a liquid asset is available to the banks, to store resources and roll them over time. This assumption is critical for two reasons: first, because it provides a precise definition of liquidity; second, because it gives content to the two main alternatives that banks face in the presence of possible liquidity shortages, namely (i) holding liquidity ex ante in excess of what the expectations of future liquidity needs would demand, or (ii) liquidating ex post part of their productive investments. Second, the concept of excess liquidity is well-defined, by comparison to a benchmark economy with perfect information, in which the banks engage in precautionary liquidity holding to insure their depositors against idiosyncratic and aggregate shocks. Third, banks' liquidity management and the probability of self-fulfilling runs are jointly and endogenously determined.

More formally, we model a Diamond and Dybvig (1983) three-date (0, 1, and 2) economy, populated by risk-averse agents/depositors. These are hit by idiosyncratic liquidity shocks, that make them willing to consume in the interim period (at date 1), and by aggregate productivity shocks, realized in the final period (at date 2), that affect the return on the available investment technology. In this framework, a banking system arises to provide insurance against both sources of uncertainty: to this end, the banks collect deposits at date 0, and offer a standard deposit contract, financed by investing in a liquid and safe asset (that we call "liquidity") and in a productive – yet risky – one, that can be liquidated in the interim period at a cost.

In the interim period, the depositors also privately observe a noisy signal about the realization of the aggregate productivity shock (which represents the state of the economy) based on which they form posterior beliefs about the true state as well as about the other depositors' signals. Because of strategic complementarities in their withdrawal decisions, this information structure implies that the depositors might all withdraw from the banks, thus triggering a self-fulfilling run, if they expect everybody else to do the same, and know that the banks do not hold sufficient liquid resources to fulfill their contractual obligations. This happens whenever the observed signal falls below some threshold, that endogenously depends on the state of the economy and on the deposit contract. Facing systemic risk due to both aggregate productivity shocks and the possibility of self-fulfilling runs, the banks decide the pecking order with which to employ their assets and serve the depositors at the interim date. Specifically, the banks can either (i) use liquidity first and then liquidate their productive assets at a given recovery rate or (ii) liquidate their productive asset first, and then use liquidity.

We characterize the equilibrium by backward induction. First, we study the equilibrium in the interim period, when liquidity shocks hit the depositors, and they decide whether to join a run or not, based on their posterior beliefs. This decision crucially depends on the deposit contract and asset portfolio chosen by the banks at date 0. On the one side, by increasing the amount of early consumption offered to those withdrawing in the interim period, the banks open themselves to the possibility of not being able to repay all depositors in the case of a run: in other words, high interim consumption induces a high probability of a self-fulfilling run. On the other side, by increasing the amount of liquidity held in portfolio, the banks are able to serve more depositors before declaring bankruptcy, thus lowering their incentives to join a run:

in other words, high liquidity induces a low probability of a self-fulfilling run.

Second, we characterize the optimal pecking order followed by the banks, given depositors' optimal decisions about whether to run or not. The equilibrium pecking order trades off the opportunity cost of financing withdrawals at a run either by liquidating the productive asset, in terms of (i) forgone resources in the interim period due to a low recovery rate and (ii) forgone consumption in the final period in the good state of the world, or by using liquidity, in terms of diminished insurance against the aggregate productivity shock. In particular, the banks choose to employ liquidity first and then liquidation as long as the productive asset is sufficiently illiquid, i.e. the recovery rate associated with its early liquidation is sufficiently low. Accordingly, the model predicts the typical chain of events that we observe in the real world during a self-fulfilling run: if the recovery rate from liquidating the productive assets is low, at first banks are liquid; then, they become illiquid but solvent, when they run out of liquidity and start liquidating the productive asset; finally, they become insolvent, thereby going into bankruptcy.

Third, taking into account the withdrawal decisions of the depositors and the optimal pecking order of the interim period, in the initial period the banks solve for the optimal composition of their asset portfolios, in terms of liquidity and productive assets, and for the deposit contract. The equilibrium is characterized by a distorted Euler equation, featuring a wedge between the marginal rate of substitution, between consumption in the interim period and in the final period, and the expected return on the productive asset (equivalent to the marginal rate of transformation of a productive technology). This wedge is due to two distinctive properties of the banking equilibrium: first, the composition of the asset portfolios and the deposit contract indirectly affect depositors' expected welfare, by altering the threshold signal below which a run is triggered; second, the amount of liquidity affects the utility of the depositors in the case of a run. The optimal composition of the asset portfolios implies that the banks hold liquidity in excess of the amount that they would choose in a benchmark no-run economy, where liquidity shocks are observable and the signals about the state of the economy are perfectly informative. More precisely, we are able to show that insurance against idiosyncratic liquidity shocks is lower (i.e. interim consumption is lower), while insurance against systemic risk due to self-fulfilling runs and aggregate productivity shocks is higher (i.e. higher liquidity) compared to the benchmark case, so that excess liquidity emerges.

The approach of connecting banks' liquidity management to aggregate productivity shocks and self-fulfilling runs is novel in the literature on banking and financial crises. In fact, in the first-generation models of bank runs, Cooper and Ross (1998) and Ennis and Keister (2006) also study banks' liquidity management, but in an environment without aggregate productivity shocks. In there, self-fulfilling runs arise as a consequence of multiple equilibria, and the depositors select an equilibrium following the realization of an extrinsic event, commonly named "sunspot", completely uncorrelated to the fundamentals of the economy. In such a framework, banks do hold excess liquidity in equilibrium, but only to be able to serve all depositors in the case of a run, thus making themselves "run-proof". In other words, contrary to the empirical evidence, these models do not exhibit excess liquidity and self-fulfilling runs simultaneously. Allen and Gale (1998) also study banks' liquidity management, but in an environment with aggregate productivity shocks and no multiple equilibria. Moreover, the banks in their model do not hold excess liquidity, but offer a standard deposit contract coupled with the possibility to default in the bad states of the world, thus allowing optimal risk sharing against idiosyncratic and aggregate real uncertainty. In the second-generation models of bank runs (Goldstein and Pauzner, 2005), instead, the economy does feature aggregate productivity shocks, and the equilibrium selection mechanism, in the presence of multiple equilibria, is endogenously determined via a "global game". Yet, in these models, there is no liquidity management, i.e. no room for either liquidity or excess liquidity, as the productive asset in which banks invest can be liquidated at zero cost. To the best of our knowledge, the work of Anhert and Elamin (2014) is the only example of a second-generation model where the banks have the possibility to invest in a liquid asset, but only ex post (i.e. during a run), to transfer the proceeds from early liquidation of the productive asset to possible bad states of the world. In other words, there is no liquidity management ex ante, to hedge against aggregate productivity shocks as well as self-fulfilling runs.

The rest of the paper is organized as follows: in section 2, we lay down the basic features of the environment; in section 3, we study the withdrawing decisions of the depositors, and characterize the optimal pecking order with which the banks employ their assets to finance depositors' withdrawals; in section 4, we solve for the banking equilibrium; finally, section 5 concludes.

## 2 Environment

The economy lives for three periods, labeled $t = 0, 1, 2$, and is populated by a continuum of ex-ante identical agents, all endowed with 1 unit of a consumption good at date 0, and 0 afterwards. At date 1, all agents are hit by a privately-observed idiosyncratic liquidity shock $\theta$, taking value 0 with probability $\lambda$ and 1 with probability $1 - \lambda$. The law of large numbers holds, hence the probability distribution of the idiosyncratic liquidity shocks is equivalent to their cross-sectional distribution: at date 1, there is a fixed fraction $\lambda$ of agents in the whole economy whose realized shock is $\theta = 0$, and a fraction $1 - \lambda$ whose realized shock is $\theta = 1$. The idiosyncratic liquidity shocks affect the point in time when the agents want to consume, according to the welfare function $U(c_1, c_2, \theta) = (1 - \theta)u(c_1) + \theta u(c_2)$. In other words, those agents receiving a shock $\theta = 0$ are only willing to consume at date 1, and those receiving a shock $\theta = 1$ are only willing to consume at date 2. Thus, in line with the literature, we refer to them as early (or impatient) consumers and late (or patient) consumers, respectively. The utility function $u(c)$ is increasing, concave and twice-continuously differentiable, and is such that $u(0) = 0$ and the coefficient of relative risk aversion is strictly larger than 1. Importantly, it also satisfies the Inada conditions: $\lim_{c \to 0} u'(c) = +\infty$ and $\lim_{c \to +\infty} u'(c) = 0$.

There are two technologies available in the economy to hedge against the idiosyncratic liquidity shocks. The first one is a storage technology, here called "liquidity", yielding 1 unit of consumption at date $t+1$ for each unit invested in $t$. The second one is instead a productive asset that, for each unit invested at date 0, yields a stochastic return $A$ at date 2. This stochastic return takes values $R > 1$ with probability $p$, and 0 with probability $1-p$. The probability of success of the productive asset $p$ represents the aggregate state of the economy, and is distributed uniformly over the interval $[0, 1]$, with $\mathbb{E}[p]R > 1$. Moreover, the productive asset can be liquidated at date 1 via a liquidation technology, that allows to recover $r < 1$ units of consumption for each unit liquidated. Intuitively, this means that the economy features a liquid asset, with low but safe yields, and a partially illiquid asset, that yields a low return in the short run, but a possible high return in the long run, which is subject to the realization of an aggregate productivity shock.

The economy is also populated by a large number of banks, operating in a perfectly-competitive market with free entry. The banks collect the endowments of the agents in the form of deposits, and invest them so as to maximize their profits, subject to agents' participation. Perfect competition and free entry ensure that the banks solve the equivalent problem of maximizing the expected welfare of the agents/depositors, subject to their budget constraint. To this end, they offer a standard deposit contract $\{c, c_L(A)\}$, stating the uncontingent amount $c$ that the depositors can withdraw at date 1, and the state-dependent amount $c_L(A)$ that they can withdraw at date 2.[1] As the realizations of the idiosyncratic types are private information, the deposit contract must be incentive compatible, i.e. the depositors must have the incentives to truthfully report their types. This implies that the deposit contract must satisfy the incentive compatibility constraint $c \leq c_L(R)$.

To finance the deposit contract, the banks invest the deposits – which are the only liability on their balance sheets – in an amount $L$ of liquidity and $1-L$ of the productive asset, respectively. Then, given the deposit contract and asset portfolio chosen at date 0, the banks commit to pay $c$ to whoever depositor comes to withdraw at date 1, until their resources are exhausted. To this end, the banks also choose the "pecking order" with which to use the assets in order to finance the early withdrawals: {Liquidation, Liquidity} or {Liquidity, Liquidation}. When resources are exhausted, and the banks are not able to fulfill their contractual obligations anymore, they instead go into bankruptcy, at which they must liquidate all the productive assets in portfolio, and serve their depositors according to an "equal service constraint", i.e. such that all depositors get an equal share of the available resources. Finally, at date 2 the depositors who have not withdrawn at date 1 are residual claimants of an equal share of the remaining resources.

We assume that depositors cannot observe the true value of the realization of the fundamental $p$, but receive at date 1 a "noisy" signal $\sigma = p + e$ about it. The term $e$ is an idiosyncratic noise, indistinguishable from the true value of $p$, that is uniformly distributed over the interval $[-\epsilon, +\epsilon]$, where $\epsilon$ is positive but small. Given the received signal, each late consumer decides whether to withdraw from her bank at date 2, as the realization of her idiosyncratic shock would command, or "run on her bank" and withdraw at date 1, in accordance to the scheme to be described in the incoming section.

[1] In order to rule out uninteresting run equilibria, the amount of early consumption $c$ must be smaller than $\min\{1/\lambda, R\}$. The fact that the banks have to offer a standard deposit contract here is assumed. However, Farhi et al. (2009) show that a standard deposit contract, with an uncontingent amount of early consumption, endogenously emerge as part of the banking equilibrium, in the presence of non-exclusive contracts.

The timing of actions is as follows: at date 0, the banks collect the deposits, and choose the deposit contract $\{c, c_L(A)\}$ and asset portfolio $\{L, 1 - L\}$; at date 1, the banks choose the pecking order with which to finance early withdrawals; then, all agents get to know their private types and private signals, and the early consumers withdraw, while the late consumers, once observed the signals, decide whether to run or not; finally, at date 2, those late consumers who have not withdrawn at date 1 withdraw an equal share of the available resources left. We solve the model by backward induction, and characterize a pure-strategy symmetric Bayesian Nash equilibrium. Hence, we focus our attention on the behavior of a representative bank. The definition of equilibrium is the following:

**Definition 1.** *Given the distributions of the idiosyncratic and aggregate shocks and of the individual signals, a banking equilibrium is a deposit contract $\{c, c_L(A)\}$, an asset portfolio $\{L, 1 - L\}$, a pecking order and depositors' decisions to run such that, for every realization of signals and idiosyncratic types $\{\sigma, \theta\}$:*

- *the depositors' decisions to run maximize their expected welfare;*
- *the pecking order, the deposit contract and the asset portfolio maximize the depositors' expected welfare, subject to budget constraints.*

## 2.1 Equilibrium with Perfect Information

As a benchmark for the results that follow, we start our analysis with the characterization of the equilibrium with perfect information, provided by a social planner who can observe the realization of the idiosyncratic liquidity shocks hitting the depositors, and maximizes their expected welfare subject to budget constraints. More formally, the social planner solves:

$$\max_{c, c_L(A), L, D} \lambda u(c) + (1 - \lambda)\mathbb{E}[u(c_L(A))], \tag{1}$$

subject to the resource constraints:

$$L + rD \geq \lambda c, \tag{2}$$

$$(1 - \lambda)c_L(A) + \lambda c = A(1 - L - D) + L + rD, \tag{3}$$

where the last constraint has to hold for any $A \in \{0, R\}$, and to the non-negativity constraint $D \geq 0$.[2] At date 0, the planner collects all endowments, and invests them in an amount $L$ of liquidity and $1 - L$ of productive assets. At date 1, the liquidity constraint (2) states that the amount of liquid assets, given by the sum of liquidity plus the extra resources generated by liquidating an amount $D$ of productive assets at rate $r$, must be sufficient to pay early consumption $c$ to the $\lambda$ early consumers. Any resources left are rolled over to date 2 and, together with the return from the remaining productive assets, pay late consumption $c_L(A)$ to the $1 - \lambda$ late consumers in any state $A$. Plugging the resource constraints in the objective function, the planner's problem can be rewritten as:

$$\max_{c,L,D} \lambda u(c) + (1-\lambda) \int_0^1 \left[ pu \left( \frac{R(1-L-D) + L + rD - \lambda c}{1-\lambda} \right) + \right. \\ \left. + (1-p) u \left( \frac{L + rD - \lambda c}{1-\lambda} \right) \right] dp, \tag{4}$$

subject to the liquidity constraint $L + rD \geq \lambda c$ and $D \geq 0$. In this framework, we can prove the following:

**Lemma 1.** *The equilibrium with perfect information exhibits no liquidation of the productive asset ($D = 0$) and excess liquidity ($L > \lambda c$). The deposit contract and asset portfolio satisfy the Euler equation:*

$$u'(c) = \mathbb{E}[p] R u' \left( \frac{R(1-L) + L - \lambda c}{1-\lambda} \right). \tag{5}$$

**Proof**. In Appendix A. ■

Intuitively, the lemma shows that liquidating the productive asset to create liquidity at date 1 is never part of an equilibrium with perfect information, because the recovery rate $r$ is too low. Moreover, the planner provides the optimal amount of insurance against the aggregate productivity shock by engaging in precautionary savings, i.e. by holding more liquidity than the one needed to cover early consumption and insure against the idiosyncratic liquidity shock. In equilibrium, this is achieved with an allocation satisfying an Euler equation, i.e. so that the marginal rate of substitution between consumption at date 1 and consumption at date 2 is equal to the marginal rate of transformation of the productive asset. Finally, the concavity of

[2] The non-negativity constraints on the other choice variables are clearly satisfied in equilibrium, given the assumption that the Inada conditions hold.

the utility function and the assumption that $\mathbb{E}[p]R > 1$ imply that the incentive compatibility constraint is satisfied, hence this allocation is equivalent to a constrained efficient allocation, in which the social planner has to induce truth-telling among the depositors.

# 3 Strategic Complementarities

We now move to the analysis of the competitive banking equilibrium. As stated above, we characterize it by backward induction, hence in this section we start by studying the withdrawing decisions of the late consumers (as the early consumers withdraw for sure at date 1) who choose whether to withdraw ("run") at date 1 or wait until date 2. Then, in the following section, we characterize the equilibrium deposit contract and portfolio allocation.

A late consumer receives a signal $\sigma$ at date 1, and takes as given the deposit contract and asset portfolio, fixed at date 0, and the pecking order, fixed at date 1 before the signal is realized. Based on these, she creates her posterior beliefs about how many depositors are withdrawing at date 1 (call this number $n$), and the probability of the realization of the aggregate state $A$, and decides whether to withdraw or not. We assume the existence of two regions of extremely high and extremely low signals, where the decision of a late consumer is independent of her posterior beliefs. In the "upper dominance region", the signal is so high that a late consumer always prefers to wait until date 2 to withdraw. Following Goldstein and Pauzner (2005), we assume that this happens above a threshold $\bar{\sigma}$, where the investment is safe, i.e. $p = 1$, and gives the same return $R$ at date 1 and 2. In this way, a late consumer is sure to get $\frac{R(1-L)+L-\lambda c}{1-\lambda}$ at date 2, irrespective of the behavior of all the other late consumers, and prefers to wait for any possible realization of the aggregate state. In the "lower dominance region", instead, the signal is so low that a late consumer always runs, irrespective of the behavior of the other depositors, thus triggering a "fundamental run". This happens below the threshold signal $\underline{\sigma}_j$, that makes her indifferent between withdrawing or not, and depends on the pecking order $j$ chosen by the bank (we characterize the thresholds in the incoming sections).

The existence of the lower and upper dominance regions, regardless of their size, ensures the existence of an equilibrium in the intermediate region $[\underline{\sigma}_j, \bar{\sigma}]$, where the late consumers decide whether to run or not based on a threshold strategy: they run if the signal is lower than a threshold signal $\sigma_j^*$.[3] Let $Prob(\sigma \leq \sigma_j^*)$ be the probability that $\sigma \leq \sigma_j^*$ under pecking order $j$.

[3] In the present environment, Goldstein and Pauzner (2005) prove that the equilibrium strategy is always a

Then, given $\sigma = p + e$, we have:

$$Prob(\sigma \leq \sigma_j^*) = \int_{-\epsilon}^{\sigma_j^* - p} \frac{1}{2\epsilon} de = \max\left(\frac{\sigma_j^* - p + \epsilon}{2\epsilon}, 0\right). \tag{6}$$

Define as $c_L(A, n)$ the amount of late consumption that a late consumer would get if the realized state is $A$ and $n$ depositors are withdrawing at date 1. Arguably, it should be the case that the higher the number of depositors who run is, the lower late consumption is, or $\partial c_L(A, n)/\partial n$. Moreover, define $n_j^{**}$ as the maximum number of depositors that a bank can serve under pecking order $j$ without breaking the deposit contract, i.e. while still being able to pay $c$ to all those depositors who withdraw at date 1. After $n_j^{**}$, the bank goes into bankruptcy: there are no more resources for late consumption, and the bank pays $c^B(n)$ according to an equal service constraint, i.e. it equally splits the available resources among the $n$ depositors who withdraw.

Define the expected utility from waiting $\mathbb{E}[u(c_L(A, n))]$, given the signal $\sigma$ and the number $n$ of depositors who withdraw, as:

$$\mathbb{E}[u(c_L(A, n))] = \int_{-\epsilon}^{\epsilon} (\sigma - e) u(c_L(R, n)) \frac{1}{2\epsilon} de + \int_{-\epsilon}^{\epsilon} (1 - \sigma + e) u(c_L(0, n)) \frac{1}{2\epsilon} de. \tag{7}$$

It is immediate to verify that this reduces to:

$$\mathbb{E}[u(c_L(A, n))] = \sigma u(c_L(R, n)) + (1 - \sigma) u(c_L(0, n)). \tag{8}$$

Then, the utility advantage of waiting versus running, for a given number $n$ of depositors who withdraw and pecking order $j$, is:

$$v_j(A, n) = \begin{cases} \sigma u(c_L(R, n)) + (1 - \sigma) u(c_L(0, n)) - u(c) & \text{if } \lambda \leq n < n_j^{**}, \\ -u(c(n)) & \text{if } n_j^{**} \leq n < 1. \end{cases} \tag{9}$$

The number of depositors who withdraw at date 1 is given by the sum of the $\lambda$ early consumers and those among the $1 - \lambda$ late consumers who receive a signal lower than the threshold signal

threshold strategy.

$\sigma^*$:

$$n = \lambda + (1-\lambda) Prob(\sigma \leq \sigma_j^*) = \lambda + (1-\lambda)\max\left(\frac{\sigma_j^* - p + \epsilon}{2\epsilon}, 0\right). \tag{10}$$

Thus, $n$ is a random variable that depends on the state of the economy. Importantly, as $\sigma$ is a random variable, the Laplacian Property (Morris and Shin, 1998) tells us that its cumulative distribution function $Prob(\sigma \leq \sigma_j^*)$ is uniformly distributed over the interval $[0,1]$, thus the number of depositors who withdraw $n$ must be uniformly distributed over the interval $[\lambda, 1]$. This allows us to calculate the expected value of waiting versus running as:

$$\mathbb{E}[v_j(A,n)|\sigma] = \int_{\lambda}^{1} \frac{v_j(A,n)}{1-\lambda} dn, \tag{11}$$

and to characterize the threshold signal $\sigma_j^*$ as the one such that $\mathbb{E}[v_j(A,n)|\sigma_j^*] = 0$.

From what said so far, it is clear that the decision of the late consumers about whether to join a run depends, in turns, on the decision of the bank about how to finance early withdrawals, i.e. on the pecking order with which it employs liquidation of the productive asset and liquidity. In what follows, we characterize and compare the withdrawing behavior of the depositors under each pecking order, in particular by studying its effects on the lower dominance region and the threshold strategies.

### 3.1 Pecking order 1: {Liquidation; Liquidity}

In this first case, the bank serves the depositors who withdraw at date 1 first by liquidating the productive asset, and then by employing the liquidity in portfolio. Under this pecking order, the threshold signal $\underline{\sigma}_1$ characterizing the lower dominance region is the one that equalizes:

$$u(c) = \underline{\sigma}_1 u\left(\frac{R\left(1 - L - \frac{\lambda c}{r}\right) + L}{1-\lambda}\right) + (1-\underline{\sigma}_1) u\left(\frac{L}{1-\lambda}\right). \tag{12}$$

This expression states that a late consumer receiving a signal $\underline{\sigma}_1$ must be indifferent between withdrawing at date 1 and getting $c$ and waiting until date 2 and getting $\frac{R\left(1-L-\frac{\lambda c}{r}\right)+L}{1-\lambda}$ with probability $\underline{\sigma}_1$ or $\frac{L}{1-\lambda}$ with probability $1-\underline{\sigma}_1$. These values come from the fact that, by liquidating the productive asset first, the bank withholds liquidity, that finances late consumption irrespective of the realization of the aggregate state. Moreover, the bank has to pay an amount of early consumption $c$ to $\lambda$ early consumers, by liquidating an amount $D$ of productive assets at rate $r$, hence $D = \lambda c/r$. Rearranging the equality above, we obtain the threshold:

$$\underline{\sigma}_1 = \frac{u(c) - u\left(\frac{L}{1-\lambda}\right)}{u\left(\frac{R\left(1-L-\frac{\lambda c}{r}\right)+L}{1-\lambda}\right) - u\left(\frac{L}{1-\lambda}\right)}, \tag{13}$$

which is clearly increasing in the amount of early consumption $c$ set in the deposit contract.

The threshold strategy in the intermediate region $[\underline{\sigma}_1, \bar{\sigma}]$, instead, depends on the late consumers' advantage of waiting versus running:

$$v_1(A,n) = \begin{cases} \sigma u\left(\frac{R\left(1-L-\frac{nc}{r}\right)+L}{1-n}\right) + (1-\sigma)u\left(\frac{L}{1-n}\right) - u(c) & \text{if } \lambda \leq n < n_1^* \\ \sigma u\left(\frac{r(1-L)+L-nc}{1-n}\right) + (1-\sigma)u\left(\frac{r(1-L)+L-nc}{1-n}\right) - u(c) & \text{if } n_1^* \leq n < n_1^{**} \\ -u\left(\frac{r(1-L)+L}{n}\right) & \text{if } n_1^{**} \leq n < 1. \end{cases} \tag{14}$$

In this expression, $n_1^* = \frac{r(1-L)}{c}$ and $n_1^{**} = \frac{r(1-L)+L}{c}$ are the maximum number of depositors that a bank can serve at date 1 without breaking the deposit contract, and either liquidating the whole amount of productive assets in portfolio (up to $n_1^*$) or using also liquidity (up to $n_1^{**}$). When the number of depositors who withdraw at date 1 lies in the interval $[\lambda, n_1^*]$, the bank fulfills its contractual obligation by retaining liquidity, and liquidating the productive asset: it needs to pay an amount of early consumption $c$ to $n$ depositors via $rD$ resources from liquidation, hence $D = \frac{nc}{r}$. Then, if $n$ depositors withdraw, the consumption of a late consumer who waits is either $c_L(R,n) = \frac{R\left(1-L-\frac{nc}{r}\right)+L}{1-n}$ or $c_L(0,n) = \frac{L}{1-n}$, depending on the realization of the aggregate state. When the number of depositors who withdraw lies in the interval $[n_1^*, n_1^{**}]$, the bank, instead, fulfills its contractual obligation by liquidating all productive assets in portfolio (thus generating resources equal to $r(1-L)$) and by employing the liquidity. Thus, if $n$ depositors withdraw, the consumption of a late consumer who waits is independent of the realization of the aggregate state (as the productive assets have all been liquidated) and equal to $c_L^L(n) = \frac{r(1-L)+L-nc}{1-n}$. Finally, when the number of depositors who withdraw lies in the interval $[n_1^{**}, 1]$, the bank goes bankrupt, as it does not hold sufficient resources to pay an amount of early consumption $c$ to all depositors. In this case, the bank is forced to liquidate all productive assets and close down, so a late consumer who waits gets zero. Moreover, the available resources (equal to $r(1-L)+L$) are equally split among all the $n$ depositors who withdraw, and each one gets $c^B(n) = \frac{r(1-L)+L}{n}$.

The sign of the strategic complementarity affecting the decision of a late consumer to run depends on how the advantage of waiting versus running depends on the number of depositors withdrawing. More formally:

$$\frac{\partial v_1(A,n)}{\partial n} = \begin{cases} \sigma u'(c_L(R,n)) \frac{-\frac{R}{r}c(1-n)+\left[R\left(1-L-\frac{nc}{r}\right)+L\right]}{(1-n)^2} + \frac{(1-\sigma)u'(c_L(0,n))L}{(1-n)^2}, & \text{if } \lambda \leq n < n_1^* \\ u'(c_L^L(n))\frac{r(1-L)+L-c}{(1-n)^2}, & \text{if } n_1^* \leq n < n_1^{**} \\ u'(c^B(n))\frac{c^B(n)}{n} & \text{if } n_1^{**} \leq n < 1. \end{cases} \tag{15}$$

On the one side, in the interval $[n_1^{**}, 1]$ the derivative is positive as, after bankruptcy, equal service prescribes total resources to be shared pro-rata to all depositors; on the other side, in the interval $[n_1^*, n_1^{**}]$ the derivative is negative by definition of $n_1^{**}$, highlighting the presence of one-sided strategic complementarities. We characterize the direction of the strategic complementarity in the interval $[\lambda, n_1^*]$ in the following lemma:

**Lemma 2.** *In the interval $[\lambda, n_1^*]$, $v_1(A,n)$ is decreasing in $n$.*

**Proof**. In Appendix A. ■

Figure 1 shows that, despite the different environment, the economy exhibits one sided strategic complementarities as in Goldstein and Pauzner (2005): the advantage of waiting versus running is decreasing in the number of depositors running before bankruptcy, and increasing after bankruptcy. However, despite not knowing the sign of $v_1(A, n_1^*)$, the function $v_1(A,n)$ crosses zero only once, because is decreasing in $n$ in both intervals $[\lambda, n_1^*]$ and $[n_1^*, n^*]$, and this guarantees the uniqueness of the equilibrium in the intermediate region $[\underline{\sigma}_1, \bar{\sigma}]$.

**Lemma 3.** *Under the pecking order {Liquidation; Liquidity}, in the intermediate region $[\underline{\sigma}_1, \bar{\sigma}]$ a late consumer runs if her signal is lower than the threshold signal:*

$$\sigma_1^* = \frac{\int_\lambda^{n_1^{**}} u(c)dn + \int_{n_1^{**}}^1 u\left(\frac{L+r(1-L)}{n}\right)dn - \int_\lambda^{n_1^*} u\left(\frac{L}{1-n}\right)dn - \int_{n_1^*}^{n_1^{**}} u\left(\frac{L+r(1-L)-nc}{1-n}\right)dn}{\int_\lambda^{n_1^*}\left[u\left(\frac{R\left(1-L-\frac{nc}{r}\right)+L}{1-n}\right) - u\left(\frac{L}{1-n}\right)\right]dn}. \tag{16}$$

*The threshold signal $\sigma_1^*$ is increasing in $c$ and decreasing in $L$.*

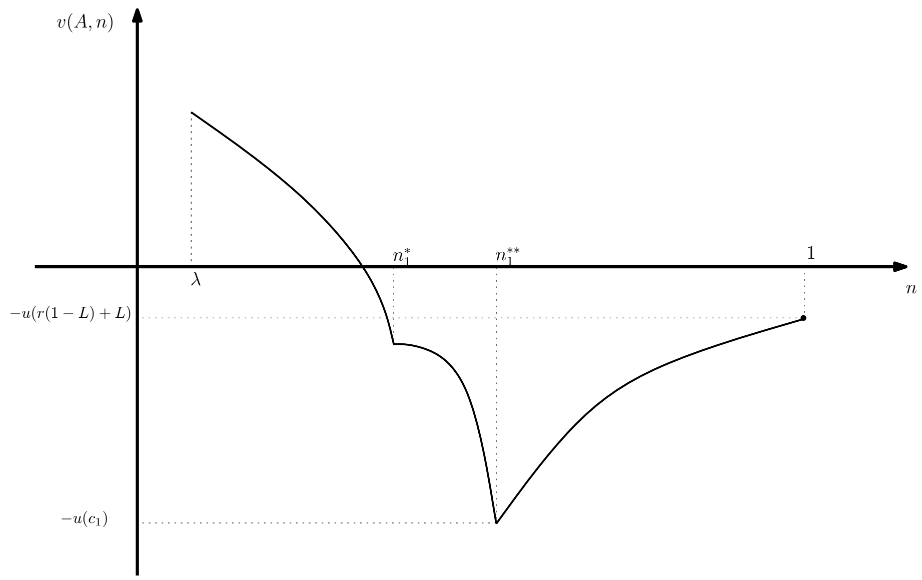


Figure 1: The advantage of waiting versus running, as a function of the number of depositors running, when the bank chooses the pecking order {Liquidation; Liquidity}.

**Proof**. In Appendix A. ■

The lemma characterizes the endogenous threshold signal in the case of pecking order {Liquidation; Liquidity}, and shows the effect that the bank's deposit contract and asset portfolio have on it. In particular, increasing early consumption $c$ has a threefold positive effect on the threshold signal $\sigma_1^*$: it directly increases the advantages for a late consumer to run, both before and after bankruptcy; it lowers the maximum fraction of depositors that a bank can serve before bankruptcy; it decreases the advantages of waiting until date 2. The effect that increasing the total amount of liquidity in the bank's portfolio has on the threshold signal $\sigma_1^*$ instead looks ambiguous: on the one side, more liquidity increases consumption after bankruptcy and lowers late consumption, thus also increasing the threshold and the incentives to run; on the other side, more liquidity also increases the amount of insurance against the aggregate productivity shock that a bank can provide, thus lowering the threshold and the incentives to run. However, the effect that one more unit of liquidity has on the marginal utility of those depositors running just before bankruptcy (i.e. at $n_1^{**}$) is large because of the Inada Conditions. Thus, the second effect dominates the first, and the threshold signal $\sigma_1^*$ turns out to be decreasing in $L$.

### 3.2 Pecking order 2: {Liquidity; Liquidation}

In this second case, we assume that the bank serves the withdrawers at date 1 first by employing the liquidity, and then by liquidating the productive asset. Under this pecking order, the threshold signal $\underline{\sigma}_2$ characterizing the lower dominance region is the one that equalizes:

$$u(c) = \underline{\sigma}_2 u\left(\frac{R(1-L)+L-\lambda c}{1-\lambda}\right) + (1-\underline{\sigma}_2)u\left(\frac{L-\lambda c}{1-\lambda}\right). \tag{17}$$

This expression states that a late consumer receiving a signal $\underline{\sigma}_2$ must be indifferent between withdrawing at date 1 and getting $c$ and waiting until date 2 and getting $\frac{R(1-L)+L-\lambda c}{1-\lambda}$ with probability $\underline{\sigma}_2$ or $\frac{L-\lambda c}{1-\lambda}$ with probability $1-\underline{\sigma}_2$. These values come from the fact that, by employing liquidity first, the bank withholds the productive asset. Hence, having to pay an amount of early consumption $c$ to $\lambda$ early consumers, it rolls over an amount $L-\lambda c$ of excess liquidity from date 1 to date 2. Rearranging the equality above, we obtain the threshold:

$$\underline{\sigma}_2 = \frac{u(c) - u\left(\frac{L-\lambda c}{1-\lambda}\right)}{u\left(\frac{R(1-L)+L-\lambda c}{1-\lambda}\right) - u\left(\frac{L-\lambda c}{1-\lambda}\right)}. \tag{18}$$

As for the previous pecking order, this value is increasing in the amount of early consumption $c$ set in the deposit contract. To see that, it suffices to calculate:

$$\frac{\partial \underline{\sigma}_2}{\partial c} = \frac{u'(c) + \frac{\lambda}{1-\lambda}u'\left(\frac{L-\lambda c}{1-\lambda}\right) + \underline{\sigma}_2 \frac{\lambda}{1-\lambda}\left[u'\left(\frac{R(1-L)+L-\lambda c}{1-\lambda}\right) - u'\left(\frac{L-\lambda c}{1-\lambda}\right)\right]}{u\left(\frac{R(1-L)+L-\lambda c}{1-\lambda}\right) - u\left(\frac{L-\lambda c}{1-\lambda}\right)}. \tag{19}$$

This expression is always positive, as $\underline{\sigma}_2$ is lower than 1.

The threshold strategy in the intermediate region $[\underline{\sigma}_1, \bar{\sigma}]$, instead, depends on the late consumers' advantage of waiting versus running:

$$v_2(A,n) = \begin{cases} \sigma u\left(\frac{R(1-L)+L-nc}{1-n}\right) + (1-\sigma)u\left(\frac{L-nc}{1-n}\right) - u(c) & \text{if } \lambda \le n < n_2^* \\ \sigma u\left(\frac{R(1-L-D)}{1-n}\right) - u(c) = \sigma u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right) - u(c) & \text{if } n_2^* \le n < n_2^{**} \\ -u\left(\frac{L+r(1-L)}{n}\right) & \text{if } n_2^{**} \le n < 1 \end{cases} \tag{20}$$

where, similarly to the previous case, $n_2^* = \frac{L}{c}$ and $n_2^{**} = \frac{r(1-L)+L}{c}$ are the maximum number of depositors that a bank can serve at date 1 without breaking the deposit contract and using liquidity (up to $n_2^*$), and also liquidating the whole amount of productive assets in portfolio (up to $n_2^{**}$). When the number of depositors who withdraw lies in the interval $[\lambda, n_2^*]$, the bank fulfills its contractual obligation by keeping the productive asset and using liquidity. Hence, if $n$ depositors are withdrawing, the consumption of a late consumer who waits is either $c_L(R,n) = \frac{R(1-L)+L-nc}{1-n}$ or $c_L(0,n) = \frac{L-nc}{1-n}$, depending on the realization of the aggregate productivity shock. When the number of depositors who withdraw lies instead in the interval $[n_2^*, n_2^{**}]$, the bank is forced to fulfill its contractual obligation also by liquidating the productive assets in portfolio (thus generating resources equal to $r(1-L)$). Hence, the total available resources to provide early consumption $c$ to $n$ depositors who withdraw are $L+rD$, meaning that the amount that the bank liquidates is equal to $D = \frac{nc-L}{r}$. Moreover, as the liquidity has been exhausted, the consumption of a late consumer who decides to wait and finds herself in the aggregate state where $A = 0$ is zero, while in the aggregate state when $A$ is positive is $c_L^D(R,n) = \frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}$. Finally, when the number of depositors who withdraw lies in the interval $[n_2^{**}, 1]$, the bank is bankrupt. Thus, by the equal service constraint, all the $n$ depositors who withdraw get $\frac{r(1-L)+L}{n}$, and those $1-n$ who do not withdraw get zero. Notice that the total number of depositors that can be served before bankruptcy is the same under the two pecking orders. Hence, to economize on notation, we write $n_1^{**} = n_2^{**} = n^{**}$.

We again study the sign of the strategic complementarities by taking the derivative of $v_2(A,n)$ with respect to $n$:

$$\frac{\partial v_2(A,n)}{\partial n} = \begin{cases} \sigma u'(c_L(R,n))\frac{c_L(R,n)-c}{1-n} - (1-\sigma)u'(c_L(0,n))\frac{c-c_L(0,n)}{1-n} & \text{if } \lambda \leq n < n_2^* \\ \sigma u'(c_L^D(R,n))\frac{c_L^D(R,n)-\frac{Rc}{r}}{1-n} & \text{if } n_2^* \leq n < n^{**} \\ u'(c^B(n))\frac{c^B(n)}{n} & \text{if } n^{**} \leq n < 1 \end{cases} \tag{21}$$

As before, in the interval $[n^{**}, 1]$ the derivative is positive, while in the interval $[n_2^*, n^{**}]$ is negative by definition of $n^{**}$. We characterize the sign of the strategic complementarity in the interval $[\lambda, n_2^*]$ in the following lemma:

**Lemma 4.** *In the interval $[\lambda, n_2^*]$, $v_2(A,n)$ is decreasing in $n$ whenever is non-positive.*

**Proof**. In Appendix A. ■

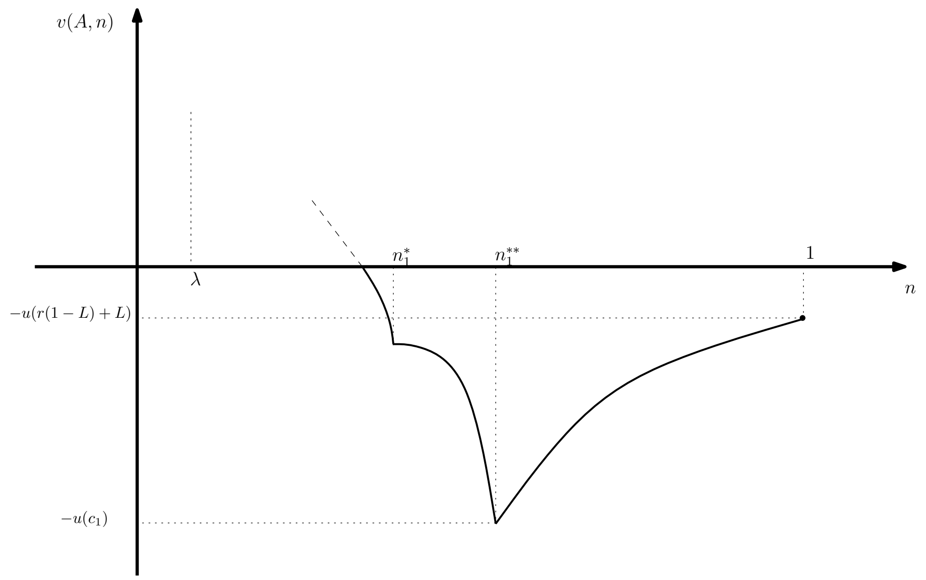


Figure 2: The advantage of waiting versus running, as a function of the number of depositors running, when the bank chooses the pecking order {Liquidity; Liquidation}.

In order to guarantee the uniqueness of the equilibrium, we need to show that $v(A, n_2^*) < 0$. To this end, notice that:

$$v_2(A, n_2^*) = \sigma u\left(\frac{R(1-L)}{c-L}c\right) - u(c). \tag{22}$$

This expression is negative if:

$$\sigma < \frac{u(c)}{u\left(\frac{R(1-L)}{c-L}c\right)} \equiv \tilde{\sigma}, \tag{23}$$

where $\tilde{\sigma} > 1$ whenever $R < \frac{c-L}{1-L}$. In the proof of lemma 6, we show that this condition holds in the banking equilibrium under the pecking order {Liquidity, Liquidation}. Hence, $v_2(A, n_2^*) < 0$, because $\sigma$ is always lower than 1 by definition. In this way, $v_2(A, n)$ crosses zero only once in the interval $[\lambda, n_2^*]$, and that is sufficient for the solution to exist and be unique.

With this result in hand, we can characterize the threshold signal that makes a late consumer indifferent between waiting or running:

**Lemma 5.** *Under the pecking order {Liquidity, Liquidation}, in the intermediate region* $[\underline{\sigma}_2, \bar{\sigma}]$ *a late consumer runs if her signal is lower than the threshold signal:*

$$\sigma_2^* = \frac{\displaystyle\int_{\lambda}^{n^{**}} u(c)dn + \int_{n^{**}}^{1} u\left(\tfrac{L+r(1-L)}{n}\right)dn - \int_{\lambda}^{n_2^*} u\left(\tfrac{L-nc}{1-n}\right)dn}{\displaystyle\int_{\lambda}^{n_2^*}\left[u\left(\tfrac{R(1-L)+L-nc}{1-n}\right) - u\left(\tfrac{L-nc}{1-n}\right)\right]dn + \int_{n_2^*}^{n^{**}} u\left(\tfrac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn}. \tag{24}$$

*The threshold signal $\sigma_2^*$ is increasing in c, and decreasing in L.*

**Proof**. In Appendix A. ■

Intuitively, the lemma shows that increasing early consumption $c$ has a positive effect on the threshold signal $\sigma_2^*$ for many concurrent reasons. First, as in the pecking order {Liquidation, Liquidity}, it directly increases the advantages of running before bankruptcy. Moreover, it decreases the advantages of waiting until date 2, either by decreasing the amount of excess liquidity rolled over to date 2 or by forcing the bank to liquidate more productive assets, whenever the liquidity has been completely exhausted. Finally, increasing $c$ has a negative effect on the amount of insurance against the aggregate productivity shock that a bank can provide, and that in turns increases the threshold signal and the incentives to run. In contrast, increasing the amount of liquidity has an ambiguous effect on the threshold signal: on the one side, more liquidity increases the available resources after bankruptcy, and lower the investment in the productive asset, thus increasing the threshold and the incentives for a late consumer to run; on the other side, it allows the bank to provide more insurance against the aggregate productivity shock, and to lower the amount of productive assets to liquidate, thus lowering the incentives to run. However, the effect that one more unit of liquidity has on the marginal utility of those depositors running just before bankruptcy (i.e. at $n_1^{**}$) is again large by the Inada Conditions. Thus, the second effect dominates the first, and the threshold probability $\sigma_1^*$ is decreasing in $L$.

### 3.3 Endogenous Pecking Order

At date $t = 1$, given the deposit contract and the asset portfolio, the bank decides the optimal pecking order with which to employ the assets in its portfolio, as a best response to the withdrawing decisions of the depositors. More formally, define:

$$V_j(c, L) \equiv \int_0^{\sigma_j^*} u(L + r(1-L))dp + \int_{\sigma_j^*}^1 \Big[\lambda u(c) + (1-\lambda)[pu(c_L(R)) + (1-p)u(c_L(0))]\Big] dp \quad (25)$$

as the expected utility of a depositor, when her bank offers an amount $c$ of early consumption, holds an amount $L$ of liquidity, and chooses the pecking order $j$. If $c \geq L + r(1-L)$ and $L < 1$, the above expression is decreasing in $\sigma_j^*$. Hence, maximizing the expected utility of a depositor is equivalent to choosing the pecking order with the lowest threshold signal $\sigma_j^*$. That will crucially depend on the recovery rate from liquidating the productive asset, as the following proposition

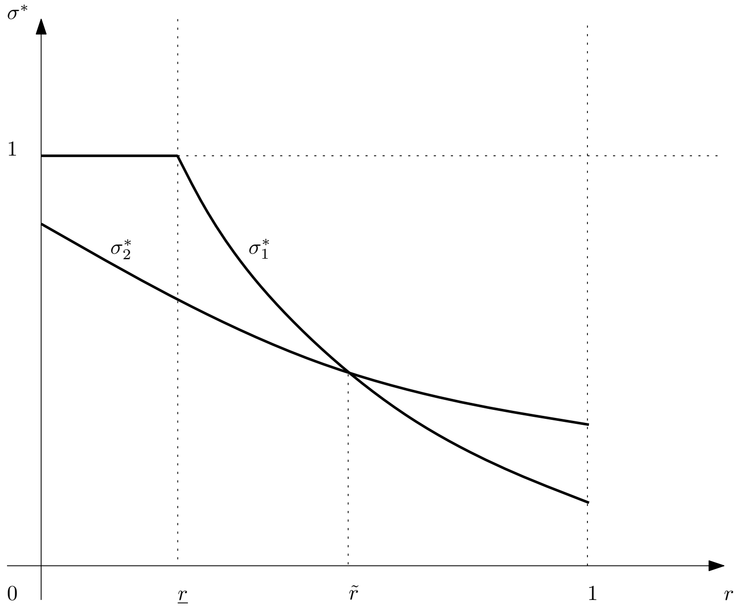


Figure 3: The threshold signals in the two pecking orders (on the y-axis), for different values of the recovery rate of the productive asset (on the x-axis).

shows:

**Proposition 1.** *Assume that the utility function belongs to the CRRA class, and that the coefficient of relative risk aversion is sufficiently high. Then, there exists a threshold* $\tilde{r} \in [0, 1]$ *such that, for any* $r \leq \tilde{r}$*, the optimal pecking order is* $\{$*Liquidity, Liquidation*$\}$*, and for any* $r \geq \tilde{r}$*, the optimal pecking order is* $\{$*Liquidation, Liquidity*$\}$*.*

**Proof**. In Appendix A. ■

The proof of this result is based on showing that the threshold signals under the two pecking orders adjust to changes in the recovery rate of the productive asset as Figure 3 shows. First, both threshold signals $\sigma_1^*$ and $\sigma_2^*$ are decreasing and convex functions of the recovery rate. This happens because, when the number of depositors who are running is $n^{**}$ (i.e. the value that triggers bankruptcy under both pecking orders) a late consumer who does not join a run gets zero. Hence, increasing the recovery rate by one marginal unit makes her consumption go from zero to a positive value. This, by the Inada conditions, has a large positive effect on the utility of waiting (although decreasing, because of the concavity of $u(c)$) and lowers both threshold signals.

Second, the comparison between the two pecking orders essentially boils down to comparing the costs associated with using either liquidation or liquidity to finance early withdrawals. On the one hand, liquidation of the productive asset at date $t = 1$ is costly in terms of (i) forgone resources at date $t = 1$ due to a low recovery rate (i.e. $r < 1$) and (ii) forgone late consumption in the good state of the world. On the other hand, using liquidity is costly in terms of forgone late consumption in the bad state of the world, i.e. in terms of diminished insurance against the aggregate productivity shock. If the depositors are sufficiently risk averse and the recovery rate $r$ is close to 1, both costs associated with liquidation become less relevant, because the depositors care relatively less about high late consumption in the good state of the word and the bank waists less resources when liquidating the productive asset. The opposite is true with respect to the cost associated with using liquidity because, being very risk averse, the depositors care a lot about late consumption in the bad state of the world. Therefore, with sufficiently high relative risk aversion and a recovery rate $r$ close to 1, {Liquidation; Liquidity} is the optimal pecking order.

If instead the recovery rate is close to zero, liquidation becomes very costly, and this is enough to ensure that {Liquidity; Liquidation} is the optimal pecking order. This happens because a late consumer who does not join a run is worse off under the pecking order {Liquidation; Liquidity} than under {Liquidity; Liquidation}: on the one side, the threshold signal $\sigma_1^*$ under the pecking order {Liquidation; Liquidity} is constant and equal to one, i.e. there exists a lower bound $\underline{r}$ for the recovery rate, below which all late consumers would rather withdraw early than wait, irrespective of the number of depositors running, hence any signal would lead to a run; on the other side, the threshold signal $\sigma_2^*$ under the pecking order {Liquidity; Liquidation} is always lower than 1 when the recovery rate is equal to zero.

To sum up, under the assumptions of Proposition 1, the graphs of the two threshold signals can meet at most once for any recovery rate in the interval $[0, 1]$. This means that the bank prefers the pecking order {Liquidation; Liquidity} only if the recovery rate of the productive asset is sufficiently high, so that it can liquidate at lower costs and roll over liquidity to the final period to ensure the depositors against the aggregate productivity shock. If instead the recovery rate of the productive asset is low, the bank prefers the pecking order {Liquidity; Liquidation}. In other words, Proposition 1 rationalizes the typical sequence of events emerging when a bank faces a self-fulfilling run, and makes it explicitly contingent on the recovery rate: if this last one is sufficiently low, a bank facing a run is first liquid, then illiquid but solvent, and finally insolvent.

# 4 Banking Equilibrium

With the results of the previous section in hand, that characterize the behavior of the depositors and the optimal pecking order, we can solve for the banking equilibrium. Assume that the recovery rate is lower than the threshold $\tilde{r}$ so that, by Proposition 1, the bank chooses the pecking order {Liquidity; Liquidation}. Then, at date 0, the bank chooses the deposit contract and asset portfolio so as to maximize the expected welfare of the depositors. More formally, it solves:

$$\max_{c,L} \int_0^{\sigma_2^*} u(L + r(1-L))dp + \int_{\sigma_2^*}^1 \left[\lambda u(c) + (1-\lambda)\left[pu\left(\frac{R(1-L)+L-\lambda c}{1-\lambda}\right)\right.\right.$$
$$\left.\left. +(1-p)u\left(\frac{L-\lambda c}{1-\lambda}\right)\right]\right] dp, \tag{26}$$

subject to the liquidity constraint $L \geq \lambda c$, and to the incentive compatibility constraint $c \leq \frac{R(1-L)+L-\lambda c}{1-\lambda}$. When the signal is below the threshold signal $\sigma_2^*$, a run happens, either fundamental or self-fulfilling: all depositors get an equal share of the available resources, given by the sum of liquidity plus the extra resources generated by completely liquidating the productive assets. When instead the signal is above the threshold signal $\sigma_2^*$, a run does not happen: a fraction $\lambda$ of depositors are early consumers, and consume $c$, while a fraction $1-\lambda$ of them are late consumers, and consume either $c_L(R) = \frac{R(1-L)+L-\lambda c}{1-\lambda}$, if the productive assets yields a positive return, or $c_L(0) = \frac{L-\lambda c}{1-\lambda}$, if it yields zero. Define the difference between the utility in the case of no-run and the utility in the case of run as:

$$\Delta U(c,L) \equiv \lambda u(c) + (1-\lambda)\left[\sigma_2^* u\left(\frac{R(1-L)+L-\lambda c}{1-\lambda}\right) + (1-\sigma_2^*)u\left(\frac{L-\lambda c}{1-\lambda}\right)\right] +$$
$$- u(L + r(1-L)), \tag{27}$$

Then, from the first-order conditions of the program, the following can be proved:

**Lemma 6.** *The banking equilibrium features excess liquidity ($L > \lambda c$). The deposit contract* *and asset portfolio satisfy the distorted Euler equation:*

$$\int_{\sigma_2^*}^{1} \left[u'(c) - pRu'(c_L(R))\right] dp + \sigma_2^*(1-r)u'(L + r(1-L)) = \left[\frac{\partial \sigma_2^*}{\partial L} + \frac{1}{\lambda}\frac{\partial \sigma_2^*}{\partial c}\right] \Delta U(c, L). \quad (28)$$

**Proof**. In Appendix A. ■

In equilibrium, the incentive compatibility constraint turns out to be slack, i.e. $c < c_L(R)$, and the amount $c_L(0)$ that the late consumers get if the productive asset yields zero is lower than early consumption $c$, as $L < c$. In addition to that, it can be proved that $\Delta U(c, L)$ is strictly positive. To see that, first notice that $c^B(n) = \frac{L+r(1-L)}{n}$ is decreasing in $n$, and its highest possible value is equal to $c$ (when $n = n^{**}$). Hence, early consumption $c$ is always higher than the amount of consumption $L + r(1-L)$ that the depositors can achieve during a run.[4] Second, by definition of the threshold of the lower dominance region in (18), we can rearrange $\underline{\sigma}_2 < \sigma_2^*$ into:

$$u(c) < \sigma_2^* u(c_L(R)) + (1 - \sigma_2^*)u(c_L(0)). \quad (29)$$

Therefore, $\Delta U(c, L)$, as the difference between a linear combination of two terms both higher than $u(L + r(1-L))$ and $u(L + r(1-L))$ itself, must be positive. Finally, in every banking equilibrium, the amount $c_L(0)$ that the late consumers get if the productive asset yields zero cannot be larger than the amount $L + r(1-L)$ that they would get during a run. To see that, argue, to the contrary, that $c_L(0) > L+r(1-L)$. By definition of $c_L(0)$, this inequality simplifies into:

$$\lambda[L + r(1-L) - c] > r(1-L). \quad (30)$$

As $c > c_L(0) > L+r(1-L)$, the left-hand side of this expression is negative. However, $r(1-L)$ is non-negative, hence we get to a contradiction, and we are left with the only possibility that $c_L(0) \leq L + r(1-L)$.

The remaining proof of the lemma is in part similar to the one of the equilibrium with perfect information: by the Inada conditions, the bank finds optimal to provide insurance against the aggregate shock by engaging in precautionary savings, i.e. by holding more liquidity than the one

[4]Incidentally, this results also allows us to compare the banking equilibrium to an autarkic equilibrium. To this end, assume that, in autarky, the agents cannot open a bank account, but are forced to invest directly in liquidity and productive assets. Thus, the amount of early consumption that they could achieve is $c^A = L^A + r(1-L^A)$, for any possible value of $L^A$. In other words, banking allows better insurance against idiosyncratic liquidity shocks (i.e. higher early consumption) than autarky.

needed to insure the depositors against the idiosyncratic liquidity shocks. However, differently from the equilibrium with perfect information, the bank imposes a wedge between the marginal rate of substitution between early and late consumption and the marginal rate of transformation of the productive asset (as represented by the first term on the left-hand side of (28)). This happens through two different channels: first, the bank takes into account that it needs higher liquidity to finance consumption in the case of a run (the second term on the left-hand side of (28)); second, it also takes into account that the equilibrium deposit contract and asset portfolio affect the endogenous threshold signal $\sigma_2^*$ and, threfore, the probability that a run is realized (the right-hand side of (28)). The direction of the distortion with respect to the equilibrium with perfect information depends on the sign of the wedge:

**Proposition 2.** *Assume that $u(c)$ is invertible, and that $\lambda > u(1)$. Then, the banking equilibrium features more excess liquidity than the equilibrium with perfect information, i.e. $c^{BE} < c^{PI}$ and $L^{BE} > L^{PI}$.*

**Proof**. In Appendix A. ■

The proof of Proposition 2, which is the main result of the paper, is based on showing that the distortion that the deposit contract and the asset portfolio impose on the banking equilibrium, through their effects on the threshold signal $\sigma_2^*$ (the right-hand side of the distorted Euler equation (28)) and on the marginal utility of consumption at a run (the second term on the left-hand side of (28)), is positive. This forces the bank to increase the marginal rate of substitution between consumption at date 1 and consumption at date 2 in the good aggregate state, by lowering early consumption and increasing the amount of liquidity, with respect to the equilibrium with perfect information. In other words, the bank reacts to the possibility of self-fulfilling runs by increasing the amount of excess liquidity in portfolio.

# 5 Concluding Remarks

With the present paper, we study a novel mechanism through which systemic risk, in the form of self-fulfilling runs, triggers excessive liquidity holdings in the banking system. To this end, we develop a positive theory of banking with three main features: a liquid asset is available to the banks, to store resources and roll them over time; the concept of excess liquidity is well-defined, by comparison to a benchmark economy with perfect information; banks' asset portfolios and depositors' expectations are jointly and endogenously determined. In such an environment, we show that an endogenous pecking order emerges, with which the banks employ their assets in order to finance withdrawals during a run: if the recovery rate from liquidating the productive asset is sufficiently low, the banks first employs liquidity, then liquidate the productive asset. In this way, our model is the first, to the best of our knowledge, to provide a rationale for the typical chain of events during a bank run: at first, banks are liquid, when they hold sufficient liquidity to honor the deposit contract with the depositors who withdraw; then, they become illiquid but solvent, when they run out of liquidity and start liquidating the productive assets held in portfolio; finally, they are insolvent, when so many depositors are withdrawing early that they do not have sufficient resources to cover their withdrawals. Moreover, we find that the deposit contract and the amount of liquidity in portfolio do have opposite effects on the probability of a run. On the one side, by increasing the amount of risk sharing against the idiosyncratic liquidity shocks (i.e., increasing early consumption), the banks open themselves to the possibility of not being able to repay all depositors in the case of a run: in other words, high early consumption induces a high probability of a run. On the other side, by increasing the amount of risk sharing against the aggregate productivity shock (i.e., increasing liquidity), the banks also lower the incentive of the depositors to join a run: in other words, high liquidity induces a low probability of a run. In turns, the banks take into account the effects that their portfolio decisions have on the expectations of the depositors, and, as a consequence, on the probability of a run, and distort the intertemporal allocation of the resources in the economy, by optimally choosing to lower the amount of early consumption and increase their holdings of liquidity, with respect to an economy with perfect information. In other words, the possibility of self-fulfilling bank runs, triggered by the contemporaneous presence of banks' risky behaviors and depositors' self-fulfilling expectations, forces the banks to hoard liquidity.

Despite our focus being mainly positive, we can easily extend the present framework, in which banks explicitly hold a portfolio of liquid and illiquid assets, to analyze policy issues. In particular, we could study the effect that ex-ante liquidity requirements and ex-post liquidity guarantees have on the probability of runs and, in turns, on banks' asset portfolios and liquidity holdings. In principle, we expect such government interventions to be considerably effective at reducing the probability of runs. However, the effect on banks' asset portfolios and liquidity holdings are not trivial, as regulation might strengthen or weaken the marginal effects of liquidity and early consumption on the probability of runs. We keep a formal analysis of these issues open to future research.

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# Appendices

## A Proofs

**Proof of lemma 1**. Attach the Lagrange multipliers $\mu$ to the liquidity constraint (2) and $\xi$ to the non-negativity constraint of $D$. The first-order conditions of the program are:

$$\begin{aligned} c: &\quad u'(c) - \int_0^1 \Big[pu'(c_L(R)) + (1-p)u'(c_L(0))\Big] dp - \mu = 0, && (31)\\ L: &\quad \int_0^1 \Big[pu'(c_L(R))(1-R) + (1-p)u'(c_L(0))\Big] dp + \mu = 0, && (32)\\ D: &\quad \int_0^1 \Big[pu'(c_L(R))(r-R) + (1-p)u'(c_L(0))r\Big] dp + \mu + \xi = 0, && (33) \end{aligned}$$

where $c_L(R)$ and $c_L(0)$ are the state-dependent amounts of late consumption in the cases when the productive assets yields a positive return or zero return, respectively. For the first part of the lemma, rewrite (32) and (33) as:

$$\begin{aligned} \int_0^1 pu'(c_L(R))Rdp &= \int_0^1 \Big[pu'(c_L(R)) + (1-p)u'(c_L(0))\Big] dp = \\ &= r\int_0^1 \Big[pu'(c_L(R)) + (1-p)u'(c_L(0))\Big] dp + \xi. \qquad (34) \end{aligned}$$

Hence $\xi > 0$, and $D = 0$ by complementary slackness. For the second part of the lemma, assume that $\mu > 0$, so that $L + rD - \lambda c = 0$ by complementary slackness. Then, $c_L(0) = 0$ and $u'(c_L(0)) \to \infty$ by the Inada conditions. Then, for the first-order condition with respect to $L$ to hold, it has to be the case that $u'(c_L(R)) \to \infty$, hence $c_L(R) = 0$ as well. As a consequence, for the first-order condition with respect to $c$ to hold, also $u'(c) \to \infty$, hence $c = 0$, implying $L = D = 0$. However, $c_L(R) = 0$ also implies that $L + D = 1$, which leads to a contradiction. Finally, use (31) and (32) to derive (5). ■

**Proof of lemma 2**. Rewrite the derivative as:

$$\frac{\partial v_1(A,n)}{\partial n} = \sigma u'(c_L(R,n))\frac{R\left(1-L-\frac{nc}{r}\right)+L-\frac{R}{r}c(1-n)}{(1-n)^2} + (1-\sigma)u'(c_L(0,n))\frac{L}{(1-n)^2} \quad (35)$$

This expression is negative whenever:

$$\sigma u'(c_L(R,n))R\left(\frac{c}{r}-1\right) > L\Big[\sigma u'(c_L(R,n))(1-R)+(1-\sigma)u'(c_L(0,n))\Big] > Lu'(c_L(R,n))(1-R), \quad (36)$$

where the last inequality follows from the term in the square bracket being a linear combination of two terms, with $u'(c_L(0,n)) > u'(c_L(R,n))(1-R)$. Hence, the derivative is negative, provided that:

$$\sigma R\left(\frac{c}{r}-1\right) > L(1-R). \quad (37)$$

As $R > 1$ by assumption, this last expression is always true if $c > r$ (as it turns out in the banking equilibrium). ■

**Proof of lemma 3**. The threshold signal $\sigma_1^*$ is the value of $\sigma$ that makes a late consumer indifferent between waiting or running, given her posterior believes:

$$\int_\lambda^{n_1^*}\Big[\sigma_1^* u(c_L(R,n)) + (1-\sigma_1^*)u(c_L(0,n))\Big]dn + \int_{n_1^*}^{n_1^{**}} u(c_L^L(n))dn =$$
$$= \int_\lambda^{n_1^{**}} u(c)dn + \int_{n_1^{**}}^1 u\left(c^B(n)\right)dn. \quad (38)$$

Rearranging this expression, we get the threshold signal in (16), below which a late consumer runs. The derivative of the threshold signal $\sigma_1^*$ with respect to $c$ reads:

$$\frac{\partial \sigma_1^*}{\partial c} = \frac{1}{\left[\int_\lambda^{n_1^*}\Big[u(c_L(R,n)) - u(c_L(0,n))\Big]dn\right]^2}\times$$
$$\times\left[\left[(n_1^{**}-\lambda)u'(c) + \int_{n_1^*}^{n_1^{**}} u'(c_L^L(n))\frac{n}{1-n}dn\right]\left[\int_\lambda^{n_1^*}\Big[u(c_L(R,n)) - u(c_L(0,n))\Big]dn\right] + \right.$$

$$+\left[\int_{\lambda}^{n_1^*} u'(c_L(R,n))\frac{Rn}{r(1-n)}dn\right]\times$$
$$\times\left[\int_{\lambda}^{n_1^{**}} u(c)dn+\int_{n_1^{**}}^{1} u\left(c^B(n)\right)dn-\int_{\lambda}^{n_1^*} u(c_L(0,n))dn-\int_{n_1^*}^{n_1^{**}} u(c_L^L(n))dn\right]\Bigg], \tag{39}$$

which is always positive as the utility function is increasing. In a similar way, the derivative of the threshold signal $\sigma_1^*$ with respect to $L$ reads:

$$\frac{\partial\sigma_1^*}{\partial L}=\frac{1}{\left[\int_{\lambda}^{n_1^*}\left[u(c_L(R,n))-u(c_L(0,n))\right]dn\right]^2}\times$$
$$\times\left[\left[\int_{n_1^{**}}^{1} u'(c^B(n))\frac{1-r}{n}dn-\int_{\lambda}^{n_1^*} u'(c_L(0,n))\frac{1}{1-n}dn-\int_{n_1^*}^{n_1^{**}} u'(c_L^L(n))\frac{1-r}{1-n}dn\right]\times\right.$$
$$\times\left[\int_{\lambda}^{n_1^*}\left[u(c_L(R,n))-u(c_L(0,n))\right]dn\right]+$$
$$-\left[\int_{\lambda}^{n_1^*}\left[u'(c_L(R,n))\frac{1-R}{1-n}-u'(c_L(0,n))\frac{1}{1-n}\right]dn\right]\times$$
$$\times\left[\int_{\lambda}^{n_1^{**}} u(c)dn+\int_{n_1^{**}}^{1} u\left(c^B(n)\right)dn-\int_{\lambda}^{n_1^*} u(c_L(0,n))dn-\int_{n_1^*}^{n_1^{**}} u(c_L^L(n))dn\right]\Bigg]. \tag{40}$$

Notice that $\lim_{n\to n_1^{**}} u'(c_L^L(n))=\lim_{c\to 0} u'(c)$, which is equal to $+\infty$ by the Inada conditions. Hence, this expression is negative. This ends the proof. ■

**Proof of lemma 4**. By definition, $u(c)$ is strictly concave on an open internal X if and only if:

$$u(x)-u(y)<u'(y)(x-y) \tag{41}$$

for all x and y in X. Hence, when $\lambda\leq n<n_2^*$, it must be the case that:

$$\begin{aligned}\frac{\partial v_2(A,n)}{\partial n}&<\sigma\frac{u(c_L(R,n))-u(c)}{1-n}-(1-\sigma)\frac{u(c)-u(c_L(0,n))}{1-n}\\&<\frac{\sigma u(c_L(R,n))+(1-\sigma)u(c_L(0,n))-u(c)}{1-n}=\frac{v_2(A,n)}{1-n}\end{aligned} \tag{42}$$

Thus, whenever $v_2(A,n)\leq 0$, the derivative is negative. This ends the proof. ■

**Proof of lemma 5**. The threshold signal $\sigma_2^*$ is the value of $\sigma$ that equalizes:

$$\int_{\lambda}^{n_2^*}\left[\sigma_2^* u\left(\frac{R(1-L)+L-nc}{1-n}\right)+(1-\sigma_2^*)u\left(\frac{L-nc}{1-n}\right)\right]dn+ \\ +\int_{n_2^*}^{n^{**}}\sigma_2^* u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn=\int_{\lambda}^{n^{**}}u(c)dn+\int_{n^{**}}^{1}u\left(\frac{L+r(1-L)}{n}\right)dn \tag{43}$$

Rearranging this expression, we get the threshold signal in (24), below which a late consumer runs. The derivative of the threshold signal $\sigma_2^*$ with respect to $c$ reads:

$$\frac{\partial\sigma_2^*}{\partial c}=\frac{1}{\left[\int_{\lambda}^{n_2^*}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right)-u\left(\frac{L-nc}{1-n}\right)\right]dn+\int_{n_2^*}^{n^{**}}u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn\right]^2}\times \\ \times\left[\left[(n^{**}-\lambda)u'(c)+\int_{\lambda}^{n_2^*}u'(c_L(0,n))\frac{n}{1-n}dn\right]\right. \\ \left[\int_{\lambda}^{n_2^*}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right)-u\left(\frac{L-nc}{1-n}\right)\right]dn+\int_{n_2^*}^{n^{**}}u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn\right]+ \\ -\left[\int_{\lambda}^{n_2^*}\left[u'(c_L(0,n))-u'(c_L(R,n))\right]\frac{n}{1-n}dn-\int_{n_2^*}^{n^{**}}u'(c_L^D(R,n))\frac{Rn}{r(1-n)}dn\right]\times \\ \left.\times\left[\int_{\lambda}^{n^{**}}u(c)dn+\int_{n^{**}}^{1}u\left(c^B(n)\right)dn-\int_{\lambda}^{n_2^*}u\left(\frac{L-nc}{1-n}\right)dn\right]\right]= \\ =\frac{1}{\left[\int_{\lambda}^{n_2^*}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right)-u\left(\frac{L-nc}{1-n}\right)\right]dn+\int_{n_2^*}^{n^{**}}u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn\right]^2}\times \\ \times\left[(n^{**}-\lambda)u'(c)\times\right. \\ \times\left[\int_{\lambda}^{n_2^*}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right)-u\left(\frac{L-nc}{1-n}\right)\right]dn+\int_{n_2^*}^{n^{**}}u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn\right]+ \\ +\left[\int_{\lambda}^{n_2^*}u'(c_L(R,n))\frac{n}{1-n}dn+\int_{n_2^*}^{n^{**}}u'(c_L^D(R,n))\frac{Rn}{r(1-n)}d\right]\times \\ \times\left[\int_{\lambda}^{n^{**}}u(c)dn+\int_{n^{**}}^{1}u\left(c^B(n)\right)dn-\int_{\lambda}^{n_2^*}u\left(\frac{L-nc}{1-n}\right)dn\right]+ \\ +\left[\int_{\lambda}^{n_2^*}u'(c_L(0,n))\frac{n}{1-n}dn\right]\times \\ \times\left[\int_{\lambda}^{n_2^*}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right)-u\left(\frac{L-nc}{1-n}\right)\right]dn+\int_{n_2^*}^{n^{**}}u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn+\right. \\ \left.\left.-\int_{\lambda}^{n^{**}}u(c)dn-\int_{n^{**}}^{1}u\left(c^B(n)\right)dn+\int_{\lambda}^{n_2^*}u\left(\frac{L-nc}{1-n}\right)dn\right]\right]. \tag{44}$$

This derivative is positive, because all terms are positive, and the expression in the last square brackets is the difference between the denominator and the numerator of $\sigma_2^*$, which must be non-negative as $\sigma_2^* \leq 1$. The derivative of the threshold signal $\sigma_2^*$ with respect to $L$ instead reads:

$$
\begin{aligned}
\frac{\partial \sigma_2^*}{\partial L} = & \frac{1}{\left[\int_\lambda^{n_2^*}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right) - u\left(\frac{L-nc}{1-n}\right)\right]dn + \int_{n_2^*}^{n^{**}} u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn\right]^2} \times \\
& \times \left[\left[\int_{n^{**}}^1 u'(c^B(n))\frac{1-r}{n}dn - \int_\lambda^{n_2^*} u'(c_L(0,n))\frac{1}{1-n}dn\right]\right. \\
& \left[\int_\lambda^{n_2^*}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right) - u\left(\frac{L-nc}{1-n}\right)\right]dn + \int_{n_2^*}^{n_2^{**}} u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn\right] + \\
& - \left[\int_\lambda^{n_2^*} u'(c_L(R,n))\frac{1-R}{1-n}dn - \int_\lambda^{n_2^*} u'(c_L(0,n))\frac{1}{1-n}dn + \right. \\
& \left. + \int_{n_2^*}^{n^{**}} u'(c_L^D(R,n))R\left(\frac{1}{r}-1\right)\frac{1}{1-n}dn\right] \times \\
& \left. \times \left[\int_\lambda^{n^{**}} u(c)dn + \int_{n^{**}}^1 u\left(c^B(n)\right)dn - \int_\lambda^{n_2^*} u\left(\frac{L-nc}{1-n}\right)dn\right]\right].
\end{aligned} \tag{45}
$$

Clearly, $\lim_{n\to n^{**}} u'(c_L^D(R,n)) = \lim_{c\to 0} u'(c) = +\infty$ by the Inada conditions. Hence, this expression is negative. This ends the proof. ■

**Proof of Proposition 1**. We study $\sigma_1^*$ and $\sigma_2^*$ as functions of the recovery rate $r$. As $r \to \underline{r} = \frac{\lambda c}{1-L}$, we have that $n_1^* \to \lambda$ and the first interval of $v_1(A,n)$ reduces to zero. Thus, the expected value of waiting versus running under the pecking order {Liquidation, Liquidity} becomes:

$$
\mathbb{E}[v_1(A,n)] = \int_\lambda^{n^{**}}\left[u\left(\frac{L+r(1-L)-nc}{1-n}\right) - u(c)\right]dn - \int_{n^{**}}^1 u\left(\frac{L+r(1-L)}{n}\right)dn. \tag{46}
$$

This expression is always negative, as the numerator of $\sigma_1^*$ must be positive. Hence, $\sigma_1^*$ is constant and equal to 1 in the interval $[0,\underline{r}]$. In the interval $[\underline{r},1]$, instead, the threshold signal $\sigma_1^*$ is a decreasing and convex function of the recovery rate $r$. To see that, calculate:

$$
\frac{\partial \sigma_1^*}{\partial r} = \frac{1}{\left[\int_\lambda^{n_1^*}\left[u(c_L(R,n)) - u(c_L(0,n))\right]dn\right]^2} \times
$$

$$\times \left[\left[\int_{n^{**}}^{1} u'(c^B(n))\frac{1-L}{n}dn - \int_{n_1^*}^{n^{**}} u'(c_L^L(n))\frac{1-L}{1-n}dn\right] \times\right.$$
$$\times \left[\int_{\lambda}^{n_1^*} \Big[u(c_L(R,n)) - u(c_L(0,n))\Big] dn\right] +$$
$$- \left[-\int_{\lambda}^{n_1^*} u'(c_L(R,n))\frac{Rnc}{r^2(1-n)}dn\right] \times$$
$$\left.\times \left[\int_{\lambda}^{n_1^{**}} u(c)dn + \int_{n_1^{**}}^{1} u\left(c^B(n)\right) dn - \int_{\lambda}^{n_1^*} u(c_L(0,n))dn - \int_{n_1^*}^{n_1^{**}} u(c_L^L(n))dn\right]\right]. \tag{47}$$

By the Inada conditions, we know that $\lim_{n\to n^{**}} u'(c_L^L(n)) = \lim_{c\to 0} u'(c) = +\infty$. Hence, the derivative must be negative.[5]

To show that the threshold signal $\sigma_1^*$ is instead a convex function of $r$, calculate:

$$\frac{\partial^2\sigma_1^*}{\partial r^2} = \frac{1}{\left[\int_{\lambda}^{n_1^*}\Big[u(c_L(R,n)) - u(c_L(0,n))\Big] dn\right]^4} \times$$
$$\times \left[\left[\left[-\frac{1-L}{c}u'(c)\frac{1-L}{n^{**}} + \int_{n^{**}}^{1} u''(c^B(n))\left(\frac{1-L}{c}\right)^2 dn +\right.\right.\right.$$
$$\left.- \int_{n_1^*}^{n^{**}} u''(c_L^L(n))\left(\frac{1-L}{1-n}\right)^2 dn + \frac{1-L}{c}u'\left(\frac{L}{1-n}\right)\frac{1-L}{1-n_1^*}\right]\left[\int_{\lambda}^{n_1^*}\Big[u(c_L(R,n)) - u(c_L(0,n))\Big] dn\right] +$$
$$+ \left[\int_{\lambda}^{n_1^*} u'(c_L(R,n))\frac{Rnc}{r^2(1-n)}dn\right]\left[\int_{n^{**}}^{1} u'(c^B(n))\frac{1-L}{n}dn - \int_{n_1^*}^{n^{**}} u'(c_L^L(n))\frac{1-L}{1-n}dn\right] +$$
$$- \left[\frac{1-L}{c}u'\left(\frac{L}{1-n_1^*}\right)\frac{R(1-L)}{r(1-n_1^*)}dn + \int_{\lambda}^{n_1^*}\left[u''(c_L(R,n))\left(\frac{Rnc}{r^2(1-n)}\right)^2 - 2u'(c_L(R,n))\frac{Rnc}{r^3(1-n)}\right] dn\right] \times$$
$$\left.\times \left[\int_{\lambda}^{n_1^{**}} u(c)dn + \int_{n_1^{**}}^{1} u\left(c^B(n)\right) dn - \int_{\lambda}^{n_1^*} u(c_L(0,n))dn - \int_{n_1^*}^{n_1^{**}} u(c_L^L(n))dn\right] +\right.$$
$$\left.- \left[\int_{\lambda}^{n_1^*} u'(c_L(R,n))\frac{Rnc}{r^2(1-n)}dn\right]\left[\int_{n^{**}}^{1} u'(c^B(n))\frac{1-L}{n}dn - \int_{n_1^*}^{n^{**}} u'(c_L^L(n))\frac{1-L}{1-n}dn\right]\right] \times$$
$$\times \left[\int_{\lambda}^{n_1^*}\Big[u(c_L(R,n)) - u(c_L(0,n))\Big] dn\right]^2 +$$
$$- \left[\left[\int_{n^{**}}^{1} u'(c^B(n))\frac{1-L}{n}dn - \int_{n_1^*}^{n^{**}} u'(c_L^L(n))\frac{1-L}{1-n}dn\right]\left[\int_{\lambda}^{n_1^*}\Big[u(c_L(R,n)) - u(c_L(0,n))\Big] dn\right] +\right.$$
$$- \left[\int_{\lambda}^{n_1^{**}} u(c)dn + \int_{n_1^{**}}^{1} u\left(c^B(n)\right) dn - \int_{\lambda}^{n_1^*} u(c_L(0,n))dn - \int_{n_1^*}^{n_1^{**}} u(c_L^L(n))dn\right] \times$$
$$\left.\times \left[\int_{\lambda}^{n_1^*} u'(c_L(R,n))\frac{Rnc}{r^2(1-n)}dn\right]\right] \times$$
$$\times\, 2\left[\int_{\lambda}^{n_1^*}\Big[u(c_L(R,n)) - u(c_L(0,n))\Big] dn\right]\int_{\lambda}^{n_1^*} u'(c_L(R,n))\frac{Rnc}{r^2(1-n)}dn\Bigg]. \tag{48}$$

[5] Notice that, for any pecking order $j$, $v_j(A,n)$ is continuous everywhere, but has kinks at $n_j^*$ $n^{**}$, so it is not differentiable at those points.

By definition of CRRA utility:

$$\frac{u''(c)}{u'(c)} = -\frac{\gamma}{c}, \tag{49}$$

where $\gamma$ is the constant coefficient of relative risk aversion. This implies that:

$$\lim_{n \to n^{**}} \frac{u''(c_L^L(n))}{u'(c_L^L(n))} = \lim_{c \to 0} \frac{u''(c)}{u'(c)} = -\lim_{c \to 0} \frac{\gamma}{c} = -\infty. \tag{50}$$

Hence, $u''(c)$ goes to $-\infty$ at a speed faster than the one at which $u'(c)$ goes to $+\infty$, when $c \to 0$. This, together with the Inada conditions, ensures that the second derivative is positive, meaning that $\sigma_1^*$ is a convex function of $r$.

In contrast, $\sigma_2^*$ at $r = 0$ is always lower than 1 if $R$ is sufficiently large. To see that, notice that:

$$\sigma_2^*(r=0) = \frac{\int_\lambda^{\frac{L}{c}} u(c)dn + \int_{\frac{L}{c}}^1 u\left(\frac{L}{n}\right)dn - \int_\lambda^{\frac{L}{c}} u\left(\frac{L-nc}{1-n}\right)dn}{\int_\lambda^{\frac{L}{c}} \left[u\left(\frac{R(1-L)+L-nc}{1-n}\right) - u\left(\frac{L-nc}{1-n}\right)\right]dn}. \tag{51}$$

This expression is lower than 1 if:

$$\int_\lambda^{\frac{L}{c}} u(c)dn + \int_{\frac{L}{c}}^1 u\left(\frac{L}{n}\right)dn < \int_\lambda^{\frac{L}{c}} u\left(\frac{R(1-L)+L-nc}{1-n}\right)dn. \tag{52}$$

This condition is true if $R$ is sufficiently high, given that $R > c$ must hold. We show that in an example with CRRA utility $u(c) = \frac{(c+\psi)^{1-\gamma}-\psi^{1-\gamma}}{1-\gamma}$, where $\psi$ is a positive but negligible constant (we assume $\psi \simeq 0$) that ensures that $u(0) = 0$, and $\gamma > 1$ is the coefficient of relative risk aversion.[6] Rewrite (52) as:

$$\int_\lambda^{\frac{L}{c}} \left[\frac{\left(\frac{R(1-L)+L-nc}{1-n}\right)^{1-\gamma}}{\gamma-1} - \frac{c^{1-\gamma}}{\gamma-1}\right]dn - \int_{\frac{L}{c}}^1 \frac{\left(\frac{L}{n}\right)^{1-\gamma}}{\gamma-1}dn < 0. \tag{53}$$

This is equivalent to:

$$\int_\lambda^{\frac{L}{c}} \left[\left(\frac{(1-L)+\frac{L-nc}{R}}{1-n}\right)^{1-\gamma} R^{1-\gamma} - c^{1-\gamma}\right]dn - \int_{\frac{L}{c}}^1 \left(\frac{L}{n}\right)^{1-\gamma} dn < 0. \tag{54}$$

[6] The parameter $\psi$ can be interpreted as a minimum level of consumption that the depositors can enjoy, for example from an extra endowment that they can consume at date 1 or 2, but not deposit in the bank at date 0. The proof would hold even for positive but non-negligible values of the $\psi$, as long as they are lower than $R$.

Multiply the previous expression by $R^{\gamma-1}$, and rewrite it as:

$$\int_{\lambda}^{\frac{L}{c}} \left[ \left( \frac{(1-L) + \frac{L-nc}{R}}{1-n} \right)^{1-\gamma} - \left( \frac{c}{R} \right)^{1-\gamma} \right] dn - \int_{\frac{L}{c}}^{1} \left( \frac{L}{n} \right)^{1-\gamma} R^{\gamma-1} dn < 0. \tag{55}$$

As $R > c$, $\frac{c}{R}$ is bounded. Therefore, this condition is always satisfied for $R \to \infty$, as the last integral goes to $-\infty$. By continuity, there must be a sufficiently large and finite value of $R$ such that this is also true.

Having proved that the threshold signal $\sigma_2^* < 1$ at $r = 0$, we want to show that it is also a decreasing and convex function of the recovery rate $r$. To this end, we first calculate:

$$\begin{aligned}
\frac{\partial \sigma_2^*}{\partial r} = & \frac{1}{\left[ \int_{\lambda}^{n_2^*} \left[ u\left( \frac{R(1-L)+L-nc}{1-n} \right) - u\left( \frac{L-nc}{1-n} \right) \right] dn + \int_{n_2^*}^{n^{**}} u\left( \frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n} \right) dn \right]^2} \times \\
& \times \left[ \left[ \int_{n^{**}}^{1} u'(c^B(n)) \frac{1-L}{n} dn \right] \times \right. \\
& \times \left[ \int_{\lambda}^{n_2^*} \left[ u\left( \frac{R(1-L)+L-nc}{1-n} \right) - u\left( \frac{L-nc}{1-n} \right) \right] dn + \int_{n_2^*}^{n^{**}} u\left( \frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n} \right) dn \right] + \\
& - \left[ \int_{n_2^*}^{n^{**}} u'\left( \frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n} \right) \frac{Rnc}{r^2(1-n)} dn \right] \times \\
& \times \left. \left[ \int_{\lambda}^{n^{**}} u(c) dn + \int_{n^{**}}^{1} u\left(c^B(n)\right) dn - \int_{\lambda}^{n_2^*} u\left( \frac{L-nc}{1-n} \right) dn \right] \right].
\end{aligned} \tag{56}$$

By the same considerations as before regarding the Inada conditions, notice that:

$$\lim_{n \to n^{**}} u'\left( \frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n} \right) = +\infty. \tag{57}$$

Hence, the derivative must be negative.

To show that the threshold signal $\sigma_2^*$ is instead a convex function of $r$, calculate:

$$
\begin{aligned}
\frac{\partial^2\sigma_2^*}{\partial r^2} = & \frac{1}{\left[\int_\lambda^{n_2^*}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right)-u\left(\frac{L-nc}{1-n}\right)\right]dn+\int_{n_2^*}^{n^{**}}u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn\right]^4}\times \\
& \times\left[\left[\left[-\frac{1-L}{c}u'(c)\frac{1-L}{n^{**}}+\int_{n^{**}}^1 u''(c^B(n))\left(\frac{1-L}{n}\right)^2 dn\right]\times\right.\right. \\
& \times\left[\int_\lambda^{n_2^*}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right)-u\left(\frac{L-nc}{1-n}\right)\right]dn+\int_{n_2^*}^{n^{**}}u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn\right]+ \\
& +\left[\int_{n^{**}}^1 u'(c^B(n))\frac{1-L}{n}dn\right]\left[\int_{n_2^*}^{n^{**}}u'\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)\frac{Rnc}{r^2(1-n)}dn\right]+ \\
& -\left[\int_\lambda^{n^{**}}u(c)dn+\int_{n^{**}}^1 u\left(c^B(n)\right)dn-\int_\lambda^{n_2^*}u\left(\frac{L-nc}{1-n}\right)dn\right]\times \\
& \times\left[\int_{n_2^*}^{n^{**}}\left[u''\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)\left(\frac{Rnc}{r^2(1-n)}\right)^2-2u'\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)\frac{Rnc}{r^3(1-n)}\right]dn\right]+ \\
& -\left[\int_{n_2^*}^{n^{**}}u'\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)\frac{Rnc}{r^2(1-n)}dn\right]\left.\left[\int_{n^{**}}^1 u'(c^B(n))\frac{1-L}{n}dn\right]\right]\times \\
& \times\left[\int_\lambda^{n_2^*}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right)-u\left(\frac{L-nc}{1-n}\right)\right]dn+\int_{n_2^*}^{n^{**}}u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn\right]^2+ \\
& -2\left[\int_\lambda^{n_2^*}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right)-u\left(\frac{L-nc}{1-n}\right)\right]dn+\int_{n_2^*}^{n^{**}}u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn\right]\times \\
& \times\left[\int_{n_2^*}^{n^{**}}u'\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)\frac{Rnc}{r^2(1-n)}dn\right]\times \\
& \times\left[\left[\int_{n^{**}}^1 u'(c^B(n))\frac{1-L}{n}dn\right]\times\right. \\
& \times\left[\int_\lambda^{n_2^*}\left[u\left(\frac{R(1-L)+L-nc}{1-n}\right)-u\left(\frac{L-nc}{1-n}\right)\right]dn+\int_{n_2^*}^{n^{**}}u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)dn\right]+ \\
& -\left[\int_{n_2^*}^{n^{**}}u'\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right)\frac{Rnc}{r^2(1-n)}dn\right]\times \\
& \times\left.\left.\left[\int_\lambda^{n^{**}}u(c)dn+\int_{n^{**}}^1 u\left(c^B(n)\right)dn-\int_\lambda^{n_2^*}u\left(\frac{L-nc}{1-n}\right)dn\right]\right]\right]. \qquad (58)
\end{aligned}
$$

By the same consideration regarding the Inada conditions, we get that this derivative is positive, meaning that $\sigma_2^*$ is a convex function of $r$.

Being the two threshold signals $\sigma_1^*$ and $\sigma_2^*$ both decreasing and convex functions of the recovery rate $r$, to prove that they cross only once in the interval $[0,1]$ it suffices to prove that

$\sigma_2^* > \sigma_1^*$ at $r = 1$:

$$\sigma_1^*(r=1) = \frac{\int_\lambda^{\frac{1}{c}} u(c)dn + \int_{\frac{1}{c}}^{1} u\left(\frac{1}{n}\right)dn - \int_\lambda^{\frac{1-L}{c}} u\left(\frac{L}{1-n}\right)dn - \int_{\frac{1-L}{c}}^{\frac{1}{c}} u\left(\frac{1-nc}{1-n}\right)dn}{\int_\lambda^{\frac{1-L}{c}} \left[u\left(\frac{R(1-L-nc)+L}{1-n}\right) - u\left(\frac{L}{1-n}\right)\right]dn}, \tag{59}$$

$$\sigma_2^*(r=1) = \frac{\int_\lambda^{\frac{1}{c}} u(c)dn + \int_{\frac{1}{c}}^{1} u\left(\frac{1}{n}\right)dn - \int_\lambda^{\frac{L}{c}} u\left(\frac{L-nc}{1-n}\right)dn}{\int_\lambda^{\frac{L}{c}} \left[u\left(\frac{R(1-L)+L-nc}{1-n}\right) - u\left(\frac{L-nc}{1-n}\right)\right]dn + \int_{\frac{L}{c}}^{\frac{1}{c}} u\left(\frac{R(1-nc)}{1-n}\right)dn}. \tag{60}$$

Define as $NUM_j$ and $DEN_j$ the numerator and denominator of $\sigma_j^*$, respectively, for any pecking order $j = \{1, 2\}$. The following relationship holds:

$$NUM_1 = NUM_2 + \int_\lambda^{n_2^*} u\left(\frac{L-nc}{1-n}\right)dn - \int_\lambda^{n_1^*} u\left(\frac{L}{1-n}\right)dn + \\ - \int_{n_1^*}^{n^{**}} u\left(\frac{r(1-L)+L-nc}{1-n}\right)dn. \tag{61}$$

As a preliminary, step, we want to show that:

$$H \equiv \int_\lambda^{n_2^*} u\left(\frac{L-nc}{1-n}\right)dn - \int_\lambda^{n_1^*} u\left(\frac{L}{1-n}\right)dn - \int_{n_1^*}^{n^{**}} u\left(\frac{r(1-L)+L-nc}{1-n}\right)dn \tag{62}$$

is negative. If $n_1^* \leq n_2^*$, the previous expression can be re-written as:

$$H = \int_\lambda^{n_1^*} u\left(\frac{L-nc}{1-n}\right)dn + \int_{n_1^*}^{n_2^*} u\left(\frac{L-nc}{1-n}\right)dn - \int_\lambda^{n_1^*} u\left(\frac{L}{1-n}\right)dn + \\ - \int_{n_1^*}^{n_2^*} u\left(\frac{r(1-L)+L-nc}{1-n}\right)dn - \int_{n_2^*}^{n^{**}} u\left(\frac{r(1-L)+L-nc}{1-n}\right)dn, \tag{63}$$

which is clearly negative. In a similar way, if $n_1^* > n_2^*$, we can re-write:

$$H = \int_\lambda^{n_2^*} u\left(\frac{L-nc}{1-n}\right)dn - \int_\lambda^{n_2^*} u\left(\frac{L}{1-n}\right)dn - \int_{n_2^*}^{n_1^*} u\left(\frac{L}{1-n}\right)dn + \\ - \int_{n_1^*}^{n^{**}} u\left(\frac{r(1-L)+L-nc}{1-n}\right)dn, \tag{64}$$

which, again, is always negative. Thus, $NUM_1 < NUM_2$. Given this result, a sufficient condition for $\sigma_2^* \geq \sigma_1^*$ is $DEN_2 \leq DEN_1$, or:

$$f(c,L) \equiv \int_{\lambda}^{\frac{L}{c}} \left[ u\left(\frac{R(1-L)+L-nc}{1-n}\right) - u\left(\frac{L-nc}{1-n}\right)\right] dn + \int_{\frac{L}{c}}^{\frac{1}{c}} u\left(\frac{R(1-nc)}{1-n}\right) dn + \\ - \int_{\lambda}^{\frac{1-L}{c}} \left[ u\left(\frac{R(1-L-nc)+L}{1-n}\right) - u\left(\frac{L}{1-n}\right)\right] dn \leq 0. \tag{65}$$

We study how $f(c,L)$ changes with $c$ and $L$. On the one side:

$$\frac{\partial f(c,L)}{\partial c} = -\int_{\lambda}^{\frac{L}{c}} \left[ u'\left(\frac{R(1-L)+L-nc}{1-n}\right) - u'\left(\frac{L-nc}{1-n}\right)\right] \frac{n}{1-n} dn + \\ - \int_{\frac{L}{c}}^{\frac{1}{c}} u'\left(\frac{R(1-nc)}{1-n}\right) \frac{Rn}{1-n} dn + \int_{\lambda}^{\frac{1-L}{c}} u'\left(\frac{R(1-L-nc)+L}{1-n}\right) \frac{Rn}{1-n} dn \tag{66}$$

The sign of this derivative is positive. To see that, notice that $\frac{R(1-nc)}{1-n} > \frac{L-nc}{1-n}$. Hence, by the fact that the coefficient of relative risk aversion is larger than 1:[7]

$$\frac{u'\left(\frac{R(1-nc)}{1-n}\right)}{u'\left(\frac{L-nc}{1-n}\right)} < \frac{L-nc}{R(1-nc)}, \tag{67}$$

and this implies that:

$$u'\left(\frac{R(1-nc)}{1-n}\right) \frac{Rn}{1-n} < u'\left(\frac{L-nc}{1-n}\right) \frac{n}{1-n} \frac{L-nc}{1-nc} < u'\left(\frac{L-nc}{1-n}\right) \frac{n}{1-n}. \tag{68}$$

On the other side:

$$\frac{\partial f(c,L)}{\partial L} = -\int_{\lambda}^{\frac{L}{c}} \left[ u'\left(\frac{R(1-L)+L-nc}{1-n}\right)(R-1) + u'\left(\frac{L-nc}{1-n}\right)\right] \frac{1}{1-n} dn + \\ + \int_{\lambda}^{\frac{1-L}{c}} \left[ u'\left(\frac{R(1-L-nc)+L}{1-n}\right)(R-1) + u'\left(\frac{L}{1-n}\right)\right] \frac{1}{1-n} dn, \tag{69}$$

which is negative, because of the Inada Conditions, that make the second integral in the first line become large and negative. Since $f(c,L)$ is increasing in $c$ and decreasing in $L$, a sufficient condition for it to be less than or equal to zero everywhere is that it is less than or equal to zero at $L^{\min} = \lambda$ and $c^{\max}$ when $L = \lambda$, which is $c^{\max} = 1$. At those points, the condition $f(c,L) \leq 0$

[7] To see that, rewrite $-\frac{u''(c)c}{u'(c)} > 1$ as $-\frac{u''(c)}{u'(c)} > \frac{1}{c}$. This, in turn, means that $-(log[u'(c)])' > (log[c])'$. Integrate between $z_1$ and $z_2 > z_1$ so as to obtain $log[u'(z_1)] - log[u'(z_2)] > log[z_2] - log[z_1]$. Once taken the exponent, the last expression gives $\frac{u'(z_1)}{u'(z_2)} > \frac{z_2}{z_1}$. If $z_1 > z_2$, the inequality is reversed.

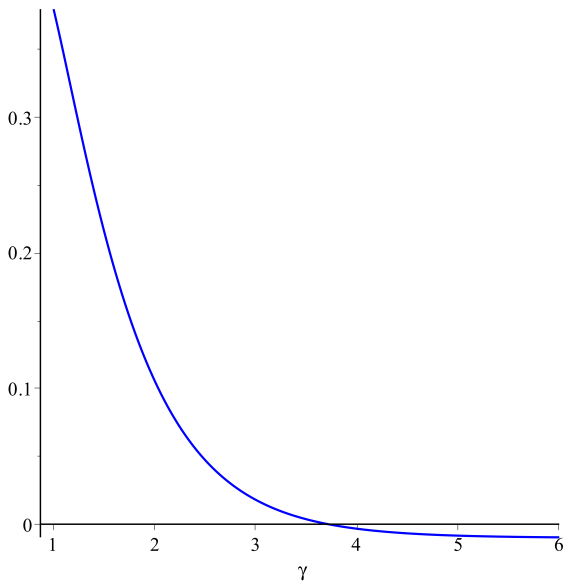


Figure 4: The condition (70) as a function of the coefficient of relative risk aversion.

reads:

$$u(R)(1-\lambda) - \int_{\lambda}^{1-\lambda} \left[ u\left(\frac{R(1-\lambda-n)+\lambda}{1-n}\right) - u\left(\frac{\lambda}{1-n}\right)\right] dn \leq 0. \tag{70}$$

We show how this expression relates to the coefficient of relative risk aversion in a numerical example. We assume that $u(c) = \frac{(c+\psi)^{1-\gamma}+\psi^{1-\gamma}}{1-\gamma}$, with $\psi = 2$ and $\gamma > 1$. Moreover, we pick $R = 2.01$ and $\lambda = .01$.[8] Figure 4 shows that condition (70) holds for high values of the coefficient of relative risk aversion. This ends the proof. ■

**Proof of Lemma 6**. Attach the Lagrange multipliers $\mu$ and $\xi$ to the liquidity constraint and the incentive compatibility constraint, respectively. The first-order conditions of the program reads:

$$c: \quad -\frac{\partial \sigma_2^*}{\partial c}\Delta U(c,L) + \lambda \int_{\sigma_2^*}^{1} \left[u'(c) - \left[pu'(c_L(R)) + (1-p)u'(c_L(0))\right]\right] dp - \lambda\mu - \xi = 0, \tag{71}$$

$$L: \quad -\frac{\partial \sigma_2^*}{\partial L}\Delta U(c,L) + \sigma_2^*(1-r)u'(L+r(1-L)) + $$
$$+ \int_{\sigma_2^*}^{1} \left[pu'(c_L(R))(1-R) + (1-p)u'(c_L(0))\right] dp + \mu + \xi(1-R) = 0. \tag{72}$$

Clearly, as in the proof for the equilibrium with perfect information, a binding liquidity constraint (with the Lagrange multiplier $\mu$ strictly positive) cannot satisfy the equilibrium condi-

[8]The results are robust to different choices of the parameters.

tions: in fact, if that was not the case, by complementary slackness we would have $L = \lambda c$ and $c_L(0) = 0$. Plugging (71) into (72) gives:

$$\int_{\sigma_2^*}^{1} \Big[u'(c) - pRu'(c_L(R))\Big] dp + \sigma_2^*(1-r)u'(L + r(1-L)) =$$
$$= \left[\frac{\partial \sigma_2^*}{\partial L} + \frac{1}{\lambda}\frac{\partial \sigma_2^*}{\partial c}\right] \Delta U(c, L) + \xi \left(\frac{1}{\lambda} + R - 1\right). \quad (73)$$

In equilibrium, either the incentive compatibility constraint is binding or it is not. However, a binding constraint would mean that $c = c_L(R)$, and this in turns would imply that $\underline{\sigma}_2 = 1$, which cannot be an equilibrium. Hence, by complementary slackness, $\xi$ must be equal to zero, and (73) boils down to the distorted Euler equation (28). A point to prove for the sign of the strategic complementarity in the interval $[\lambda, n_2^*]$ was that $\frac{R(1-L)}{c-L} > 1$. Using the previous inequality and the concavity of $u(c)$, this is satisfied by the incentive compatibility constraint. Moreover, this also implies that $c < R$, thus confirming the condition for the existence of the upper dominance region. This ends the proof. ■

**Proof of Proposition 2**. In order to characterize the sign of the distortion in the Euler equation, we start by deriving the sign of the sum of the marginal effects, for the pecking order {Liquidity; Liquidation}:

$$\left[\frac{\partial \sigma_2^*}{\partial L} + \frac{1}{\lambda}\frac{\partial \sigma_2^*}{\partial c}\right] = \frac{1}{\left[\int_{\lambda}^{n_2^*} \left[u\left(\frac{R(1-L)+L-nc}{1-n}\right) - u\left(\frac{L-nc}{1-n}\right)\right] dn + \int_{n_2^*}^{n^{**}} u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right) dn\right]^2} \times$$
$$\times \left[\left[\int_{n^{**}}^{1} u'(c^B(n)) \frac{1-r}{n} dn + \frac{1}{\lambda}(n^{**} - \lambda) u'(c)\right] \times\right.$$
$$\times \left[\int_{\lambda}^{n_2^*} \left[u\left(\frac{R(1-L)+L-nc}{1-n}\right) - u\left(\frac{L-nc}{1-n}\right)\right] dn + \int_{n_2^*}^{n^{**}} u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right) dn\right] +$$
$$+ \left[\int_{\lambda}^{n^{**}} u(c) dn + \int_{n^{**}}^{1} u\left(c^B(n)\right) dn - \int_{\lambda}^{n_2^*} u\left(\frac{L-nc}{1-n}\right) dn\right] \times$$
$$\times \left[\int_{\lambda}^{n_2^*} u'(c_L(R,n)) \frac{R-1+\frac{n}{\lambda}}{1-n} dn + \int_{n_2^*}^{n^{**}} u'(c_L^D(R,n)) \frac{R}{r(1-n)} \left(\frac{n}{\lambda} - 1 + r\right) dn\right] +$$
$$+ \left[\int_{\lambda}^{n_2^*} \left[u\left(\frac{R(1-L)+L-nc}{1-n}\right) - u\left(\frac{L-nc}{1-n}\right)\right] dn + \int_{n_2^*}^{n^{**}} u\left(\frac{R\left(1-L-\frac{nc-L}{r}\right)}{1-n}\right) dn +\right.$$
$$\left.\left.- \int_{\lambda}^{n^{**}} u(c) dn - \int_{n^{**}}^{1} u\left(c^B(n)\right) dn + \int_{\lambda}^{n_2^*} u\left(\frac{L-nc}{1-n}\right) dn\right] \int_{\lambda}^{n_2^*} u'(c_L(0,n)) \frac{\frac{n}{\lambda} - 1}{1-n} dn\right]. \quad (74)$$

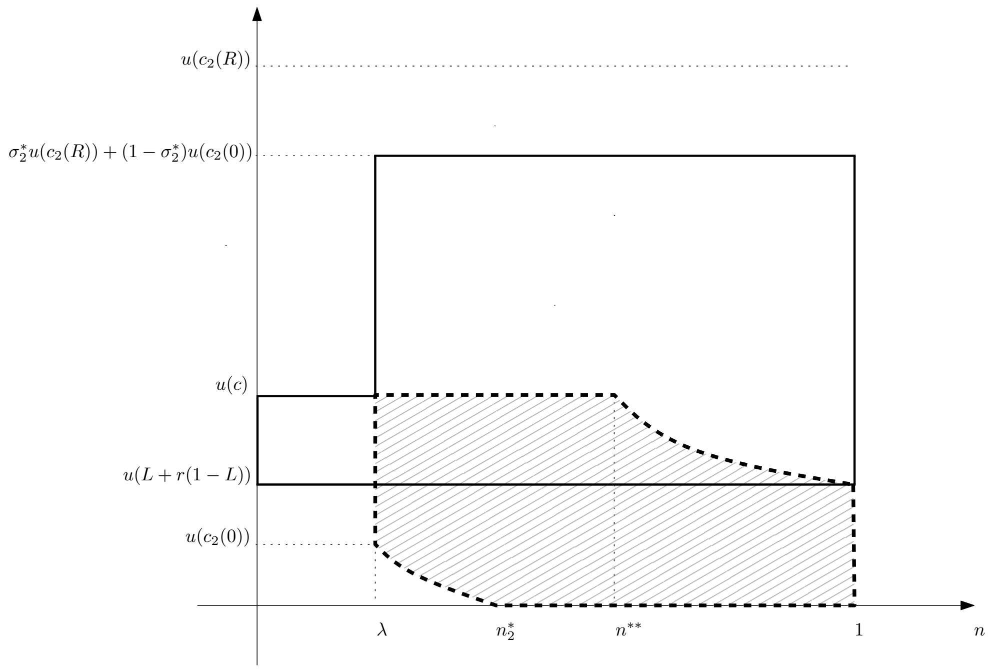


Figure 5: The condition under which the inequality (76) holds.

This expression is positive because of the Inada conditions, and because $n \geq \lambda$. We rearrange the distorted Euler equation (28) and rewrite:

$$\begin{aligned}
\left[\frac{\partial \sigma_2^*}{\partial L} + \frac{1}{\lambda}\frac{\partial \sigma_2^*}{\partial c}\right] - \sigma_2^*(1-r)u'(L+r(1-L)) &= \frac{\Delta U(c,L)}{DEN_2}\int_{n^{**}}^{1} u'\left(\frac{L+r(1-L)}{n}\right)\frac{1-r}{n}dn + \\
&- \sigma_2^*(1-r)u'(L+r(1-L)) + \cdots = \\
&= \lim_{\epsilon\to 0}\frac{\Delta U(c,L)}{DEN_2}\left[\int_{n^{**}}^{1-\epsilon} u'\left(\frac{L+r(1-L)}{n}\right)\frac{1-r}{n}dn + \int_{\epsilon}^{1} u'\left(\frac{L+r(1-L)}{n}\right)\frac{1-r}{n}dn\right] + \\
&- \sigma_2^*(1-r)u'(L+r(1-L)) + \cdots = \\
&= \lim_{\epsilon\to 0}\frac{\Delta U(c,L)}{DEN_2}\int_{n^{**}}^{1-\epsilon} u'\left(\frac{L+r(1-L)}{n}\right)\frac{1-r}{n}dn + (1-r)u'(L+r(1-L))\left[\frac{\Delta U(c,L)}{DEN_2} - \sigma_2^*\right] + \dots,
\end{aligned} \tag{75}$$

where the remaining terms are positive, as proved in (74). Hence, the expression in (75) is positive if $\Delta U(c,L) \geq NUM_2$, or:

$$\begin{aligned}
&\lambda u(c) + (1-\lambda)\left[\sigma_2^* u(c_L(R)) + (1-\sigma_2^*)u(c_L(0))\right] - u(L+r(1-L)) \geq \\
&\geq \int_{\lambda}^{n^{**}} u(c)dn + \int_{n^{**}}^{1} u\left(\frac{L+r(1-L)}{n}\right)dn - \int_{\lambda}^{n_2^*} u\left(\frac{L-nc}{1-n}\right)dn.
\end{aligned} \tag{76}$$

We show the condition under which this inequality holds in Figure 5. The grey area represents $NUM_2$. As it is clear from the Figure, $(1-\lambda)u(c) > NUM_2$. Hence, to prove that $\Delta U(c,L) \geq$

$NUM_2$, it is sufficient to prove that $\Delta U(c, L) \geq (1-\lambda)u(c)$. We do it by contradiction. Assume that the opposite is true, or:

$$(1-\lambda)u(c) > \lambda u(c) + (1-\lambda)\left[\sigma_2^* u(c_L(R)) + (1-\sigma_2^*)u(c_L(0))\right] - u(L + r(1-L)). \tag{77}$$

As $u(c) < \sigma_2^* u(c_L(R)) + (1-\sigma_2^*)u(c_L(0))$, this inequality implies:

$$(1-\lambda)u(c) > \lambda u(c) + (1-\lambda)u(c) - u(L + r(1-L)), \tag{78}$$

or:

$$u(L + r(1-L)) > \lambda u(c). \tag{79}$$

Since the utility function $u(c)$ is increasing and invertible:

$$L + r(1-L) > u^{-1}(\lambda)c, \tag{80}$$

where $u^{-1}(\lambda) > 1$. Hence, we get $L + r(1-L) > c$, which is a contradiction.

To sum up, this result shows that:

$$\left[\frac{\partial \sigma_2^*}{\partial L} + \frac{1}{\lambda}\frac{\partial \sigma_2^*}{\partial c}\right] \Delta U(c, L) - \sigma_2^*(1-r)u'(L + r(1-L)) > 0. \tag{81}$$

For this to be consistent with distorted Euler equation (28), it must also be the case that:

$$\int_{\sigma_2^*}^{1} \Big[u'(c) - pRu'(c_L(R))\Big]\, dp > 0. \tag{82}$$

This can be rewritten as:

$$(1-\sigma_2^*)u'(c) - \frac{1-\sigma_2^{*2}}{2} Ru'(c_L(R)) > 0, \tag{83}$$

which can be rearranged as:

$$\frac{u'(c)}{u'(c_L(R))} > R\frac{1+\sigma_2^*}{2} \geq \mathbb{E}[p]R = \frac{u'(c^{PI})}{u'(c_L^{PI}(R))} > 1, \tag{84}$$

where the second inequality holds as $\mathbb{E}[p] = 1/2$ and $\sigma_2^* \geq 0$, and $\{c^{PI}, c_L^{PI}(R)\}$ is the deposit contract in the equilibrium with perfect information. By concavity of the utility function, for the ratio $\frac{u'(c)}{u'(c_L(R))}$ to be higher in the banking equilibrium than in the equilibrium with perfect information, it must be the case that $\frac{c}{c_L(R)} < \frac{c^{PI}}{c_L^{PI}(R)}$. To this end, we calculate:

$$\frac{\partial}{\partial L}\left[\frac{c}{c_L(R)}\right] = -\frac{(1-\lambda)c}{[R(1-L)+L-\lambda c]^2}(1-R) > 0, \tag{85}$$

$$\frac{\partial}{\partial c}\left[\frac{c}{c_L(R)}\right] = \frac{(1-\lambda)[R(1-L)+L-\lambda c] + \lambda(1-\lambda)c}{[R(1-L)+L-\lambda c]^2} > 0. \tag{86}$$

We take the total differential of the ratio $\frac{c}{c_L(R)}$, evaluated at the equilibrium with perfect information, and look for the condition that makes it negative:

$$\frac{\partial}{\partial L}\left[\frac{c}{c_L(R)}\right] dL + \frac{\partial}{\partial c}\left[\frac{c}{c_L(R)}\right] dc < 0. \tag{87}$$

This implies that:

$$\frac{dL}{dc} < -\frac{\frac{\partial}{\partial c}\left[\frac{c}{c_L(R)}\right]}{\frac{\partial}{\partial L}\left[\frac{c}{c_L(R)}\right]}. \tag{88}$$

As the right-hand side is negative, it must be the case that $dL/dc < 0$. Finally, evaluate the first-order condition with respect to $c$ in (71) at the equilibrium with perfect information. As the term in the integral has to go up when moving from $c^{PI}$ to $c^{BE}$, then it must be the case that $c^{BE} < c^{PI}$, hence $dc < 0$. This, together with $\frac{dL}{dc} < 0$, implies that $dL > 0$, or $L^{BE} > L^{PI}$. This ends the proof. ■