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The Impact of Persistent Cycles on Zero Frequency Unit Root Tests

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Abstract

In this paper we investigate the impact of non-stationary cycles on the asymptotic and finite sample properties of standard unit root tests. Results are presented for the augmented Dickey-Fuller normalised bias and t-ratio-based tests (Dickey and Fuller, 1979, and Said and Dickey, 1984), the variance ratio unit root test of Breitung (2002) and the M class of unit-root tests introduced by Stock (1999) and Perron and Ng (1996). The limiting distributions of these statistics are derived in the presence of non-stationary cycles. We show that while the ADF statistics remain pivotal (provided the test regression is properly augmented), this is not the case for the other statistics considered and show numerically that the size properties of the tests based on these statistics are too unreliable to be used in practice. We also show that the t-ratios associated with lags of the dependent variable of order greater than two in the ADF regression are asymptotically normally distributed. This is an important result as it implies that extant sequential methods (see Hall, 1994 and Ng and Perron, 1995) used to determine the order of augmentation in the ADF regression remain valid in the presence of non-stationary cycles.

Keywords: Nonstationary cycles; unit root tests; lag augmentation order selection.
JEL classifications: C20, C22

1 Introduction

Peaks at low non-zero frequencies imply the existence of cycles in time series, a feature present in many macroeconomic, financial and other time series; see, *inter alia*, Conway and Frame (2000), Birgean and Kilian (2002) and Priestley (1981). The importance and interest in trend growth, cyclicality and seasonal fluctuations in economic and financial time series dates back several decades; see Burns and Mitchell (1946). For instance, Canova (1996) discusses the literature on Bayesian learning (Nyarko, 1992) and on noisy traders in financial markets (Campbell and Kyle, 1993) where models which generate irregularly spaced but significant cycles in economic activity and asset prices are proposed. The presence of cycles is also documented in the political economy literature (electoral cycles in government variables, Alesina and Roubini, 1992), and naturally arises in the business cycle literature.

Interestingly, the existence of both complex and real unit roots can induce growth cycles similar to those observed in economic data; see Allen (1997). Autoregressive (AR) processes with roots on the complex unit circle are non-stationary and display persistent cyclical behavior similar to that of persistent business cycles (Pagan, 1999 and Bierens, 2001). Bierens (2001) finds evidence that business cycles may indeed be due to complex unit roots. Shibayama (2008) studies inventories and monetary policy by estimating VAR models and also detects complex roots that generate cycles of around 55 to 70 months, which are close to business cycle lengths.

Given that cycles are an important feature of economic and financial variables it is important to evaluate their implications on the performance of pre-testing procedures, in particular on the limiting null distributions and finite sample properties of zero frequency unit root test statistics. Consequently, following the body of empirical evidence summarised above, in this paper we will focus on the case where the cyclical component is characterised by a second order autoregressive component with complex roots in the neighbourhood of unity, although generalisations to higher-order non-stationary factors will also be discussed.

The asymptotic distributions of the least-squares estimates of an AR(2) with complex unit roots has been addressed in Ahtola and Tiao (1987), Gregoir (2004) and Tanaka (2008), the latter generalises the results in Ahtola and Tiao (1987) by allowing the error term in the AR(2) model to follow a stationary process. For asymptotic results for general AR processes see, *inter alia*, Chan and Wei (1988). Chan and Wei (1988) considered the limiting distributions of the least squares estimate of a general nonstationary AR model of order p (AR(p)) with characteristic roots at different frequencies on or outside the unit circle, each of which may have different multiplicities (see also Nielsen, 2001). This was the first comprehensive treatment of least squares estimates for a general nonstationary AR(p) model. Chan and Wei (1988) showed that the location of the roots of the time series played an important role in characterizing the limiting distributions. Jeganathan (1991) generalized this idea to the near-integrated context where the limiting distributions of the least-squares estimators are expressed in terms of Ornstein-Uhlenbeck processes. Extensions to vector AR processes are provided in Tsay and Tiao (1990) and to processes with deterministic trends in Chan (1989).

We will focus attention on the conventional augmented Dickey-Fuller [ADF] tests (Dickey and Fuller, 1979, Said and Dickey, 1984, Hamilton, 1994), the variance ratio unit root test of Breitung (2002) and the trinity of so-called M unit-root tests introduced by Stock (1999) and popularised by Perron and Ng (1996). The M tests have been increasingly popular in the unit root literature; indeed, in discussing the ADF tests Haldrup and Jansson (2006,p.267) argue that "... practitioners ought to abandon the use of these tests..." in favour of the M tests because of "... the excellent size properties and 'nearly' optimal power properties" of the latter. The results we present in this paper, however, suggest quite the opposite conclusion holds in cases where the time series process admits (near-) unit roots at cyclical frequencies. In particular we show that while the limiting distributions of the two ADF statistics remain pivotal (provided the test regression is properly augmented), this is not the case for the variance ratio or the three M statistics, all of which become too unreliable to be used in practice. We also show that the normalized bias and the t-ratio tests associated with the lags of the dependent variable (of order greater than two) in the ADF regression are asymptotically normally distributed. This is an important result as it implies that the standard sequential methods (see Hall, 1994, and Ng and Perron, 1995) used to determine the order of augmentation in the ADF regression remain valid when cyclical (near-) unit roots are present in the data.

The remainder of the paper is organized as follows. In section 2 we outline our reference time series model which allows for cyclical unit roots and briefly outline the ADF, variance ratio and M unit root tests. In Section 3, in the case where cyclical unit roots are present in the data, we establish the large sample behaviour of these tests together with those of the conventional t-ratios for testing the significance of lagged dependent variables in the ADF test regression. Extensions to allow for near unit roots (both at the zero and cyclical frequencies) and deterministic variables are discussed in section 4. Finite sample simulations are reported in section 5. Section 6 concludes. All proofs are collected in an mathematical appendix.

In the following '$\lfloor\cdot\rfloor$' denotes the integer part of its argument, '$\Rightarrow$' denotes weak convergence, '$\xrightarrow{p}$' convergence in probability, and '$x := y$' ('$x =: y$') indicates that x is defined by y (y is defined by x).

2 The Model and Unit Root Tests

2.1 The Time Series Model

We consider a univariate time series $\{x_t\}$ generated according to the following data generation process [DGP],

$$\Psi(L)(1-aL)x_t = \varepsilon_t, \quad \varepsilon_t \sim IID(0,\sigma^2), \qquad t = 1,2,\ldots,n. \tag{1}$$

We assume throughout that the process is initialised at $x_{-2} = x_{-1} = x_0 = 0$, although weakening this to allow these starting values to be of $o_p(n^{1/2})$ would not change any of the asymptotic results which follow. In (1) the autoregressive polynomial $\Psi(L) = \left(1 - 2b\cos(\phi)L + b^2L^2\right)$, with $\phi \in (0,\pi)$, and where L denotes the usual lag operator. Consequently, when $b = 1$ ($|b| < 1$), $\Psi(L)$ admits the complex conjugate pair of unit (stable) roots, $\exp(\pm i\phi) \equiv \cos(\phi) \pm i\sin(\phi)$, at the spectral frequency ϕ. We do not assume that the value of ϕ is known to the practitioner. The process additionally admits a zero frequency unit root when $a = 1$. In the case where $a = b = 1$, $\{x_t\}$ is therefore integrated of order one at both the zero and ϕ spectral frequencies, denoted $I_0(1)$ and $I_\phi(1)$, respectively. In this case, it follows that $z_t := \Delta x_t$, where $\Delta := (1-L)$, will be $I_\phi(1)$ but $I_0(0)$, while $u_t := \Delta_\phi x_t$, where $\Delta_\phi := \left(1 - 2\cos(\phi)L + L^2\right)$, will be $I_0(1)$ but $I_\phi(0)$.

Our focus in this paper is on testing the standard zero frequency unit root null hypothesis that $x_t \sim I_0(1)$, $H_0 : a = 1$, against the alternative that $x_t \sim I_0(0)$, $H_1 : |a| < 1$, in the case where $\Psi(L)$ admits the pair of complex unit roots at frequency ϕ.

Remark 1. The model in (1) is arguably quite simple. However, it can be extended to allow for: near unit roots at the zero and/or ϕ frequencies; deterministic components; weak dependence in $\{\varepsilon_t\}$, and unit roots at other cyclical frequencies lying in $(0,\pi)$ and/or at the Nyquist (π) frequency, without altering the qualitative conclusions which can be drawn from the analysis of (1). For expositional purposes we will therefore focus on (1) but we will discuss how our results generalise to these cases.

Remark 2. Notice that $\Delta_\phi = \left(1 - 2\cos(\phi)L + L^2\right)$ generates a (non-stationary) cycle of $2\pi/\phi$ periods. Consequently, in the case of data observed with a seasonal periodicity of S, $\phi = 2\pi j/S$, $j \in \{1,2,\ldots,S^*\}$, where $S^* := \lfloor(S-1)/2\rfloor$, Δ_ϕ generates non-stationary seasonal cycles.

2.2 Zero Frequency Unit Root Tests

A large number of procedures have been proposed to test for zero frequency unit roots; see, for example, Stock (1994), Maddala and Kim (1998) and Phillips and Xiao (1998) for excellent overviews. In this Section we review three popular classes of such tests.

The first tests we consider are the augmented Dickey-Fuller [ADF] normalised bias and t-ratio tests. These are computed from the auxiliary test regression,

$$\Delta x_t = \rho x_{t-1} + \sum_{j=1}^{k} \alpha_j \Delta x_{t-j} + \varepsilon_{k,t}. \tag{2}$$

In (2), k denotes the lag truncation order chosen to account (parametrically) for $\Psi(L)$ and any weak dependence in $\{\varepsilon_t\}$; in the simplest form of the DGP given in (1), where ε_t is IID, $k = 2$. More generally, where ε_t is a linear process satisfying standard summability and moment conditions (see Chang and Park, 2002), k needs to be such that $1/k + k^3/n \to \infty$ as $n \to \infty$; see Said and Dickey (1984) and Chang and Park (2002). Based on OLS estimation of (2), the ADF t-ratio for testing H_0 against H_1 will be denoted $t_{\widehat{\rho}} := \widehat{\rho}/se(\widehat{\rho})$ and the associated normalised bias statistic as $Z_{\widehat{\rho}} := n\widehat{\rho}/(1 - \sum_{i=1}^{k} \widehat{\alpha}_i)$. The ADF tests remain the most popularly applied unit root tests due in part to their ease of construction. At this point we also outline the sequential lag selection method due to Hall (1994) and Ng and Perron (1995). Here one starts from a maximum lag length, $k_{\max}$ say in (2), satisfying the rate condition above. One then runs the standard t-test for the significance of the terminal lag using critical values from the standard normal distribution. If the null is rejected the ADF statistics are computed from (2) with lag length

$k_{\max}$; otherwise, the lag length is reduced to $k_{\max} - 1$ and the procedure repeated until the lag length cannot be reduced further, or if a pre-specified value for the minimum lag length is attained, at which point the ADF statistics are computed for that lag length.

Secondly we will consider the trinity of so-called M unit root tests due to Stock (1999) and Perron and Ng (1996), *inter alia*:

$$MSB := \left(\hat{\lambda}^{-2} n^{-2} \sum_{t=1}^{n} x_{t-1}^2\right)^{1/2} \tag{3}$$

$$MZ_\alpha := \left(2n^{-2} \sum_{t=1}^{n} x_{t-1}^2\right)^{-1} \left(n^{-1} x_n^2 - \hat{\lambda}^2\right) \tag{4}$$

$$MZ_t := MSB \times MZ_\alpha \tag{5}$$

where $\hat{\lambda}^2$ is an estimator of the long run variance of $\{\varepsilon_t\}$. Following Perron and Ng (1996) we can consider two alternative estimators for the long-run variance. Firstly, a non-parametric kernel estimator based on the sample autocovariances, $\hat{\lambda}^2 = s_{WA}^2$, with $s_{WA}^2 := \sum_{h=-n+1}^{n-1} \omega(h/m)\hat{\gamma}_h$, $\hat{\gamma}_h := n^{-1} \sum_{t=1}^{n-|h|} \hat{\varepsilon}_{0,t}\hat{\varepsilon}_{0,t+|h|}$, where $\hat{\varepsilon}_{0,t}$ are the OLS residuals from regressing x_t on x_{t-1}, with kernel function $\omega(\cdot)$ satisfying e.g. the general conditions reported in Jansson (2002, Assumption A3) and the bandwidth parameter $m \in (0, \infty)$ satisfying $1/m + m^2/n \to 0$ as $n \to \infty$ (which corresponds to Assumption A4 of Jansson, 2002). Secondly, a parametric autoregressive spectral density estimator, $\hat{\lambda}^2 := s_{AR}^2$, of the form suggested by Berk (1974), where $s_{AR}^2 := \hat{\sigma}_k^2 / \left(1 - \sum_{i=1}^{k} \widehat{\alpha}_i\right)^2$ where $\hat{\sigma}_k^2 := n^{-1} \sum \widehat{e}_t^2$, $\widehat{\alpha}_i$, $i = 1, ..., k$, and $\widehat{e}_t$ are obtained from the OLS estimation of (2) for a given value of k satisfying the same conditions as given above in the context of the ADF tests. As noted in section 2, it has been suggested by some authors (e.g. Haldrup and Jansson, 2006) that the M tests, when coupled with the modified AIC lag selection method of Ng and Perron (2001), are preferable to the standard ADF tests outlined above due to their superior size properties, relative to the latter, in the presence of weak dependence in $\{\varepsilon_t\}$.

Finally, we will consider the variance ratio test (VRT) proposed by Breitung (2002),

$$VRT := n^{-2} \left(\sum_{t=1}^{n} x_t^2\right)^{-1} \sum_{t=1}^{n} \left(\sum_{j=1}^{t} x_j\right)^2. \tag{6}$$

The variance ratio test has some appealing properties. First of all it requires no correction, parametric or non-parametric, for serial correlation from $\{\varepsilon_t\}$ and/or $\Psi(L)$. Second, by virtue of its lack of such a correction factor, it has been advocated by some authors (see, for example, Müller, 2008) as a unit root test which avoids the criticisms of Faust (1996) regarding the (theoretical) uncontrollability of the size of unit root tests based around (parametric or non-parametric) corrections for general weak dependence in $\{\varepsilon_t\}$.

3 Asymptotic Behaviour under Non-Stationary Cycles

For the purpose of analysing the impact of non-stationary cycles on the limit distributions of the zero frequency unit root tests discussed in section 2.2 it will prove useful to first consider frequency specific orthogonal decompositions of x_t and Δx_t. These results are collected together in Lemma 1.

Lemma 1 *Let the time series process $\{x_t\}$ be generated by (1) with $a = b = 1$. Then for any $\phi \in (0, \pi)$ the following decompositions hold:*

$$\begin{aligned} x_t &= \delta_0 S_0(t) + \frac{\delta_\phi^\alpha}{\sin\phi}\left[\sin(\phi(t+1)) S_{\alpha,\phi}(t) - \cos(\phi(t+1)) S_{\beta,\phi}(t)\right] \\ &\quad + \frac{\delta_\phi^\beta}{\sin\phi}\left[\cos(\phi(t+1)) S_{\alpha,\phi}(t) + \sin(\phi(t+1)) S_{\beta,\phi}(t)\right] + O_p(1) \end{aligned} \tag{7}$$

$$\Delta x_t = \frac{1}{\sin\phi}\left[\sin(\phi(t+1)) S_\alpha(t) - \cos(\phi(t+1)) S_\beta(t)\right] \tag{8}$$

where $S_0(t) := \sum_{j=1}^{t} \varepsilon_j$, $S_{\alpha,\phi}(t) := \sum_{j=1}^{t} \varepsilon_j \cos(j\phi)$, $S_{\beta,\phi}(t) := \sum_{j=1}^{t} \varepsilon_j \sin(j\phi)$, $\delta_0 := 1/2\,(1 - \cos\phi)$, $\delta_\phi^\alpha := (1 - \cos(\phi))/2\,(1 - \cos\phi)$, and $\delta_\phi^\beta := -\sin(\phi)/2\,(1 - \cos\phi)$.

Remark 3. It is straightforward to show that the results stated in Lemma 1 continue to hold under weaker linear process conditions on $\{\varepsilon_t\}$ provided Assumptions 1.1-1.2 of Gregoir (2004), adapted slightly to our situation, are satisfied. Precisely, these conditions entail that $\varepsilon_t = d(L)e_t$, where $(\varepsilon_t, \mathcal{F}_t)$ is a martingale difference sequence, with filtration $(\mathcal{F}_t)$, such that $E\left(\varepsilon_t^2|\mathcal{F}_{t-1}\right) = \sigma^2$ and $\sup_t E\left(|\varepsilon_t|^{2+\delta}|\mathcal{F}_{t-1}\right) < \infty$ a.s. for some $\delta > 0$, and where $d(L) := 1 + \sum_{j=1}^{\infty} d_j z^j$ is such that $d(z) \neq 0$ for $z = 0$ and $z = \phi$, and $\sum_{j=1}^{\infty} j |d_j| < \infty$.

Using the decompositions provided in Lemma 1, we are now in a position to state the large sample behaviour of the elements which form the unit root tests under analysis. These results are collected together in Lemma 2.

Lemma 2 *Let the conditions of Lemma 1 hold. Then, for any $\phi \in (0, \pi)$, the following results hold as $n \to \infty$,*

$$\begin{aligned}
\frac{1}{\sqrt{n}} x_t &\Rightarrow \sigma \delta_0 W_0(r) + \frac{\sigma \delta_\phi^\alpha}{\sqrt{2} \sin \phi} \left[\sin(\phi(t+1)) W_\phi^\alpha(r) - \cos(\phi(t+1)) W_\phi^\beta(r) \right] \\
&\quad + \frac{\sigma \delta_\phi^\beta}{\sqrt{2} \sin \phi} \left[\cos(\phi(t+1)) W_\phi^\alpha(r) + \sin(\phi(t+1)) W_\phi^\beta(r) \right] := \sigma \mathbf{b}(r) \qquad (9) \\
\frac{1}{n^2} \sum_{t=1}^n x_{t-1}^2 &\Rightarrow \frac{\sigma^2}{4(1-\cos\phi)^2} \int_0^1 W_0^2(r)\, dr \\
&\quad + \frac{1}{2(1-\cos\phi)} \frac{\sigma^2}{4 \sin^2 \phi} \int_0^1 \left(\left[W_\phi^\alpha(r) \right]^2 + \left[W_\phi^\beta(r) \right]^2 \right) dr \qquad (10) \\
\frac{1}{n^2} \sum_{t=1}^n \Delta x_t^2 &= \frac{1}{n^2} \sum_{t=1}^n \Delta x_{t-1}^2 + o_p(1) = \frac{1}{n^2} \sum_{t=1}^n \Delta x_{t-2}^2 + o_p(1) \\
&\Rightarrow \frac{\sigma^2}{4 \sin^2 \phi} \int_0^1 \left(\left[W_\phi^\alpha(r) \right]^2 + \left[W_\phi^\beta(r) \right]^2 \right) dr \qquad (11) \\
\frac{1}{n^2} \sum_{t=1}^n x_{t-1} \Delta x_{t-1} &\Rightarrow \frac{\sigma^2}{8 \sin^2 \phi} \int_0^1 \left(\left[W_\phi^\alpha(r) \right]^2 + \left[W_\phi^\beta(r) \right]^2 \right) dr \qquad (12) \\
\frac{1}{n^2} \sum_{t=1}^n x_{t-1} \Delta x_{t-2} &\Rightarrow \frac{\sigma^2 \left(\cos\phi + \frac{1}{2} \right)}{4 \sin^2 \phi} \int_0^1 \left(\left[W_\phi^\alpha(r) \right]^2 + \left[W_\phi^\beta(r) \right]^2 \right) dr \qquad (13) \\
\frac{1}{n} \sum_{t=1}^n x_{t-1} \varepsilon_t &\Rightarrow \frac{\sigma^2}{2(1-\cos\phi)} \int_0^1 W_0(r)\, dW_0(r) \\
&\quad + \frac{\sigma^2}{4 \sin\phi} \int_0^1 \left(W_\phi^\alpha(r)\, dW_\phi^\beta(r) - W_\phi^\beta(r)\, dW_\phi^\alpha(r) \right) \\
&\quad - \frac{\sigma^2}{4(1-\cos(\phi))} \int_0^1 \left(W_\phi^\alpha(r)\, dW_\phi^\alpha(r) + W_\phi^\beta(r)\, dW_\phi^\beta(r) \right) \qquad (14) \\
\frac{1}{n} \sum_{t=1}^n \Delta x_{t-1} \varepsilon_t &\Rightarrow \frac{\sigma^2}{2 \sin\phi} \int_0^1 \left(W_\phi^\alpha(r)\, dW_\phi^\beta(r) - W_\phi^\beta(r)\, dW_\phi^\alpha(r) \right) \qquad (15)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{n} \sum_{t=1}^n \Delta x_{t-2} \varepsilon_t &\Rightarrow \frac{\sigma^2 \cos\phi}{2 \sin\phi} \int_0^1 \left(W_\phi^\alpha(r)\, dW_\phi^\beta(r) - W_\phi^\beta(r)\, dW_\phi^\alpha(r) \right) \\
&\quad - \frac{\sigma^2}{2} \int_0^1 \left(W_\phi^\alpha(r)\, dW_\phi^\alpha(r) + W_\phi^\beta(r)\, dW_\phi^\beta(r) \right) \qquad (16)
\end{aligned}$$

where δ_0, δ_ϕ^α and δ_ϕ^β are as defined in Lemma 1, and $W_0(r)$, $W_\phi^\alpha(r)$ and $W_\phi^\beta(r)$ are independent standard Brownian motion processes.

Using the results in Lemma 2, we are now in a position in Theorem 1 to detail the asymptotic null distributions of the unit root tests from section 2.2 when the DGP contains non-stationary cycles. Subsequently, in Theorem 2, we will establish corresponding results for the t-ratios on the lagged dependent variables appearing in the ADF regression (2).

Theorem 3 *Let the conditions of Lemma 1 hold. Then for any $\phi \in (0, \pi)$, the following results hold as $n \to \infty$, and for $k \geq 2$ in (2)*

$$t_{\widehat{\rho}} \Rightarrow \frac{\int_0^1 W_0(r)\, dW_0(r)}{\sqrt{\int_0^1 \left[W_0(r) \right]^2 dr}} := \mathcal{DF}_1 \qquad (17)$$

$$Z_{\widehat{\rho}} \Rightarrow \frac{\int_0^1 W_0(r)\, dW_0(r)}{\int_0^1 \left[W_0(r) \right]^2 dr} := \mathcal{DF}_2 \qquad (18)$$

$$VRT \Rightarrow \frac{\int_0^1 \left[\int_0^r \mathbf{b}(s)\, ds \right]^2 dr}{\left(\delta_0^2 \frac{\int_0^1 W_0(r)^2 dr}{4(1-\cos(\phi))^2} + \frac{\left(\int_0^1 \left[W_\phi^\alpha(r) \right]^2 dr + \int_0^1 \left[W_\phi^\beta(r) \right]^2 dr \right)}{2(1-\cos(\phi))\, 4 \sin(\phi)^2} \right)} \qquad (19)$$

for $\hat{\lambda}^2 = s^2_{WA}$,

$$MSB = O_p\left([mn]^{-1/2}\right), \quad MZ_\alpha = O_p\left(mn\right), \quad MZ_t = O_p\left([mn]^{1/2}\right) \tag{20}$$

while for $\hat{\lambda}^2 = s^2_{AR}$

$$MSB \quad \Rightarrow \quad \left(\int_0^1 W_0\left(r\right)^2 dr + \frac{\left(1-\cos\left(\phi\right)\right)}{2\sin\left(\phi\right)^2}\left(\int_0^1 \left[W^\alpha_\phi\left(r\right)\right]^2 dr + \int_0^1 \left[W^\beta_\phi\left(r\right)\right]^2 dr\right)\right)^{1/2} \tag{21}$$

$$MZ_\alpha \quad \Rightarrow \quad \frac{\mathbf{b}\left(1\right)^2 - \left(4\left[1-\cos\left(\phi\right)\right]^2\right)^{-1}}{\left(\frac{\int_0^1 W_0(r)^2 dr}{2(1-\cos(\phi))^2} + \frac{\left(\int_0^1\left[W^\alpha_\phi(r)\right]^2 dr + \int_0^1\left[W^\beta_\phi(r)\right]^2 dr\right)}{(1-\cos(\phi))4\sin(\phi)^2}\right)} \tag{22}$$

$$MZ_t \quad \Rightarrow \quad \frac{\sqrt{2}\left[1-\cos\left(\phi\right)\right]\mathbf{b}\left(1\right)^2 - \left(2\sqrt{2}\left[1-\cos\left(\phi\right)\right]\right)^{-1}}{\sqrt{\frac{\int_0^1 W_0(r)^2 dr}{2(1-\cos(\phi))^2} + \frac{\left(\int_0^1\left[W^\alpha_\phi(r)\right]^2 dr + \int_0^1\left[W^\beta_\phi(r)\right]^2 dr\right)}{(1-\cos(\phi))4\sin(\phi)^2}}} \tag{23}$$

where m *is the bandwidth used to compute* s^2_{WA}, *and where* $W_0\left(r\right)$, $W^\alpha_\phi\left(r\right)$ *and* $W^\beta_\phi\left(r\right)$ *and* $\mathbf{b}\left(r\right)$ *are as defined in Lemma 2.*

Remark 4. In the case where $|b| < 1$ in $\Psi(L)$ in (1), so that no non-stationary cycles are present, it is well known that both $t_{\widehat{\rho}}$ and MZ_t weakly converge to $\mathcal{DF}_1$, while $Z_{\widehat{\rho}}$ and MZ_α weakly converge to $\mathcal{DF}_2$. Moreover, in this case $MSB \Rightarrow \left(\int_0^1 W_0\left(r\right)^2\right)^{1/2} =: \mathcal{MSB}$ and $VRT \Rightarrow \frac{\int_0^1\left[\int_0^r W_0(s)ds\right]^2 dr}{\int_0^1 W_0(r)^2 dr} =: \mathcal{VRT}$. Comparing these representations with those given in Theorem 1, it is seen that only the two ADF statistics, $t_{\widehat{\rho}}$ and $Z_{\widehat{\rho}}$, computed from (2) retain their usual pivotal limiting null distributions in the presence of non-stationary cycles. In contrast, the limiting null distributions of VRT and the trinity of M statistics with autoregressive spectral density estimators of the long-run variance are non-pivotal, their functional forms depending in a complicated way on both non-stochastic and stochastic functions, while the results in (20) show that the trinity of M statistics computed using a kernel-based estimator of the long-run variance have degenerate limiting null distributions in the presence of non-stationary cycles. This result obtains because Δx_t is non-stationary which causes the kernel-based long-run variance estimator to diverge to $+\infty$ at rate $O_p(mn)$; see Taylor (2003). Since both $n^{-1}x_n^2$ and $2n^{-2}\sum_{t=1}^n x_{t-1}^2$ are of $O_p(1)$, the divergence of $\hat{\lambda}^2_{WA}$ to $+\infty$ implies that both the MZ_α and MZ_t statistics will diverge to $-\infty$, while MSB will converge in probability to zero, in each case at the rates stated in (20). Consequently, the three M tests based on these statistics will therefore all have asymptotic size of unity in the presence of non-stationary cycles. The impact on the finite sample size of the unit root tests based on each of the statistics discussed in Theorem 1 when using the standard asymptotic critical values (appropriate to the case where non-stationary cycles are not present) will be investigated in section 5.

Remark 5. In this paper we have not included the Z_α and Z_t unit root tests of Phillips (1987) and Phillips and Perron (1988) in the set of tests under discussion. However, noting from expressions (2.7) and (2.10) of Perron and Ng (1996,p.437) that $Z_\alpha = MZ_\alpha - (n/2)(\widehat{\rho}-1)^2$ and $Z_t = MZ_t - 0.5(\sum_{t=1}^n x^2_{t-1}/s^2_{WA})^{1/2}(\widehat{\rho}-1)^2$, in each case for the version of the M test using the kernel-based long run variance estimator s^2_{WA}, we see immediately from the discussion in Remark 4 that both Z_α and Z_t will also diverge to $-\infty$ at the same rates as are given for MZ_α and MZ_t, respectively, in (20).

Remark 6. The result in (17) for $t_{\widehat{\rho}}$ has previously been given in Nielsen (2001), and is also shown to hold for case of seasonally integrated data in Ghysels et al. (1994).

Remark 7. It is straightforward but tedious, using Lemma 2 of Bierens (2000), to show that the results given in (17), (18) and (20) of Theorem 1 will not alter if we allow for weak dependence in $\{\varepsilon_t\}$ of the form given in Remark 3. The limiting null distributions for VRT and the trinity of M statistics with autoregressive spectral density estimators of the long-run variance will now depend on additional nuisance parameters arising from the MA parameters, $\{d_j\}_{j=1}^\infty$. Moreover, the results in Theorem 1 are qualitatively unchanged if we allow $\Psi(L)$ to be a pth order polynomial containing additional unit roots with frequencies in the range $(0, \pi]$, provided $k \geq p$ in (2). Specifically, in this case the results in (17) and (18) will continue to hold, as will the order results in (20), while the limiting null distributions for VRT

and the trinity of M statistics with autoregressive spectral density estimators of the long-run variance will now depend on stochastic and non-stochastic functions relating to these frequencies but will remain non-degenerate.

In Theorem 2 we now establish the large sample behaviour of the t-ratios on the lagged dependent variables in (2).

Theorem 4 *Let the conditions of Lemma 1 hold and define the vector of parameters from (2) as* $\boldsymbol{\beta}' := [\rho, \alpha_1, \alpha_2, \alpha_3 \cdots \alpha_k] =: [\rho, \alpha_1, \alpha_2, \Phi']$. *For* $k \geq 2$ *in (2), then as* $n \to \infty$,

$$n(\hat{\alpha}_1 - \alpha_1) \Rightarrow \frac{A\cos(\phi) + B}{\mathcal{V}^2(\mathcal{A}^2 + \mathcal{B}^2)} \tag{24}$$

$$n(\hat{\alpha}_2 - \alpha_2) \Rightarrow \frac{A}{\mathcal{V}^2(\mathcal{A}^2 + \mathcal{B}^2)} \tag{25}$$

$$\sqrt{n}\left(\hat{\Phi} - \Phi\right) \Rightarrow N\left(0, \sigma^2 \mathbf{H}_2 \boldsymbol{\Gamma}^{-1} \mathbf{H}_2\right) \tag{26}$$

where

$$\begin{aligned}
A &:= 2\left[\mathcal{V}^2 \int_0^1 \left(W_\phi^\alpha(r)\, dW_\phi^\alpha(r) + W_\phi^\beta(r)\, dW_\phi^\beta(r)\right) + \mathcal{V}^2(\mathcal{A}^2 + \mathcal{B}^2)\right] \\
B &:= 2\left[\mathcal{V}^2 \int_0^1 \left(W_\phi^\alpha(r)\, dW_\phi^\beta(r) - W_\phi^\beta(r)\, dW_\phi^\alpha(r)\right) \sin(\phi) - \mathcal{V}^2(\mathcal{A}^2 + \mathcal{B}^2)\right] \\
\mathcal{V}^2 &:= \int_0^1 W_0^2(r) dr, \quad \mathcal{A}^2 := \int_0^1 \left[W_\phi^\alpha(r)\right]^2 dr, \quad \mathcal{B}^2 := \int_0^1 \left[W_\phi^\beta(r)\right]^2 dr
\end{aligned}$$

where $W_0(r)$, $W_\phi^\alpha(r)$ *and* $W_\phi^\beta(r)$ *are as defined in Lemma 2, and where* $\mathbf{H}_2$ *and* $\boldsymbol{\Gamma}$ *are defined in (A.29) and (A.22), respectively, in the Appendix. Moreover, for* $2 < i \leq k$,

$$t_{\widehat{\alpha}_i} := (\widehat{\alpha}_i - \alpha_i)/s.e.(\widehat{\alpha}_i) \Rightarrow N(0,1). \tag{27}$$

Remark 8. The results in (A.26) and (A.27) imply that $\hat{\alpha}_1$ and $\hat{\alpha}_2$, like $\widehat{\rho}$, are super-consistent. From (26), it is seen that the parameters on the lagged difference terms from lag three onwards are root-n consistent asymptotically normal [CAN]. Under the conditions of Theorem 2, $\Phi = \mathbf{0}$, since $\varepsilon_t \sim IID(0, \sigma^2)$; however, the stated results also hold when ε_t is a stationary $AR(k^*)$ process provided $k \geq k^*$. Moreover, where ε_t displays the general weak dependence of the form given in Remark 3 the foregoing results still remain valid although here, for a given lag truncation k, the parameters α_j, $j \geq 3$, take the role of pseudo-parameters in the same sense as in, for example, Chang and Park (2002) and Ng and Perron (1995).

Remark 9. The key result in Theorem 2 is that given in (27) which establishes that standard t-tests for the significance of the lagged dependent variables of order three and above can be conducted using standard normal critical values. This is an important result in that it implies that the sequential lag specification methods of Hall (1994) and Ng and Perron (1995), as outlined in section 2.2, remain valid in the presence of non-stationary cycles. In contrast, it follows straightforwardly from the representations in (A.26) and (A.27) that the t-ratios associated with $\hat{\alpha}_1$ and $\hat{\alpha}_2$ have non-standard limiting distributions which are functionals of the independent standard Brownian motion processes, $W_0(r)$, $W_\phi^\alpha(r)$ and $W_\phi^\beta(r)$. In the scenario considered in Remark 7 where $\Psi(L)$ is a pth order polynomial of non-stationary factors then it is straightforward but tedious to show that the OLS estimators associated with the first p lagged dependent variables in (2) will have non-standard limiting distributions (now being functionals of p independent standard Brownian motion processes) while those for $p+1$ onwards will again be root-n CAN with their associated t-ratios having standard normal limiting null distributions.

4 Extensions to Near-Integration and Deterministics

4.1 Near-Integration

In this section we generalise the results given in section 3 to the case where the data are generated according to the near-integrated process,

$$(1 - \varphi_n L)\left(1 - 2\cos(\phi)\varphi_n L + \varphi_n^2 L^2\right) x_t = \varepsilon_t, \quad \varepsilon_t \sim IID(0, \sigma^2), \quad t = 1, 2, \ldots, n \tag{28}$$

where $\varphi_n := \exp\left(\frac{c}{n}\right) \simeq \left(1+\frac{c}{n}\right)$, and $\phi \in (0,\pi)$. This process is near-integrated at the zero and ϕ frequencies with a common non-centrality parameter c. Under this setting we can establish the following Lemma.

Lemma 4.1 *Let the time series process $\{x_t\}$ be generated by (28) with $x_{-2} = x_{-1} = x_0 = 0$. Then for any $\phi \in (0,\pi)$ the following results hold:*

$$\begin{aligned}\frac{1}{\sigma\sqrt{n}}x_t &= \delta_0 J_{c,0,n}(t/n) + \frac{\delta_\phi^\alpha}{\sin(\phi)\sqrt{2}}\left\{\sin(\phi[t+1])J^\alpha_{c,\phi,n}(t/n) - \cos(\phi[t+1])J^\beta_{c,\phi,n}(t/n)\right\} \\ &\quad + \frac{\delta_\phi^\beta}{\sin(\phi)\sqrt{2}}\left\{\cos(\phi[t+1])J^\alpha_{c,\phi,n}(t/n) + \sin(\phi[t+1])J^\alpha_{c,\phi,n}(t/n)\right\} + O_p\left(1/\sqrt{n}\right)\end{aligned} \tag{29}$$

where

$$\begin{aligned} J_{c,0,n}(x) &:= \frac{1}{\sigma\sqrt{n}}\sum_{j=1}^{\lfloor xn\rfloor}\varphi_n^{\lfloor xn\rfloor-j}\varepsilon_j \\ J^\alpha_{c,\phi,n}(x) &:= \frac{\sqrt{2}}{\sigma\sqrt{n}}\sum_{j=1}^{\lfloor xn\rfloor}\varphi_n^{\lfloor xn\rfloor-j}\varepsilon_j\cos(j\phi), \quad J^\beta_{c,\phi,n}(x) := \frac{\sqrt{2}}{\sigma\sqrt{n}}\sum_{j=1}^{\lfloor xn\rfloor}\varphi_n^{\lfloor xn\rfloor-j}\varepsilon_j\sin(j\phi)\end{aligned}$$

and where, as $n\to\infty$,

$$\begin{pmatrix} J_{c,0,n}(t/n) \\ J^\alpha_{c,\phi,n}(t/n) \\ J^\alpha_{c,\phi,n}(t/n)\end{pmatrix} \Rightarrow \begin{pmatrix} J_{c,0}(r) \\ J^\alpha_{c,\phi}(r) \\ J^\beta_{c,\phi}(r)\end{pmatrix}$$

where $J_{c,0}(r)$, $J^\alpha_{c,\phi}(r)$ and $J^\beta_{c,\phi}(r)$ are independent standard Ornstein-Uhlenbeck processes such that $dJ_{c,0}(r) = cJ_{c,0}(r)\,dr + dW_0(r)$ and $dJ^j_{c,\phi}(r) = cJ^j_{c,\phi}(r)\,dr + dW^j_\phi(r)$, $j = \alpha,\ \beta$, with $W_0(r)$, $W^\alpha_k(r)$ and $W^\beta_k(r)$ the standard Brownian motions defined in Lemma 2.

Using the results in Lemma 3, it is then straightforward to show that the results given in Theorems 1 and 2 carry over to this context, substituting $W_0(r)$, $W^\alpha_k(r)$ and $W^\beta_k(r)$ by $J_{c,0}(r)$, $J^\alpha_{c,\phi}(r)$ and $J^\beta_{c,\phi}(r)$, respectively, throughout. The comments in Remark 7 again apply in this case.

Remark 10. The assumption of a common non-centrality parameter to the zero and ϕ frequencies, as embodied in (28), can be relaxed. To that end, consider the case where the DGP admits a different non-centrality parameter at the zero and ϕ frequencies; viz,

$$\left(1-\varphi^0_n L\right)\left(1 - 2\cos(\phi)\varphi^\phi_n L + \left(\varphi^\phi_n\right)^2 L^2\right)x_t = \varepsilon_t, \quad \varepsilon_t \sim IID(0,\sigma^2), \quad t = 1,2,\ldots,n \tag{30}$$

where $\varphi^0_n := \exp\left(\frac{c_0}{n}\right) \simeq \left(1+\frac{c_0}{n}\right)$ and $\varphi^\phi_n := \exp\left(\frac{c_\phi}{n}\right) \simeq \left(1+\frac{c_\phi}{n}\right)$. For data generated by (28) rather than (30), the result given in Lemma 3 still hold, provided, $J_{c,0,n}(x)$, $J^\alpha_{c,\phi,n}(x)$ and $J^\beta_{c,\phi,n}(x)$ are replaced by $J_{c_0,0,n}(x) := \frac{1}{\sigma\sqrt{n}}\sum_{j=1}^{\lfloor xn\rfloor}\left(\varphi^0_n\right)^{\lfloor xn\rfloor-j}\varepsilon_j$, $J^\alpha_{c_\phi,\phi,n}(x) := \frac{\sqrt{2}}{\sigma\sqrt{n}}\sum_{j=1}^{\lfloor xn\rfloor}\left(\varphi^\phi_n\right)^{\lfloor xn\rfloor-j}\varepsilon_j\cos(j\phi)$ and $J^\beta_{c_\phi,\phi,n}(x) := \frac{\sqrt{2}}{\sigma\sqrt{n}}\sum_{j=1}^{\lfloor xn\rfloor}\left(\varphi^\phi_n\right)^{\lfloor xn\rfloor-j}\varepsilon_j\sin(j\phi)$, respectively, and $J_{c,0}(r)$, $J^\alpha_{c,\phi}(r)$ and $J^\beta_{c,\phi}(r)$ are similarly replaced by $J_{c_0,0}(r)$, $J^\alpha_{c_\phi,\phi}(r)$ and $J^\beta_{c_\phi,\phi}(r)$, respectively, where $dJ_{c_0,0}(r) = c_0J_{c_0,0}(r)\,dr + dW_0(r)$ and $dJ^j_{c_\phi,\phi}(r) = c_\phi J^j_{c_\phi,\phi}(r)\,dr + dW^j_\phi(r)$, $j = \alpha,\ \beta$. The results given in Theorems 1 and 2 again carry over, *mutatis mutandis*, as do the comments in Remarks 7 and 9.

4.2 Deterministic Components

Where deterministic components are present in the DGP, the previous results can be extended in a straightforward fashion. More specifically, we consider the cases where (2) is constructed from de-trended data, denoted $\hat{x}_t$, obtained as the OLS residuals from the regression of x_t onto either a constant (de-meaned data) or a constant and linear trend (linear de-trended data); i.e.,

$$\Delta\hat{x}_t = \rho\hat{x}_{t-1} + \sum_{j=1}^{k}\alpha_j\Delta\hat{x}_{t-j} + \hat{\varepsilon}_{k,t},$$

and similarly the variance ratio and M unit root tests are constructed using the de-trended data, although in the definition of the MZ_α statistic the term $-n^{-1}\hat{x}_0^2$ needs to be added to the numerator of (4); see, for example, Müller and Elliott (2003). In both these cases the results in Theorems 1 and 2 (together with the corresponding results in section 4.1 for the near-integrated case) remain valid provided the standard Brownian motion process, $W_0(r)$, and the standard Ornstein-Uhlenbeck process, $J_{c_0,0}$, in (17) and (18) are replaced by their OLS de-trended counterparts; e.g., $\widetilde{W}_0(r) := W_0(r) - \int_0^1 W_0(s)\,ds$, for the OLS de-meaned case, and $\widehat{W}_0(r) := W_0(r) - (4-6r)\int_0^1 W_0(s)\,ds - (12r-6)\int_0^1 sW_0(s)\,ds$, for the OLS linear de-trended case. Finally note also, that for the corresponding unit root tests based on local GLS de-trending (see, *inter alia*, Elliott, Rothenberg and Stock, 1996, and Ng and Perron, 2001) then the previous results again hold but now replacing the standard Brownian motion, $W_0(r)$, and standard Ornstein-Uhlenbeck processes, $J_{c_0,0}$ by their local GLS de-meaned or linear de-trended counterparts; see Elliott *et al.* (1996,pp 824-825) for precise details.

5 Monte Carlo Experiments

In this section, we use simulation methods to investigate the finite sample properties of the unit root tests discussed in section 2.2 when (near-) non-stationary cycles are present in the data. All results reported in this section are based on 20,000 Monte Carlo replications using the RNDN function of Gauss 9.0. Unless otherwise stated, results are presented for both de-meaned and linear de-trended data.

Our first set of experiments, reported in Tables 1a (OLS de-meaned data) and Table 1b (linear OLS de-trended data), relate to data generated according to the DGP

$$\left(1-\left(1+\frac{c_0}{n}\right)L\right)\left(1-2\cos(\phi)\left(1+\frac{c_\phi}{n}\right)L+\left(1+\frac{c_\phi}{n}\right)^2L^2\right)x_t=\epsilon_t\sim NIID(0,1) \tag{31}$$

with $c_0 = 0$, so that the unit root null hypothesis holds, and with $\phi \in \{\pi/7, \pi/6, \pi/5, \pi/4, \pi/2\}$ and $c_\phi \in \{0, 5, 10\}$, in each case for a sample size of $n = 200$, initialised at $x_{-2} = x_{-1} = x_0 = 0$. Although the frequency $\pi/2$ would not be considered a low frequency component, it is nonetheless a seasonal frequency component for any case where the number of seasons is even (e.g. monthly or quarterly data) and therefore seems worth including. For comparative purposes, results for the conventional random walk, $(1-L)x_t = \epsilon_t \sim NIID(0,1)$, initialised at $x_0 = 0$, are also provided in the rows labelled '0'.

The results in Tables 1a and 1b report the empirical (null) rejection frequencies of the unit root tests from section 2.2 in each case for a nominal 5% significance level using the asymptotic critical values appropriate to the case where (near-) non-stationary cycles are not present in the data; that is, from $\mathcal{DF}_1$, $\mathcal{DF}_2$, $\mathcal{MSB}$ or $\mathcal{VRT}$, as appropriate. In Table 1, the ADF $t_{\hat{\rho}}$ and $Z_{\hat{\rho}}$ tests were computed from the ADF regression (2) for the true lag length, $k = 2$. In the context of the trinity of M tests, the superscript used in the nomenclature of Tables 1a and 1b denotes the long run variance estimator used; the subscripts b and q indicate that $\hat{\lambda}^2 = s_{WA}^2$ with the Barlett and quadratic spectral kernels, respectively, while the subscript AR indicates that $\hat{\lambda}^2 := s_{AR}^2$. For $\hat{\lambda}^2 = s_{WA}^2$, results are reported for the Bartlett and quadratic spectral kernel, using the data-dependent bandwidth formulations for these kernels suggested in Newey and West (1994, equations (3.8) to (3.15) and Table 1). For $\hat{\lambda}^2 := s_{AR}^2$, we again set $k = 2$. Also reported in Tables 1a and 1b are the empirical rejection frequencies for the conventional t-ratio tests on the lagged dependent variables Δx_{t-1} and Δx_{t-2} from (2), denoted $t_{\hat{\alpha}_1}$ and $t_{\hat{\alpha}_2}$, in each case compared to the 0.05 level critical values from the standard normal distribution (a 5% rule).

Insert Tables 1a and 1b about here

As predicted by the asymptotic distribution theory in Theorem 1 and section 4.1, it is only the ADF $t_{\hat{\rho}}$ and $Z_{\hat{\rho}}$ tests which display finite sample size properties which are robust to the presence of (near-) non-stationary cycles in the data. The size properties of the $t_{\hat{\rho}}$ test are somewhat better than those of $Z_{\hat{\rho}}$ which is a little over-sized, most notably so in the case of linear de-trended data, but both show no significant variations in size from the random walk base case under non-stationary or near-non-stationary cycles for all of the frequencies considered. Again in line with the predictions from the asymptotic theory, we see that this is not the case for the other tests considered. Again as predicted by the results in Theorem 1 and section 4.1, the empirical sizes of the VRT test vary considerably across both ϕ and c_ϕ. As might be expected, for a given frequency ϕ, the size distortions in VRT decrease as c_ϕ increases; this is because the cyclical component at frequency ϕ moves further away from the non-stationarity boundary as c_ϕ increases. The size distortions for the three M tests with the autoregressive spectral density estimator show even greater variation across ϕ than the VRT test. Overall though, even though the VRT and M

tests with $\hat{\lambda}^2 = s^2_{WA}$ are not asymptotically degenerate under (near-) non-stationary cycles the results in Tables 1a and 1b suggest that none of these tests could reliably be used in practice. The degeneracy of the kernel-based M tests is clearly reflected in Tables 1a and 1b, although again as with the VRT test there is some amelioration of this in the case of the MSB test for a given value of ϕ as c_ϕ is increased. Finally, we observe that the empirical rejection frequency of the $t_{\hat{\alpha}_2}$ test with a 5% rule is unity throughout (except in the random walk case where it is essentially correctly sized), implying that in the presence of (near-) non-stationary cycles the sequential method of Hall (1994) and Ng and Perron (2005) will always retain both lagged dependent variables in (2), as would be hoped. Indeed, we obtained the same outcome when a tighter 1% rule was used. More generally, in unreported simulations we found that in the case outlined in the latter part of Remark 6, where $\Psi(L)$ is a pth order polynomial of non-stationary factors, the same finding holds for the t-test on the pth lag, so that p lagged dependent variables will always be included in the ADF regression.

Insert Tables 2 – 5 about here

In a second experiment, we now investigate in detail the impact of a number of conventional lag selection methods on the empirical size and power properties of the $t_{\hat{\rho}}$ test (again run at the nominal asymptotic 5% level). Corresponding results for $Z_{\hat{\rho}}$ are available on request. Results are reported for the sequential method (denoted SQ) outlined in section 2.2 using a 10% rule (as is commonly done in practice), the standard AIC rule, and also the modified AIC (denoted $MAIC$) rule of Ng and Perron (2001). For all of these methods, results are reported for maximum lag lengths of $k_{\max 4} := \lfloor 4[\frac{n}{100}]^{1/4} \rfloor$ and $k_{\max 12} := \lfloor 12[\frac{n}{100}]^{1/4} \rfloor$; with the subscript 4 or 12 on each method denoting which of these maximum lag lengths was used. No minimum lag length was set for any of the selection methods. The data were again generated according to (31) with the parameter settings as were considered for the results in Tables 1a and 1b, but augmented to include results for $c_0 \in \{0, 5, 10, 13, 20, 25, 30\}$, rather than just $c_0 = 0$. For the case of $c_0 = 0$ (Table 2) results are presented only for $c_\phi = 0$. To ensure comparability with the other cases of ϕ, the results reported for $\phi = 0$ in Tables 2 and 3 pertain to the DGP $\left(1 - \left(1 + \frac{c_0}{n}\right) L\right)(1 - 0.5L^2)x_t = \epsilon_t \sim NIID(0, 1)$ initialised at $x_{-2} = x_{-1} = x_0 = 0$; this process, like the cases considered where $\phi \neq 0$, has a true lag length of $k = 2$ in (2), but here the cyclical pair of roots are stable (both have modulus $\sqrt{0.5}$ and lie at frequency $\pi/2$). Finally, for comparative purposes, results are presented both for tests based on OLS de-trending and the corresponding tests based on local GLS de-trending.

Table 2 reports the empirical size ($c_0 = c_\phi = 0$) obtained with the three lag-order selection methods and also reports the average order of lag augmentation selected by each approach. The results in Table 2 indicate that the empirical size of the ADF test with OLS de-trending is reasonably close to the nominal level throughout for both SEQ and AIC, although both methods yield slightly over-sized tests in the case of $k_{\max 12}$ with linear de-trending. As regards the tests which are based on the $MAIC$ rule, here a degree of under-sizing is seen throughout; this is perhaps not unexpected given that the penalty function on which the MAIC rule is based is in fact misspecified when non-stationary cycles are present in the DGP, as is the case here. Similar comments apply when local GLS de-trending is used, although here it is noteworthy that the tests based on $MAIC$ can be very badly undersized in the case of linear de-trending; indeed all of the tests display a tendency to undersize here. As regards the average lag length chosen, it is seen that SQ_4, AIC_4, AIC_{12} and $MAIC_4$ get reasonably close to the true order (recall that this is two throughout), while both $MAIC_{12}$ and SQ_{12} over-fit the lag order, most notably so in the case of SQ_{12}. Such over-fitting will of course necessarily lead to power losses under the alternative, as will be seen in the results Tables 3-5. In the case of SQ_{12} it should be noted that this result is not attributable to the presence of non-stationary cycles because it happens to the same degree in the $\phi = 0$ case where non-stationary cycles are not present in the data. In contrast for $MAIC_{12}$ (and to a lesser extent $MAIC_4$) we see that the degree of over-fitting is higher when non-stationary cycles are present relative to the case where they are not.

Table 3 ($c_\phi = 0$), Table 4 ($c_\phi = 2.5$) and Table 5 ($c_\phi = 10$) report the empirical power of the $t_{\hat{\rho}}$ test for $c_0 = 5, 10, 13, 20, 25, 30$ under the various lag selection rules. Results in these tables are reported only for the case of linear de-trended data; qualitatively similar results were seen for the case of de-meaned data and may be obtained from the authors on request. Overall, for a given value of c_ϕ, the results are qualitatively very similar regardless of the frequency at which the non-stationary cyclical roots occur. The best power performance for both OLS and local GLS de-trending is obtained when either SQ_4 or AIC_4 is used to specify the lag augmentation length, consistent with the findings in Table 2 on average lag length fitted by the various rules. The ramifications of the over-fitting seen in Table 2 for the $MAIC_{12}$ and SQ_{12} rules is clearly seen in Tables 3-5 with the tests based on these rules showing considerably lower power throughout than the tests based on the other lag selection methods, other things being equal.

Another interesting aspect of the power results is seen most clearly in the pure non-stationary cycles case in Table 3. Here we see that the finite sample power of the OLS de-trended tests are, for a given lag selection rule, fairly insensitive to frequency at which the non-stationary cycle occurs, and indeed as to whether a non-stationary cycle occurs or not. The same cannot be said for the local GLS de-trended tests. To illustrate, in the $\phi = 0$ case, where no non-stationary cycles are present, the power of the local GLS de-trended tests are clearly superior to those of the corresponding OLS de-trended tests; for example, with SQ_4 and $c_0 = 20$ the local GLS test has power 70.2 %, while the OLS test has power 57.4 %. However, for $\phi \neq 0$ the converse tends to be the case with the OLS de-trended tests now the more powerful; for example, for $\phi = \pi/7(\pi/2)$ using SQ_4, for $c_0 = 13$ the OLS test now has power of 61.0% (62.7%) and the local GLS test 54.9% (33.9%). This reflects the fact that the local GLS de-trending method is based on the assumption that, aside from the possible zero frequency unit root, the process is stationary, and it is clear that where this assumption is violated the finite sample power of the local GLS de-trended tests suffers considerably relative to their OLS de-trended counterparts. This is most likely attributable to the fact that the local GLS estimates of the parameters characterising the deterministic trend component will be highly inefficient, relative to the corresponding OLS estimates, in this case. As the cyclical component becomes less persistent (i.e. as c_ϕ increases away from zero) then so we would expect the finite sample power of the local GLS de-trended tests to recover, and a comparison of the results in Tables 4 and 5 with those in Table 3 shows that this is indeed the case; in the foregoing example when $c_\phi = 10$ the local GLS test has power 77.2% and 63.6% for $\pi/7$ and $\pi/2$, respectively, while the OLS test has power 57.3% and 61.1% for $\pi/7$ and $\pi/2$, respectively.

6 Conclusions

In this paper we have shown that among popularly applied unit root test statistics, only the ADF t-ratio and normalised bias statistics have pivotal limiting null distributions in the presence of (near-) non-stationary cycles in the data. Other commonly employed unit root test statistics, such as the variance ratio statistic of Breitung (2002) and the trinity of M statistics due to Stock (1999) and Perron and Ng (1996), were shown either to admit non-pivotal limiting null distributions or to have non-degenerate limiting null distributions, in the latter case yielding tests with an asymptotic size of one, when (near-) non-stationary cycles are present. Additionally, we have shown that the t-ratios on the lagged dependent variables within the ADF test regression also retain standard normal limiting null distributions such that sequential lag specification also remains valid under (near-) non-stationary cycles. Consequently, we strongly recommend the use of ADF unit root tests coupled with sequential lag selection in cases where it is suspected that (near-) non-stationary cycles may be present in the data. In such cases our results also suggest that the finite sample power advantages of local GLS de-trended ADF-type tests over their OLS de-trended counterparts seen when any cyclical behaviour is stationary are likely to be overturned when non-stationary cycles are present.

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A Appendix

Before presenting the proofs of the main text, we note the following trigonometric identities which will be used in the sequel:

$$\begin{aligned}
\sin t\phi &\equiv \cos(\phi)\sin(\phi(t+1)) - \sin(\phi)\cos(\phi(t+1)) && \text{(A.1)}\\
\cos t\phi &\equiv \cos(\phi)\cos(\phi(t+1)) + \sin(\phi)\sin(\phi(t+1)) && \text{(A.2)}\\
\sin\phi(t-1) &\equiv \cos^2(\phi)\sin(\phi(t+1)) - 2\cos(\phi)\sin(\phi)\cos(\phi(t+1))\\
&\quad - \sin^2(\phi)\sin(\phi(t+1)) && \text{(A.3)}\\
\cos\phi(t-1) &\equiv \cos^2(\phi)\cos(\phi(t+1)) + 2\cos(\phi)\sin(\phi)\sin(\phi(t+1))\\
&\quad - \sin^2(\phi)\cos(\phi(t+1)) && \text{(A.4)}\\
\cos(2\phi) &\equiv \cos^2(\phi) - \sin^2(\phi) && \text{(A.5)}
\end{aligned}$$

where in each case $\phi \in (0, \pi)$.

Moreover, for the DGP in (1), and using the representation of the partial sum of an AR(2) process with complex unit roots given in Equation (2) of Bierens (2001,p.963), we can write the following spectral decompositions of Δx_t, Δx_{t-1} and Δx_{t-2},

$$\begin{aligned}
\Delta x_t &= \sum_{j=1}^{t} \frac{\sin[\phi(t+1-j)]}{\sin\phi}\varepsilon_j\\
&= \frac{1}{\sin\phi}[\sin(\phi(t+1))S_\alpha(t) - \cos(\phi(t+1))S_\beta(t)]; && \text{(A.6)}\\
\Delta x_{t-1} &= L\sum_{j=1}^{t} \frac{\sin[\phi(t+1-j)]}{\sin\phi}\varepsilon_j\\
&= \frac{1}{\sin\phi}[\sin(\phi t)S_\alpha(t-1) - \cos(\phi t)S_\beta(t-1)]\\
&= \frac{\cos\phi}{\sin\phi}[\sin(\phi(t+1))S_\alpha(t-1) - \cos(\phi(t+1))S_\beta(t-1)]\\
&\quad - \frac{\sin\phi}{\sin\phi}[\cos(\phi(t+1))S_\alpha(t-1) + \sin(\phi(t+1))S_\beta(t-1)] && \text{(A.7)}
\end{aligned}$$

and

$$\begin{aligned}
\Delta x_{t-2} &= L^2\sum_{j=1}^{t} \frac{\sin[\phi(t+1-j)]}{\sin\phi}\varepsilon_j\\
&= \frac{1}{\sin\phi}[\sin(\phi(t-1))S_\alpha(t-2) - \cos(\phi(t-1))S_\beta(t-2)]\\
&= \frac{\cos(2\phi)}{\sin\phi}[\sin(\phi(t+1))S_\alpha(t-2) - \cos(\phi(t+1))S_\beta(t-2)]\\
&\quad - \frac{2\cos\phi\sin\phi}{\sin\phi}[\cos(\phi(t+1))S_\alpha(t-2) - \sin(\phi(t+1))S_\beta(t-2)] && \text{(A.8)}
\end{aligned}$$

where we have defined the ϕ-frequency partial sum processes, $S_\alpha(t) := \sum_{j=1}^{t}\varepsilon_j\cos(j\phi)$ and $S_\beta(t) := \sum_{j=1}^{t}\varepsilon_j\sin(j\phi)$.

Proof of Lemma 1: From Property 2.5 in Gregoir (1999,p.440), which makes uses of the identity

$$1 = \frac{1}{2(1-\cos(\phi))}\left(1 - 2\cos(\phi)L + L^2\right) + \frac{(1 - 2\cos(\phi) + L)}{2(1-\cos(\phi))}(1-L) \tag{A.9}$$

see Gregoir (1999,p.462), it follows that if $\{x_t\}$ is generated by (1) then,

$$\begin{aligned}
x_t &= \frac{1}{2(1-\cos\phi)}\sum_{j=1}^{t}\varepsilon_j + \frac{1-2\cos\phi+L}{2(1-\cos\phi)}\sum_{j=1}^{t}\frac{\sin[\phi(t+1-j)]}{\sin\phi}\varepsilon_j && \text{(A.10)}\\
&=: \mathcal{C}_{0t} + \mathcal{C}_{\phi t}. && \text{(A.11)}
\end{aligned}$$

Consequently, noting that the ϕ-frequency component, $\mathcal{C}_{\phi t}$, of (A.10) is,

$$\begin{aligned}\mathcal{C}_{\phi t} &= \frac{1-2\cos\phi}{2(1-\cos\phi)}\frac{1}{\sin\phi}\left(\sin(\phi(t+1))S_\alpha(t)-\cos(\phi(t+1))S_\beta(t)\right)\\ &\quad+\frac{1}{2(1-\cos\phi)}\frac{1}{\sin\phi}\left(\sin(\phi t)\,S_\alpha(t-1)-\cos(\phi t)\,S_\beta(t-1)\right)\end{aligned}$$

it follows from (A.7) that, after simplification,

$$\begin{aligned}\mathcal{C}_{\phi t} &= \frac{\delta_\phi^\alpha}{\sin\phi}\left[\sin(\phi(t+1))S_\alpha(t)-\cos(\phi(t+1))S_\beta(t)\right]\\ &\quad+\frac{\delta_\phi^\beta}{\sin\phi}\left[\cos(\phi\,(t+1))\,S_\alpha(t)+\sin(\phi\,(t+1))\,S_\beta(t)\right]+O_p(1)\end{aligned}$$

where $\delta_\phi^\alpha := \frac{1-\cos\phi}{2(1-\cos\phi)}$ and $\delta_\phi^\beta := -\frac{\sin\phi}{2(1-\cos\phi)}$. As a consequence, we therefore have, on defining the zero-frequency partial sum process $S_0(t) := \sum_{j=1}^t \varepsilon_j$, that

$$\begin{aligned}x_t &= \frac{1}{2(1-\cos\phi)}S_0(t)+\frac{\delta_\phi^\alpha}{\sin\phi}\left[\sin(\phi(t+1))S_\alpha(t)-\cos(\phi(t+1))\,S_\beta(t)\right]\\ &\quad+\frac{\delta_\phi^\beta}{\sin\phi}\left[\cos(\phi\,(t+1))\,S_\alpha(t)+\sin(\phi\,(t+1))\,S_\beta(t)\right]+O_p(1) \qquad \text{(A.12)}\end{aligned}$$

which establishes the result in (7). The result in (8) follows directly from (A.6). ■

Before proving the results in Lemma 2, we first provide some additional results in a preparatory lemma, relevant to the computation of the unit root statistics from section 2.2.

Lemma A.1 *Let the conditions of Lemma 1 hold. Then*

$$\begin{aligned} i)\ \sum_{t=1}^n x_{t-1}^2 &= \frac{1}{4(1-\cos\phi)^2}\sum_{t=1}^n S_0^2(t)\\ &\quad+\frac{1}{2(1-\cos\phi)}\frac{1}{2\sin^2\phi}\sum_{t=1}^n\left(S_\alpha^2(t-1)+S_\beta^2(t-1)\right)+o_p(n^2)\\ ii)\ \sum_{t=1}^n \Delta x_t^2 &= \sum_{t=1}^n\frac{S_\alpha^2(t)+S_\beta^2(t)}{2\sin^2\phi}+o_p(n^2)\end{aligned}$$

$$\begin{aligned} iii)\ \sum_{t=1}^n \Delta x_{t-1}^2 &= \sum_{t=1}^n\frac{S_\alpha^2(t-1)+S_\beta^2(t-1)}{2\sin^2\phi}+o_p(n^2)\\ iv)\ \sum_{t=1}^n \Delta x_{t-2}^2 &= \sum_{t=1}^n\frac{S_\alpha^2(t-2)+S_\beta^2(t-2)}{2\sin^2\phi}+o_p(n^2)\\ v)\ \sum_{t=1}^n x_{t-1}\Delta x_{t-1} &= \frac{1}{4\sin^2\phi}\sum_{t=1}^n\left(S_\alpha^2(t-1)+S_\beta^2(t-1)\right)+o_p(n^2)\\ vi)\ \sum_{t=1}^n x_{t-1}\Delta x_{t-2} &= \frac{\cos\phi+\frac{1}{2}}{2\sin^2\phi}\sum_{t=1}^n\left(S_\alpha^2(t-1)+S_\beta^2(t-1)\right)+o_p(n^2).\end{aligned}$$

Proof of Lemma A.1: The decompositions in (i)-(vi) obtain using the results in Lemma 1, the trigonometric identities in (A.1)-(A.5), and invoking the result that the different partial sums that compose the moments expressed in (i)-(vi) are asymptotically uncorrelated (see Chan and Wei, 1988, Theorem 3.4.1, p.393). Note that the asymptotic uncorrelatedness of these components also holds in the near integrated context considered in Section 4; see Jeganathan (1991, Proposition 5, p.281). ■

Proof of Lemma 2: First define

$$\begin{aligned} W_{\phi n}^{\alpha}(t/n) &:= \frac{\sqrt{2}}{\sigma\sqrt{n}}\sum_{j=1}^{\lfloor rn\rfloor}\cos(j\phi)\,\varepsilon_j = \frac{\sqrt{2}}{\sigma\sqrt{n}}S_{\alpha}(r) \\ W_{\phi n}^{\beta}(t/n) &:= \frac{\sqrt{2}}{\sigma\sqrt{n}}\sum_{j=1}^{\lfloor rn\rfloor}\sin(j\phi)\,\varepsilon_j = \frac{\sqrt{2}}{\sigma\sqrt{n}}S_{\beta}(r) \end{aligned}$$

and $W_{0n}(t/n) := \frac{1}{\sigma\sqrt{n}}\sum_{j=1}^{\lfloor rn\rfloor}\varepsilon_j$. Using these definitions, we then have from Lemma 1, and noting that $\Delta_{\phi}x_t = S_0(t) = \sum_{j=1}^{t}\varepsilon_j$, that

$$\begin{aligned} \frac{1}{\sqrt{n}}x_t &= \sigma\left[\delta_0 W_{0,n}(t/n) + \frac{\delta_{\phi}^{\alpha}}{\sqrt{2}\sin\phi}\left[\sin(\phi(t+1))W_{\phi,n}^{\alpha}(t/n) - \cos(\phi(t+1))\,W_{\phi,n}^{\beta}(t/n)\right]\right. \\ &\quad \left. + \frac{\delta_{\phi}^{\beta}}{\sqrt{2}\sin\phi}\left[\cos(\phi(t+1))\,W_{\phi,n}^{\alpha}(t/n) + \sin(\phi(t+1))\,W_{\phi,n}^{\beta}(t/n)\right]\right] + o_p(1) \\ &\Rightarrow \sigma\delta_0 W_0(r) + \frac{\sigma\delta_{\phi}^{\alpha}}{\sqrt{2}\sin\phi}\left[\sin(\phi(t+1))W_{\phi}^{\alpha}(r) - \cos(\phi(t+1))\,W_{\phi}^{\beta}(r)\right] \\ &\quad + \frac{\sigma\delta_{\phi}^{\beta}}{\sqrt{2}\sin\phi}\left[\cos(\phi(t+1))\,W_{\phi}^{\alpha}(r) + \sin(\phi(t+1))\,W_{\phi}^{\beta}(r)\right] := \sigma\mathbf{b}(r) \end{aligned} \tag{A.13}$$

$$\begin{aligned} \frac{1}{\sqrt{n}}\Delta x_t &= \frac{\sigma}{\sqrt{2}\sin\phi}\left[\sin(\phi(t+1))W_{\phi,n}^{\alpha}(t/n) - \cos(\phi(t+1))\,W_{\phi,n}^{\beta}(t/n)\right] \\ &\Rightarrow \frac{\sigma}{\sqrt{2}\sin\phi}\left[\sin(\phi(t+1))W_{\phi}^{\alpha}(r) - \cos(\phi(t+1))\,W_{\phi}^{\beta}(r)\right] \end{aligned} \tag{A.14}$$

$$\frac{1}{\sqrt{n}}\Delta_{\phi}x_t = \sigma W_{0,n}(t/n) \Rightarrow \sigma W_0(r). \tag{A.15}$$

Next observe that

$$\begin{aligned} \sum_{t=1}^{n}x_{t-1}\varepsilon_t &= \frac{1}{2(1-\cos\phi)}\sum_{t=1}^{n}S_0(t-1)\varepsilon_t - \frac{1}{2\sin\phi}\left(\sum_{t=1}^{n}S_{\alpha}(t-1)\Delta S_{\beta}(t) - \sum_{t=1}^{n}S_{\beta}(t-1)\Delta S_{\alpha}(t)\right) \\ &\quad - \frac{1}{2(1-\cos(\phi))}\left(\sum_{t=1}^{n}S_{\alpha}(t-1)\Delta S_{\alpha}(t) + \sum_{t=1}^{n}S_{\beta}(t-1)\Delta S_{\beta}(t)\right) + o_p(n) \end{aligned}$$

and, hence,

$$\begin{aligned} \frac{1}{n}\sum_{t=1}^{n}x_{t-1}\varepsilon_t &= \frac{\sigma^2}{2(1-\cos\phi)}\sum_{t=1}^{n}W_{0n}((t-1)/n)\,\Delta W_{0n}(t/n) \\ &\quad + \frac{\sigma^2}{4\sin\phi}\left(\sum_{t=1}^{n}W_{\phi n}^{\alpha}((t-1)/n)\,dW_{\phi n}^{\beta}(t/n) - \sum_{t=1}^{n}W_{\phi n}^{\beta}((t-1)/n)\,dW_{\phi n}^{\alpha}(t/n)\right) \\ &\quad - \frac{\sigma^2}{4(1-\cos(\phi))}\left(\sum_{t=1}^{n}W_{\phi n}^{\alpha}((t-1)/n)\,dW_{\phi n}^{\alpha}(t/n) + \sum_{t=1}^{n}W_{\phi n}^{\beta}((t-1)/n)\,dW_{\phi n}^{\beta}(t/n)\right) + o_p(1) \\ &\Rightarrow \frac{\sigma^2}{2(1-\cos\phi)}\int_0^1 W_0(r)\,dW_0(r) + \frac{\sigma^2}{4\sin\phi}\left(\int_0^1 W_{\phi}^{\alpha}(r)\,dW_{\phi}^{\beta}(r) - \int_0^1 W_{\phi}^{\beta}(r)\,dW_{\phi}^{\alpha}(r)\right) \\ &\quad - \frac{\sigma^2}{4(1-\cos(\phi))}\left(\int_0^1 W_{\phi}^{\alpha}(r)\,dW_{\phi}^{\alpha}(r) + \int_0^1 W_{\phi}^{\beta}(r)\,dW_{\phi}^{\beta}(r)\right). \end{aligned}$$

The proof of the results for $n^{-1}\sum_{t=1}^{n}\Delta x_{t-1}\varepsilon_t$ and $n^{-1}\sum_{t=1}^{n}\Delta x_{t-2}\varepsilon_t$ follow along similar lines and are therefore omitted. Furthermore, the stated convergence results for $n^{-2}\sum_{t=1}^{n}x_{t-1}^2$, $n^{-2}\sum_{t=1}^{n}\Delta x_t^2$, $n^{-2}\sum_{t=1}^{n}x_{t-1}\Delta x_{t-1}$ and $n^{-2}\sum_{t=1}^{n}x_{t-1}\Delta x_{t-2}$ follow straightforwardly from (i), (ii), (v) and (vi) of Lemma A.1, respectively, and applications of the CMT. ■

Proof of Theorems 1 and 2: First we define $\mathbf{z}_{t-1} := [x_{t-1}\ \Delta x_{t-1}\ \Delta x_{t-2}\ \Delta x_{t-3}\ \ldots\ \Delta x_{t-k}]'$ and $\boldsymbol{\beta} := (\rho, \alpha_1, \alpha_2, \alpha_3, ..., \alpha_k)'$. Hence, to analyse the convergence of the OLS parameter estimates from (2) we will make use of the following expression for the OLS estimation error

$$\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta} = \left[\sum_{t=1}^{n} \mathbf{z}_{t-1}\mathbf{z}_{t-1}'\right]^{-1} \left[\sum_{t=1}^{n} \mathbf{z}_{t-1}\varepsilon_t\right]. \tag{A.16}$$

Notice that under the conditions of Theorems 1 and 2, the true values of the parameters are given by $\boldsymbol{\beta} = (0,\ -2\cos\phi,\ 1, 0, ..., 0)'$.

Using a similar approach to Choi (1993), we define the $(k+1)\times(k+1)$ matrix

$$\mathbf{A} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -2\cos\phi & 1 & \cdots & 0 \\ 0 & 0 & 0 & 1 & -2\cos\phi & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & -2\cos\phi & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2\cos\phi \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} =: [\mathbf{A}_1\ \mathbf{A}_2] \tag{A.17}$$

where $\mathbf{A}_1$ is a $(k+1)\times 3$ matrix and $\mathbf{A}_2$ a $(k+1)\times(k-2)$ matrix. Notice that $\mathbf{A}_2$ is a filtration matrix since, $\mathbf{A}_2'\mathbf{z}_{t-1} = (\Delta_\phi\Delta x_{t-1}, \Delta_\phi\Delta x_{t-2}, ..., \Delta_\phi\Delta x_{t-k})'$. Using the matrix $\mathbf{A}$, and introducing the scaling matrix $\Upsilon_n := diag\{\boldsymbol{\Upsilon}_{1n}, \boldsymbol{\Upsilon}_{2n}\}$ where $\boldsymbol{\Upsilon}_{1n} := diag\{n,n,n\}$ and $\boldsymbol{\Upsilon}_{2n} := diag\{\sqrt{n}, ..., \sqrt{n}\}$, the latter a $(k-2)\times(k-2)$ matrix, we can rewrite the scaled estimator from (A.16) as

$$\Upsilon_n\left(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) = \mathbf{A}\left\{\Upsilon_n^{-1}\begin{bmatrix} \sum_{t=1}^{n}\mathbf{A}_1'\mathbf{z}_{t-1}\mathbf{z}_{t-1}'\mathbf{A}_1 & \sum_{t=1}^{n}\mathbf{A}_1'\mathbf{z}_{t-1}\mathbf{z}_{t-1}'\mathbf{A}_2 \\ \sum_{t=1}^{n}\mathbf{A}_2'\mathbf{z}_{t-1}\mathbf{z}_{t-1}'\mathbf{A}_1 & \sum_{t=1}^{n}\mathbf{A}_2'\mathbf{z}_{t-1}\mathbf{z}_{t-1}'\mathbf{A}_2 \end{bmatrix}^{-1}\Upsilon_n^{-1}\right\}\Upsilon_n^{-1}\begin{bmatrix} \sum_{t=1}^{n}\mathbf{A}_1'\mathbf{z}_{t-1}\varepsilon_t \\ \sum_{t=1}^{n}\mathbf{A}_2'\mathbf{z}_{t-1}\varepsilon_t \end{bmatrix}$$

$$= \mathbf{A}\left\{\begin{bmatrix} \Upsilon_{1n}^{-1}\sum_{t=1}^{n}\mathbf{A}_1'\mathbf{z}_{t-1}\mathbf{z}_{t-1}'\mathbf{A}_1\Upsilon_{1n}^{-1} & o_p(1) \\ o_p(1) & \Upsilon_{2n}^{-1}\sum_{t=1}^{n}\mathbf{A}_2'\mathbf{z}_{t-1}\mathbf{z}_{t-1}'\mathbf{A}_2\Upsilon_{2n}^{-1} \end{bmatrix}^{-1}\right\}\begin{bmatrix} \Upsilon_{1n}^{-1}\sum_{t=1}^{n}\mathbf{A}_1'\mathbf{z}_{t-1}\varepsilon_t \\ \Upsilon_{2n}^{-1}\sum_{t=1}^{n}\mathbf{A}_2'\mathbf{z}_{t-1}\varepsilon_t \end{bmatrix}. \tag{A.18}$$

We now establish convergence results for the elements in (A.18). First we observe that

$$\Upsilon_{1n}^{-1}\sum_{t=1}^{n}\mathbf{A}_1'\mathbf{z}_{t-1}\mathbf{z}_{t-1}'\mathbf{A}_1\Upsilon_{1n}^{-1} = \begin{bmatrix} \frac{1}{n^2}\sum_{t=1}^{n}x_{t-1}^2 & \frac{1}{n^2}\sum_{t=1}^{n}x_{t-1}\Delta x_{t-1} & \frac{1}{n^2}\sum_{t=1}^{n}x_{t-1}\Delta x_{t-2} \\ \frac{1}{n^2}\sum_{t=1}^{n}x_{t-1}\Delta x_{t-1} & \frac{1}{n^2}\sum_{t=1}^{n}(\Delta x_{t-1})^2 & \frac{1}{n^2}\sum_{t=1}^{n}\Delta x_{t-1}\Delta x_{t-2} \\ \frac{1}{n^2}\sum_{t=1}^{n}x_{t-1}\Delta x_{t-2} & \frac{1}{n^2}\sum_{t=1}^{n}\Delta x_{t-1}\Delta x_{t-2}' & \frac{1}{n^2}\sum_{t=1}^{n}(\Delta x_{t-2})^2 \end{bmatrix}$$

which, as $n\to\infty$, will converge, using results in Lemma 2, to

$$\Upsilon_{1n}^{-1}\sum_{t=1}^{n}\mathbf{A}_1'\mathbf{z}_{t-1}\mathbf{z}_{t-1}'\mathbf{A}_1\Upsilon_{1n}^{-1} \Rightarrow \sigma^2\int_0^1 \Xi(r)dr \tag{A.19}$$

where

$$\int_0^1 \Xi(r)dr := \begin{bmatrix} \frac{\mathcal{V}_0^2}{4(1-\cos\phi)^2} + \frac{1}{2(1-\cos\phi)}\frac{\left(\mathcal{A}^2+\mathcal{B}^2\right)}{4\sin^2\phi} & \frac{\left(\mathcal{A}^2+\mathcal{B}^2\right)}{8\sin^2\phi} & \frac{\left(\cos\phi+\frac{1}{2}\right)\left(\mathcal{A}^2+\mathcal{B}^2\right)}{4\sin^2\phi} \\ \frac{\left(\mathcal{A}^2+\mathcal{B}^2\right)}{8\sin^2\phi} & \frac{\left(\mathcal{A}^2+\mathcal{B}^2\right)}{4\sin^2\phi} & \frac{\cos\phi\left(\mathcal{A}^2+\mathcal{B}^2\right)}{4\sin^2\phi} \\ \frac{\left(\cos\phi+\frac{1}{2}\right)\left(\mathcal{A}^2+\mathcal{B}^2\right)}{4\sin^2\phi} & \frac{\cos\phi\left(\mathcal{A}^2+\mathcal{B}^2\right)}{4\sin^2\phi} & \frac{\left(\mathcal{A}^2+\mathcal{B}^2\right)}{4\sin^2\phi} \end{bmatrix}$$

with $\mathcal{V}_0^2$, $\mathcal{A}^2$ and $\mathcal{B}^2$ as defined in Theorem 1. Consequently, for the inverse of (A.19) we have that

$$\left(\Upsilon_{1n}^{-1}\sum_{t=1}^{n}\mathbf{A}_1'\mathbf{z}_{t-1}\mathbf{z}_{t-1}'\mathbf{A}_1\Upsilon_{1n}^{-1}\right)^{-1}\Rightarrow\left[\sigma^2\int_0^1\Xi(r)dr\right]^{-1} \tag{A.20}$$

where $\left[\int_0^1\Xi(r)dr\right]^{-1}=$

$$\begin{bmatrix} \frac{4(1-\cos(\phi))^2}{\mathcal{V}_0^2} & \frac{2(2\cos\phi-1)(1-\cos\phi)}{\mathcal{V}_0^2} & -\frac{2(1-\cos\phi)}{\mathcal{V}_0^2} \\ \frac{2(2\cos\phi-1)(1-\cos\phi)}{\mathcal{V}_0^2} & \frac{4\mathcal{V}_0^2+[3-4\cos(\phi)](\mathcal{A}^2+\mathcal{B}^2)}{\mathcal{V}_0^2(\mathcal{A}^2+\mathcal{B}^2)} & \frac{(\mathcal{A}^2+\mathcal{B}^2)-2(2\mathcal{V}_0^2+\mathcal{A}^2+\mathcal{B}^2)\cos(\phi)}{\mathcal{V}_0^2(\mathcal{A}^2+\mathcal{B}^2)} \\ -\frac{2(1-\cos\phi)}{\mathcal{V}_0^2} & \frac{(\mathcal{A}^2+\mathcal{B}^2)-2(2\mathcal{V}_0^2+\mathcal{A}^2+\mathcal{B}^2)\cos(\phi)}{\mathcal{V}_0^2(\mathcal{A}^2+\mathcal{B}^2)} & \frac{4\mathcal{V}_0^2+(\mathcal{A}^2+\mathcal{B}^2)}{\mathcal{V}_0^2(\mathcal{A}^2+\mathcal{B}^2)} \end{bmatrix}.$$

Next, using results in Lemma 2, we have that

$$\Upsilon_{1n}^{-1}\sum_{t=1}^{n}\mathbf{A}_1'\mathbf{z}_{t-1}\varepsilon_t=\begin{bmatrix}\frac{1}{n}\sum_{t=1}^{n}x_{t-1}\varepsilon_t\\ \frac{1}{n}\sum_{t=1}^{n}\Delta x_{t-1}\varepsilon_t\\ \frac{1}{n}\sum_{t=1}^{n}\Delta x_{t-2}\varepsilon_t\end{bmatrix}\Rightarrow\sigma^2Q(\phi)\Psi(r) \tag{A.21}$$

where $Q(\phi)=\begin{bmatrix}\frac{1}{2(1-\cos\phi)} & \frac{1}{4\sin\phi} & -\frac{1}{4(1-\cos(\phi))}\\ 0 & \frac{1}{2\sin\phi} & 0\\ 0 & \frac{\cos\phi}{2\sin\phi} & -\frac{1}{2}\end{bmatrix}$ and $\Psi(r)=\begin{bmatrix}\int_0^1 W_0(r)\,dW_0(r)\\ \int_0^1\left(W_\phi^\alpha(r)\,dW_\phi^\beta(r)-W_\phi^\beta(r)\,dW_\phi^\alpha(r)\right)\\ \int_0^1\left(W_\phi^\alpha(r)\,dW_\phi^\alpha(r)+W_\phi^\beta(r)\,dW_\phi^\beta(r)\right)\end{bmatrix}.$

Next we have $\Upsilon_{2n}^{-1}\sum\mathbf{A}_2'\mathbf{z}_{t-1}\mathbf{z}_{t-1}'\mathbf{A}_2\Upsilon_{2n}^{-1}=$

$$\begin{bmatrix}\frac{1}{n}\sum_{t=1}^{n}(\Delta_\phi\Delta x_{t-1})^2 & \frac{1}{n}\sum_{t=1}^{n}\Delta_\phi\Delta x_{t-1}\Delta_\phi\Delta x_{t-2} & \cdots & \frac{1}{n}\sum_{t=1}^{n}\Delta_\phi\Delta x_{t-1}\Delta_\phi\Delta x_{t-k}\\ \frac{1}{n}\sum_{t=1}^{n}\Delta_\phi\Delta x_{t-2}\Delta_\phi\Delta x_{t-1} & \frac{1}{n}\sum_{t=1}^{n}(\Delta_\phi\Delta x_{t-2})^2 & \cdots & \frac{1}{n}\sum_{t=1}^{n}\Delta_\phi\Delta x_{t-2}\Delta_\phi\Delta x_{t-k}\\ \vdots & \vdots & \vdots & \vdots\\ \frac{1}{n}\sum_{t=1}^{n}\Delta_\phi\Delta x_{t-k}\Delta_\phi\Delta x_{t-1} & \frac{1}{n}\sum_{t=1}^{n}\Delta_\phi\Delta x_{t-k}\Delta_\phi\Delta x_{t-2} & \cdots & \frac{1}{n}\sum_{t=1}^{n}(\Delta_\phi\Delta x_{t-k})^2\end{bmatrix}$$

and therefore by a standard law of large numbers result we have that

$$\Upsilon_{2n}^{-1}\sum_{t=1}^{n}\mathbf{A}_2'\mathbf{z}_{t-1}\mathbf{z}_{t-1}'\mathbf{A}_2\Upsilon_{2n}^{-1}\Rightarrow\mathbf{\Gamma} \tag{A.22}$$

where $\mathbf{\Gamma}$ is the $(k-2)\times(k-2)$ variance-covariance matrix

$$\mathbf{\Gamma}:=\begin{bmatrix}\gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{k-3}\\ \gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{k-2}\\ \gamma_2 & \gamma_1 & \gamma_0 & \cdots & \gamma_{k-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \gamma_{k-3} & \gamma_{k-2} & \gamma_{k-1} & \cdots & \gamma_0\end{bmatrix}$$

with γ_j denoting the jth order autocovariance of $\{\Delta\Delta_\phi x_t\}$. Moreover, from the CLT it follows that,

$$\Upsilon_{2n}^{-1}\sum_{t=1}^{n}\mathbf{A}_2'\mathbf{z}_{t-1}\varepsilon_t\Rightarrow N(0,\sigma^2\mathbf{\Gamma}). \tag{A.23}$$

Consequently, putting the results in (A.20), (A.21), (A.22) and (A.23) together we obtain that,

$$\Upsilon_n\left(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}\right)\Rightarrow\mathbf{A}\left\{\begin{bmatrix}\left[\int_0^1\Xi(r)dr\right]^{-1} & 0\\ 0 & \mathbf{\Gamma}^{-1}\end{bmatrix}\right\}\begin{bmatrix}(Q(\phi)\Psi(r))\\ N(0,\sigma^2\mathbf{\Gamma})\end{bmatrix}. \tag{A.24}$$

Before progressing we need to introduce some further notation. Define the $(k+1) \times 1$ selection vector $\mathbf{e}_i$ to have a 1 at the ith position and zeros elsewhere. Moreover, partition this selection vector as $\mathbf{e}_i = (\mathbf{e}'_{1j}, \mathbf{e}'_{2v})'$ where $\mathbf{e}_{1j}$ is a 3×1 vector and $\mathbf{e}_{2v}$ is a $(k-2) \times 1$ vector, where the indices j and v serve to indicate the position at which the one occurs. Note that, if one of the first three parameters, i.e. ρ, α_1 or α_2, is to be selected then a one will occur in the $\mathbf{e}_{1j}$ vector and $\mathbf{e}_{2v}$ will be a vector of zeros, and vice versa.

For each element of $\widehat{\boldsymbol{\beta}}$ we therefore have from (A.24) that

$$n\widehat{\rho} \Rightarrow 2(1-\cos(\phi))\frac{\int_0^1 W_0(r)\, dW_0(r)}{\mathcal{V}_0^2} \tag{A.25}$$

$$n(\widehat{\alpha}_1 - \alpha_1) \Rightarrow \frac{A\cos(\phi) + B}{\mathcal{V}_0^2(\mathcal{A}^2 + \mathcal{B}^2)} \tag{A.26}$$

$$n(\widehat{\alpha}_2 - \alpha_2) \Rightarrow \frac{A}{\mathcal{V}_0^2(\mathcal{A}^2 + \mathcal{B}^2)} \tag{A.27}$$

$$\sqrt{n}(\widehat{\alpha}_{v+2} - \alpha_{v+2}) \Rightarrow N(0, \sigma^2 \mathbf{e}'_{2v}\mathbf{H}_2\boldsymbol{\Gamma}^{-1}\mathbf{H}'_2\mathbf{e}_{2v}), \quad \text{for } 1 \le v \le k-2 \tag{A.28}$$

where

$$\mathbf{H}_2 := \begin{bmatrix} 1 & -2\cos\phi & \cdots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -2\cos\phi & 1 \\ 0 & 0 & 1 & -2\cos\phi \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{A.29}$$

and where A and B are as defined in Theorem 1.

Then from the result in (A.25) coupled with the consistency of the α_j, $j = 1, ..., k$, estimators, we obtain that $Z_{\hat{\rho}} \Rightarrow \int_0^1 W_0(r)\, dW_0(r) \,/ \int_0^1 W_0^2(r)dr$. This completes the proof for $Z_{\hat{\rho}}$. Turning to $t_{\widehat{\rho}}$, the appropriate selection vector is given by $\mathbf{e}_{11} := (1, 0, 0)'$, and hence

$$t_{\widehat{\rho}} = \frac{n\widehat{\rho}}{\sqrt{\widehat{\sigma}_k^2 \mathbf{e}'_{11}\left[\Upsilon_{1n}^{-1}\sum_{t=1}^{n}\mathbf{A}'_1\mathbf{z}_{t-1}\mathbf{z}'_{t-1}\mathbf{A}_1\Upsilon_{1T}^{-1}\right]^{-1}\mathbf{e}_{11}}}$$

$$\Rightarrow \frac{2(1-\cos(\phi))\int_0^1 W_0(r)\, dW_0(r)}{\int_0^1 W_0^2(r)dr}\sqrt{\frac{\int_0^1 W_0^2(r)dr}{4(1-\cos(\phi))^2}} = \frac{\int_0^1 W_0(r)\, dW_0(r)}{\sqrt{\int_0^1 W_0^2(r)dr}}$$

as required, using the fact that $\widehat{\sigma}_k^2 \xrightarrow{p} \sigma^2$.

The result for the VRT statistic in (19) follows directly from the results in Lemma 2 and applications of the CMT. For the M statistics, in addition to previous results and noting that $n^{-1}x_n^2 \Rightarrow \sigma^2\mathbf{b}(1)^2$ from (A.13) and the CMT, we also need to establish the behaviour of the long-run variance estimators used in constructing these statistics. In the case of the autoregressive-based estimator, $\hat{\lambda}^2 = s_{AR}^2$ the consistency of the OLS parameter estimates from (2), established above, yields that $s_{AR}^2 \xrightarrow{p} \sigma^2/\left(1 - 2\cos(\phi) + 1\right)^2 = \sigma^2/(4[1-\cos(\phi)]^2)$. In contrast, for $\hat{\lambda}^2 = s_{WA}^2$ it follows, as shown in the proof of Theorem 2 in Taylor (2003), that $s_{WA}^2 = O_p(mn)$, since $z_t := (1-L)x_t$ is non-stationary.

We now turn to establishing the results in Theorem 2. Here for $1 \le v \le k-2$ we have that

$$t_{\widehat{\alpha}_{v+2}} = \frac{\sqrt{n}\left(\widehat{\alpha}_{v+2} - \alpha_{v+2}\right)}{\widehat{\sigma}_k \mathbf{e}'_{2v}\left\{\mathbf{H}_2\left[\Upsilon_{2n}^{-1}\sum_{t=1}^{n}\mathbf{A}'_2\mathbf{z}_{t-1}\mathbf{z}'_{t-1}\mathbf{A}_2\Upsilon_{2n}^{-1}\right]^{-1/2}\right\}\mathbf{e}_{2v}}$$

$$= \left\{\widehat{\sigma}_k \mathbf{e}'_{2v}\left[\mathbf{H}_2\left(\Upsilon_{2n}^{-1}\sum_{t=1}^{n}\mathbf{A}'_2\mathbf{z}_{t-1}\mathbf{z}'_{t-1}\mathbf{A}_2\Upsilon_{2n}^{-1}\right)^{-1/2}\right]\mathbf{e}_{2v}\right\}^{-1} N(0, \sigma^2\mathbf{e}'_{2v}\mathbf{H}_2\boldsymbol{\Gamma}^{-1}\mathbf{H}'_2\mathbf{e}_{2v}) + o_p(1)$$

$$\Rightarrow N\left(0, \frac{\mathbf{e}'_{2v}\mathbf{H}_2\boldsymbol{\Gamma}^{-1}\mathbf{H}'_2\mathbf{e}_{2v}}{\mathbf{e}'_{2v}\mathbf{H}_2\boldsymbol{\Gamma}^{-1}\mathbf{H}'_2\mathbf{e}_{2v}}\right) = N(0,1). \quad \blacksquare$$

Proof of Lemma 3: The proof of Lemma 3 follows along the same lines as the proof of Lemma 1, on replacing the identity in (A.9) with the identity

$$1 \equiv \frac{1}{2\left(1-\cos\left(\phi\right)\right)}\left(1-2\cos(\phi)\varphi_n L+\varphi_n^2 L^2\right)+\frac{\left(1-2\cos(\phi)+\varphi_n L\right)}{2\left(1-\cos\left(\phi\right)\right)}\left(1-\varphi_n L\right). \quad \blacksquare$$

Proof of Remark 9: In this case we have to use the identity

$$\begin{aligned}
& \frac{1}{2\left(1-\cos\left(\phi\right)\right)}\left(1-2\cos(\phi)\varphi_n^{\phi} L+\left(\varphi_n^{\phi}\right)^2 L^2\right)+\frac{\left(1-2\cos(\phi)+\varphi_n^0 L\right)}{2\left(1-\cos\left(\phi\right)\right)}\left(1-\varphi_n^0 L\right) \\
= \; & 1+\frac{\cos\left(\phi\right)}{\left(1-\cos\left(\phi\right)\right)}\left(\frac{c_0-c_\phi}{n}\right)L+\frac{1}{2\left(1-\cos\left(\phi\right)\right)}\left[\left(\frac{c_\phi^2-c_0^2}{n^2}\right)+\left(\frac{2c_\phi-2c_0}{n}\right)\right]L^2
\end{aligned}$$

and that the terms $\left(\frac{c_0-c_\phi}{n}\right)$ and $\left(\frac{2c_\phi-2c_0}{n}\right)$ are $O\left(n^{-1}\right)$ and $\left(\frac{c_\phi^2-c_0^2}{n^2}\right)$ is $O\left(n^{-2}\right)$. Hence, it follows that,

$$\frac{1}{2\left(1-\cos\left(\phi\right)\right)}\left(1-2\cos(\phi)\varphi_n^{\phi} L+\left(\varphi_n^{\phi}\right)^2 L^2\right)+\frac{\left(1-2\cos(\phi)+\varphi_n^0 L\right)}{2\left(1-\cos\left(\phi\right)\right)}\left(1-\varphi_n^0 L\right)=1+o\left(1\right).$$

Table 1a: Empirical Rejection Frequencies, OLS De-meaned Data. DGP (33).

		$t_{\widehat{\rho}}$	$t_{\widehat{\alpha}_1}$	$t_{\widehat{\alpha}_2}$	$Z_{\widehat{\rho}}$	VRT	MSB_b	MSB_q	MSB_{AR}	$MZ\alpha_b$	$MZ\alpha_q$	$MZ\alpha_{AR}$	MZt_b	MZt_q	MZt_{AR}
	$c_\phi = 0$														
	0	0.045	0.051	0.049	0.056	0.047	0.040	0.041	0.036	0.048	0.048	0.045	0.047	0.047	0.044
	$\frac{\pi}{7}$	0.045	1.000	1.000	0.057	0.344	0.948	0.951	0.000	0.943	0.946	0.000	0.913	0.918	0.001
	$\frac{\pi}{6}$	0.049	1.000	1.000	0.062	0.355	0.967	0.976	0.000	0.961	0.970	0.000	0.939	0.954	0.000
ϕ	$\frac{\pi}{5}$	0.049	1.000	1.000	0.060	0.359	0.975	0.987	0.000	0.972	0.986	0.000	0.955	0.978	0.001
	$\frac{\pi}{4}$	0.049	1.000	1.000	0.063	0.367	0.967	0.994	0.000	0.967	0.993	0.000	0.947	0.988	0.000
	$\frac{\pi}{2}$	0.049	0.231	1.000	0.059	0.473	1.000	1.000	0.000	1.000	1.000	0.000	1.000	1.000	0.000
	$c_\phi = 2.5$														
	$\frac{\pi}{7}$	0.049	1.000	1.000	0.064	0.195	0.814	0.821	0.000	0.798	0.806	0.000	0.728	0.740	0.002
	$\frac{\pi}{6}$	0.051	1.000	1.000	0.063	0.196	0.865	0.893	0.000	0.857	0.885	0.000	0.798	0.835	0.002
ϕ	$\frac{\pi}{5}$	0.047	1.000	1.000	0.058	0.202	0.893	0.941	0.000	0.891	0.939	0.000	0.840	0.909	0.002
	$\frac{\pi}{4}$	0.049	1.000	1.000	0.061	0.215	0.884	0.970	0.000	0.873	0.968	0.000	0.815	0.949	0.001
	$\frac{\pi}{2}$	0.050	0.180	1.000	0.061	0.286	0.999	0.999	0.000	0.998	0.998	0.000	0.997	0.997	0.001
	$c_\phi = 10$														
	$\frac{\pi}{7}$	0.048	1.000	1.000	0.060	0.094	0.406	0.409	0.001	0.384	0.387	0.002	0.301	0.304	0.009
	$\frac{\pi}{6}$	0.048	1.000	1.000	0.061	0.099	0.499	0.545	0.001	0.479	0.525	0.001	0.390	0.437	0.008
ϕ	$\frac{\pi}{5}$	0.049	1.000	1.000	0.061	0.097	0.565	0.671	0.001	0.545	0.657	0.001	0.460	0.576	0.008
	$\frac{\pi}{4}$	0.048	1.000	1.000	0.058	0.101	0.561	0.781	0.001	0.552	0.772	0.001	0.461	0.704	0.008
	$\frac{\pi}{2}$	0.047	0.117	1.000	0.060	0.139	0.971	0.972	0.000	0.969	0.970	0.000	0.948	0.949	0.005

Table 1b: Empirical Rejection Frequencies. OLS Linear De-trended Data. DGP (33).

		$t_{\widehat{\rho}}$	$t_{\widehat{\alpha}_1}$	$t_{\widehat{\alpha}_2}$	$Z_{\widehat{\rho}}$	VRT	MSB_b	MSB_q	MSB_{AR}	$MZ\alpha_b$	$MZ\alpha_q$	$MZ\alpha_{AR}$	MZt_b	MZt_q	MZt_{AR}
	$c_\phi = 0$														
	0	0.050	0.050	0.052	0.078	0.051	0.076	0.076	0.063	0.044	0.045	0.036	0.036	0.037	0.030
	$\frac{\pi}{7}$	0.048	1.000	1.000	0.079	0.649	0.990	0.991	0.000	0.980	0.981	0.000	0.965	0.968	0.000
	$\frac{\pi}{6}$	0.045	1.000	1.000	0.074	0.659	0.995	0.996	0.000	0.988	0.993	0.000	0.981	0.987	0.000
ϕ	$\frac{\pi}{5}$	0.047	1.000	1.000	0.077	0.677	0.997	0.999	0.000	0.995	0.999	0.000	0.990	0.997	0.000
	$\frac{\pi}{4}$	0.050	1.000	1.000	0.080	0.691	0.997	1.000	0.000	0.992	0.999	0.000	0.986	0.999	0.000
	$\frac{\pi}{2}$	0.047	0.477	1.000	0.077	0.814	1.000	1.000	0.000	1.000	1.000	0.000	1.000	1.000	0.000
	$c_\phi = 2.5$														
	$\frac{\pi}{7}$	0.045	1.000	1.000	0.078	0.383	0.942	0.945	0.000	0.907	0.913	0.000	0.855	0.865	0.000
	$\frac{\pi}{6}$	0.048	1.000	1.000	0.078	0.393	0.972	0.981	0.000	0.950	0.967	0.000	0.920	0.944	0.000
ϕ	$\frac{\pi}{5}$	0.047	1.000	1.000	0.077	0.407	0.983	0.994	0.000	0.967	0.988	0.000	0.943	0.979	0.000
	$\frac{\pi}{4}$	0.046	1.000	1.000	0.077	0.435	0.980	0.998	0.000	0.960	0.996	0.000	0.933	0.993	0.000
	$\frac{\pi}{2}$	0.046	0.382	1.000	0.078	0.582	1.000	1.000	0.000	1.000	1.000	0.000	1.000	1.000	0.000
	$c_\phi = 10$														
	$\frac{\pi}{7}$	0.046	1.000	1.000	0.075	0.157	0.604	0.607	0.000	0.477	0.482	0.000	0.378	0.386	0.002
	$\frac{\pi}{6}$	0.048	1.000	1.000	0.077	0.161	0.723	0.769	0.000	0.620	0.682	0.000	0.523	0.594	0.001
ϕ	$\frac{\pi}{5}$	0.047	1.000	1.000	0.076	0.175	0.799	0.890	0.000	0.721	0.842	0.000	0.629	0.778	0.001
	$\frac{\pi}{4}$	0.045	1.000	1.000	0.078	0.186	0.799	0.952	0.000	0.711	0.926	0.000	0.620	0.889	0.001
	$\frac{\pi}{2}$	0.044	0.228	1.000	0.075	0.281	0.999	0.999	0.000	0.999	0.999	0.000	0.997	0.998	0.000

Table 2: Empirical Size and Average Lag Order Selected, OLS and Local GLS De-trended Data. DGP (33).

		OLS						GLS					
		SQ_4	SQ_{12}	AIC_4	AIC_{12}	$MAIC_4$	$MAIC_{12}$	SQ_4	SQ_{12}	AIC_4	AIC_{12}	$MAIC_4$	$MAIC_{12}$
							De-meaned						
0	size	0.051	0.055	0.053	0.057	0.036	0.030	0.065	0.064	0.067	0.069	0.056	0.052
	$mean_k$	2.224	7.249	2.185	2.782	2.317	3.052	2.228	7.195	2.189	2.746	2.198	2.527
$\frac{\pi}{7}$	size	0.050	0.055	0.051	0.055	0.038	0.028	0.059	0.065	0.059	0.062	0.046	0.031
	$mean_k$	2.290	7.323	2.306	2.893	2.501	3.740	2.288	7.251	2.301	2.843	2.370	3.026
$\frac{\pi}{6}$	size	0.051	0.057	0.050	0.054	0.038	0.028	0.062	0.066	0.062	0.064	0.049	0.034
	$mean_k$	2.286	7.257	2.302	2.920	2.497	3.723	2.282	7.196	2.295	2.872	2.369	3.038
$\frac{\pi}{5}$	size	0.049	0.055	0.049	0.053	0.036	0.027	0.058	0.062	0.058	0.060	0.046	0.031
	$mean_k$	2.288	7.330	2.296	2.901	2.493	3.751	2.283	7.255	2.292	2.858	2.361	3.023
$\frac{\pi}{4}$	size	0.049	0.059	0.050	0.053	0.036	0.028	0.058	0.063	0.057	0.061	0.044	0.031
	$mean_k$	2.283	7.358	2.301	2.888	2.492	3.721	2.282	7.307	2.293	2.839	2.364	3.026
$\frac{\pi}{2}$	size	0.047	0.054	0.048	0.052	0.036	0.027	0.058	0.061	0.058	0.061	0.043	0.027
	$mean_k$	2.295	7.279	2.311	2.909	2.516	3.978	2.293	7.212	2.307	2.861	2.400	3.224
							Linear de-trended						
0	size	0.063	0.070	0.066	0.074	0.037	0.028	0.058	0.065	0.063	0.068	0.039	0.034
	$mean_k$	2.202	7.318	2.162	2.774	2.509	4.100	2.209	7.235	2.161	2.719	2.217	2.539
$\frac{\pi}{7}$	size	0.056	0.071	0.056	0.064	0.027	0.018	0.028	0.035	0.028	0.033	0.017	0.009
	$mean_k$	2.291	7.465	2.311	2.988	2.621	4.561	2.287	7.353	2.300	2.915	2.503	3.726
$\frac{\pi}{6}$	size	0.056	0.071	0.057	0.066	0.038	0.031	0.029	0.035	0.029	0.033	0.018	0.009
	$mean_k$	2.297	7.397	2.312	2.984	2.610	4.568	2.291	7.315	2.302	2.910	2.497	3.714
$\frac{\pi}{5}$	size	0.060	0.075	0.061	0.071	0.049	0.043	0.029	0.035	0.030	0.033	0.017	0.009
	$mean_k$	2.296	7.373	2.314	2.989	2.614	4.576	2.289	7.293	2.299	2.901	2.511	3.777
$\frac{\pi}{4}$	size	0.053	0.069	0.054	0.065	0.045	0.041	0.027	0.035	0.028	0.033	0.015	0.008
	$mean_k$	2.299	7.428	2.312	3.000	2.614	4.622	2.292	7.319	2.298	2.898	2.510	3.783
$\frac{\pi}{2}$	size	0.060	0.074	0.061	0.069	0.049	0.043	0.022	0.026	0.023	0.026	0.013	0.006
	$mean_k$	2.299	7.427	2.318	2.989	2.608	4.794	2.288	7.272	2.302	2.883	2.543	4.177

Note: $mean_k$ denotes the average lag length selected.

Table 3: Empirical Power ($\mathbf{c}_\phi = 0$), OLS and Local GLS De-trended Data. DGP (33).

											OLS									
c_0	ϕ	SQ_4	SQ_{12}	AIC_4	AIC_{12}	$MAIC_4$	$MAIC_{12}$	ϕ	SQ_4	SQ_{12}	AIC_4	AIC_{12}	$MAIC_4$	$MAIC_{12}$	ϕ	SQ_4	SQ_{12}	AIC_4	AIC_{12}	$MAIC_4$	$MAIC_{12}$
5	0	0.105	0.120	0.111	0.127	0.064	0.050	$\frac{\pi}{6}$	0.089	0.106	0.091	0.104	0.054	0.037	$\frac{\pi}{4}$	0.096	0.108	0.098	0.108	0.060	0.044
10		0.210	0.201	0.222	0.236	0.136	0.110		0.190	0.196	0.191	0.208	0.117	0.079		0.194	0.199	0.194	0.210	0.126	0.088
13		0.299	0.267	0.312	0.329	0.203	0.164		0.302	0.275	0.305	0.318	0.198	0.132		0.300	0.271	0.301	0.312	0.200	0.138
20		0.574	0.455	0.583	0.587	0.440	0.355		0.602	0.482	0.605	0.604	0.446	0.297		0.617	0.489	0.618	0.616	0.465	0.327
25		0.756	0.573	0.762	0.748	0.630	0.497		0.800	0.608	0.800	0.783	0.636	0.426		0.807	0.611	0.807	0.792	0.674	0.482
30		0.879	0.662	0.880	0.862	0.782	0.612		0.916	0.689	0.916	0.894	0.783	0.532		0.925	0.698	0.925	0.903	0.830	0.608
5	$\frac{\pi}{7}$	0.091	0.110	0.093	0.108	0.051	0.033	$\frac{\pi}{5}$	0.091	0.106	0.091	0.105	0.056	0.040	$\frac{\pi}{2}$	0.093	0.107	0.094	0.107	0.067	0.053
10		0.194	0.196	0.196	0.209	0.110	0.069		0.198	0.198	0.200	0.215	0.129	0.087		0.202	0.201	0.201	0.216	0.147	0.117
13		0.304	0.282	0.304	0.318	0.180	0.111		0.298	0.275	0.301	0.314	0.203	0.136		0.302	0.276	0.302	0.315	0.228	0.176
20		0.610	0.476	0.610	0.604	0.396	0.238		0.62	0.479	0.623	0.618	0.471	0.321		0.627	0.484	0.628	0.619	0.518	0.396
25		0.806	0.613	0.806	0.791	0.551	0.335		0.808	0.616	0.809	0.793	0.666	0.463		0.81	0.608	0.810	0.789	0.717	0.561
30		0.918	0.695	0.918	0.893	0.634	0.379		0.923	0.702	0.923	0.902	0.829	0.594		0.927	0.698	0.926	0.903	0.862	0.698
											GLS									
c_0	ϕ	SQ_4	SQ_{12}	AIC_4	AIC_{12}	$MAIC_4$	$MAIC_{12}$	ϕ	SQ_4	SQ_{12}	AIC_4	AIC_{12}	$MAIC_4$	$MAIC_{12}$	ϕ	SQ_4	SQ_{12}	AIC_4	AIC_{12}	$MAIC_4$	$MAIC_{12}$
5	0	0.126	0.132	0.133	0.143	0.089	0.078	$\frac{\pi}{6}$	0.063	0.068	0.064	0.07	0.037	0.02	$\frac{\pi}{4}$	0.062	0.065	0.062	0.067	0.035	0.018
10		0.290	0.258	0.299	0.307	0.214	0.182		0.173	0.154	0.174	0.179	0.108	0.058		0.161	0.141	0.163	0.164	0.101	0.051
13		0.416	0.343	0.427	0.429	0.324	0.277		0.276	0.218	0.277	0.276	0.185	0.096		0.255	0.200	0.256	0.256	0.169	0.084
20		0.702	0.529	0.706	0.692	0.605	0.488		0.537	0.37	0.537	0.517	0.406	0.199		0.489	0.327	0.488	0.471	0.361	0.170
25		0.834	0.615	0.836	0.811	0.753	0.579		0.673	0.435	0.671	0.637	0.546	0.25		0.607	0.379	0.605	0.576	0.477	0.204
30		0.908	0.665	0.910	0.881	0.846	0.629		0.744	0.468	0.743	0.7	0.634	0.266		0.687	0.410	0.685	0.642	0.572	0.230
5	$\frac{\pi}{7}$	0.065	0.070	0.065	0.070	0.038	0.022	$\frac{\pi}{5}$	0.064	0.065	0.065	0.07	0.039	0.021	$\frac{\pi}{2}$	0.049	0.053	0.050	0.054	0.029	0.015
10		0.172	0.152	0.173	0.174	0.109	0.059		0.175	0.151	0.175	0.18	0.112	0.059		0.118	0.102	0.119	0.122	0.073	0.038
13		0.284	0.227	0.284	0.283	0.187	0.096		0.273	0.211	0.273	0.27	0.182	0.092		0.175	0.135	0.176	0.174	0.114	0.057
20		0.549	0.370	0.547	0.526	0.418	0.200		0.525	0.346	0.524	0.502	0.398	0.189		0.339	0.214	0.339	0.324	0.244	0.109
25		0.692	0.449	0.692	0.661	0.568	0.258		0.654	0.420	0.652	0.62	0.528	0.238		0.422	0.248	0.422	0.395	0.321	0.130
30		0.762	0.483	0.760	0.715	0.654	0.283		0.730	0.446	0.728	0.687	0.617	0.256		0.491	0.268	0.489	0.451	0.387	0.146

Table 4: Empirical Power ($c_\phi = 2.5$), OLS and Local GLS De-trended Data. DGP (33).

								OLS													
c_0	ϕ	SQ_4	SQ_{12}	AIC_4	AIC_{12}	$MAIC_4$	$MAIC_{12}$	ϕ	SQ_4	SQ_{12}	AIC_4	AIC_{12}	$MAIC_4$	$MAIC_{12}$	ϕ	SQ_4	SQ_{12}	AIC_4	AIC_{12}	$MAIC_4$	$MAIC_{12}$
5	$\frac{\pi}{7}$	0.096	0.096	0.096	0.106	0.056	0.038	$\frac{\pi}{5}$	0.094	0.096	0.092	0.105	0.057	0.039	$\frac{\pi}{2}$	0.095	0.099	0.095	0.107	0.057	0.040
10		0.200	0.165	0.200	0.213	0.127	0.084		0.198	0.169	0.198	0.214	0.126	0.085		0.201	0.169	0.200	0.216	0.128	0.084
13		0.299	0.227	0.302	0.316	0.197	0.132		0.300	0.223	0.302	0.313	0.198	0.132		0.301	0.222	0.305	0.315	0.198	0.129
20		0.595	0.373	0.603	0.601	0.455	0.299		0.606	0.377	0.616	0.613	0.462	0.308		0.613	0.382	0.622	0.617	0.468	0.309
25		0.775	0.463	0.790	0.776	0.642	0.412		0.784	0.470	0.803	0.789	0.662	0.435		0.799	0.477	0.814	0.800	0.677	0.444
30		0.892	0.539	0.908	0.886	0.797	0.514		0.902	0.546	0.916	0.898	0.815	0.537		0.910	0.553	0.924	0.901	0.825	0.556
5	$\frac{\pi}{6}$	0.098	0.102	0.098	0.111	0.061	0.040	$\frac{\pi}{4}$	0.094	0.095	0.093	0.105	0.059	0.041							
10		0.202	0.172	0.201	0.217	0.128	0.087		0.205	0.173	0.204	0.219	0.131	0.089							
13		0.298	0.224	0.298	0.309	0.200	0.136		0.303	0.225	0.304	0.317	0.200	0.135							
20		0.597	0.373	0.608	0.604	0.457	0.299		0.602	0.376	0.615	0.611	0.466	0.308							
25		0.776	0.463	0.792	0.777	0.652	0.423		0.790	0.474	0.807	0.788	0.666	0.439							
30		0.901	0.551	0.919	0.899	0.815	0.538		0.905	0.550	0.920	0.896	0.822	0.534							
								GLS													
c_0		SQ_4	SQ_{12}	AIC_4	AIC_{12}	$MAIC_4$	$MAIC_{12}$	ϕ	SQ_4	SQ_{12}	AIC_4	AIC_{12}	$MAIC_4$	$MAIC_{12}$	ϕ	SQ_4	SQ_{12}	AIC_4	AIC_{12}	$MAIC_4$	$MAIC_{12}$
5	$\frac{\pi}{7}$	0.104	0.095	0.104	0.110	0.064	0.040	$\frac{\pi}{5}$	0.101	0.092	0.099	0.107	0.061	0.038	$\frac{\pi}{2}$	0.085	0.078	0.084	0.091	0.051	0.029
10		0.273	0.196	0.272	0.275	0.187	0.111		0.258	0.190	0.260	0.266	0.175	0.105		0.198	0.141	0.200	0.204	0.131	0.069
13		0.412	0.269	0.418	0.417	0.298	0.176		0.397	0.258	0.404	0.400	0.288	0.169		0.288	0.181	0.295	0.290	0.196	0.104
20		0.737	0.426	0.754	0.734	0.622	0.355		0.719	0.401	0.737	0.716	0.598	0.333		0.534	0.272	0.552	0.531	0.419	0.202
25		0.875	0.500	0.894	0.862	0.795	0.441		0.854	0.466	0.875	0.843	0.768	0.412		0.655	0.294	0.680	0.647	0.551	0.236
30		0.940	0.540	0.953	0.918	0.895	0.481		0.919	0.503	0.936	0.901	0.868	0.444		0.726	0.307	0.755	0.709	0.635	0.243
5	$\frac{\pi}{6}$	0.108	0.095	0.107	0.114	0.069	0.044	$\frac{\pi}{4}$	0.096	0.091	0.097	0.105	0.060	0.035							
10		0.266	0.199	0.267	0.273	0.179	0.110		0.257	0.185	0.259	0.262	0.172	0.104							
13		0.409	0.269	0.415	0.412	0.301	0.181		0.385	0.252	0.391	0.390	0.276	0.162							
20		0.727	0.408	0.747	0.723	0.613	0.345		0.680	0.366	0.699	0.680	0.564	0.304							
25		0.866	0.479	0.885	0.854	0.787	0.429		0.821	0.435	0.844	0.812	0.731	0.379							
30		0.936	0.535	0.951	0.917	0.890	0.472		0.898	0.460	0.917	0.875	0.835	0.395							

Table 5: Empirical Power ($\mathbf{c}_\phi = 10$), OLS and Local GLS De-trended Data. DGP (33).

											OLS										
c_0	ϕ	SQ_4	SQ_{12}	AIC_4	AIC_{12}	$MAIC_4$	$MAIC_{12}$	ϕ	SQ_4	SQ_{12}	AIC_4	AIC_{12}	$MAIC_4$	$MAIC_{12}$	ϕ	SQ_4	SQ_{12}	AIC_4	AIC_{12}	$MAIC_4$	$MAIC_{12}$
5	$\frac{\pi}{7}$	0.095	0.112	0.096	0.110	0.059	0.042	$\frac{\pi}{5}$	0.091	0.107	0.091	0.103	0.056	0.040	$\frac{\pi}{2}$	0.094	0.106	0.095	0.105	0.058	0.041
10		0.194	0.197	0.194	0.210	0.128	0.093		0.200	0.201	0.202	0.217	0.131	0.095		0.199	0.196	0.201	0.215	0.132	0.094
13		0.287	0.262	0.290	0.301	0.199	0.145		0.298	0.274	0.300	0.312	0.207	0.147		0.301	0.277	0.303	0.316	0.199	0.139
20		0.573	0.459	0.574	0.571	0.437	0.315		0.595	0.469	0.596	0.591	0.452	0.322		0.611	0.479	0.612	0.609	0.468	0.319
25		0.753	0.570	0.754	0.741	0.621	0.434		0.781	0.595	0.782	0.767	0.647	0.459		0.811	0.613	0.811	0.796	0.677	0.456
30		0.870	0.659	0.871	0.852	0.759	0.530		0.900	0.678	0.899	0.878	0.798	0.553		0.923	0.698	0.923	0.902	0.829	0.556
5	$\frac{\pi}{6}$	0.091	0.107	0.092	0.105	0.057	0.042	$\frac{\pi}{4}$	0.094	0.108	0.095	0.107	0.058	0.041							
10		0.195	0.195	0.197	0.209	0.129	0.092		0.194	0.198	0.196	0.212	0.126	0.091							
13		0.290	0.269	0.290	0.304	0.197	0.142		0.295	0.269	0.298	0.313	0.203	0.146							
20		0.581	0.463	0.584	0.581	0.442	0.317		0.598	0.473	0.599	0.594	0.459	0.324							
25		0.771	0.588	0.772	0.756	0.637	0.446		0.798	0.598	0.798	0.782	0.663	0.464							
30		0.893	0.680	0.894	0.876	0.787	0.557		0.912	0.689	0.912	0.892	0.813	0.564							
											GLS										
c_0		SQ_4	SQ_{12}	AIC_4	AIC_{12}	$MAIC_4$	$MAIC_{12}$	ϕ	SQ_4	SQ_{12}	AIC_4	AIC_{12}	$MAIC_4$	$MAIC_{12}$	ϕ	SQ_4	SQ_{12}	AIC_4	AIC_{12}	$MAIC_4$	$MAIC_{12}$
5	$\frac{\pi}{7}$	0.117	0.123	0.119	0.128	0.080	0.059	$\frac{\pi}{5}$	0.114	0.120	0.116	0.124	0.077	0.058	$\frac{\pi}{2}$	0.103	0.108	0.104	0.110	0.068	0.046
10		0.290	0.262	0.291	0.298	0.213	0.156		0.294	0.265	0.296	0.303	0.217	0.157		0.244	0.217	0.245	0.252	0.174	0.113
13		0.439	0.364	0.441	0.441	0.340	0.246		0.437	0.362	0.437	0.436	0.333	0.235		0.363	0.302	0.364	0.363	0.265	0.172
20		0.772	0.587	0.772	0.753	0.662	0.470		0.773	0.579	0.774	0.752	0.658	0.456		0.636	0.454	0.635	0.616	0.518	0.305
25		0.902	0.682	0.901	0.879	0.828	0.570		0.902	0.679	0.902	0.878	0.824	0.553		0.772	0.525	0.769	0.740	0.663	0.359
30		0.960	0.746	0.960	0.935	0.915	0.617		0.961	0.725	0.961	0.933	0.916	0.588		0.843	0.562	0.842	0.804	0.758	0.380
5	$\frac{\pi}{6}$	0.114	0.125	0.115	0.123	0.078	0.058	$\frac{\pi}{4}$	0.113	0.117	0.113	0.120	0.077	0.055							
10		0.290	0.259	0.290	0.295	0.214	0.152		0.282	0.253	0.284	0.288	0.204	0.148							
13		0.444	0.366	0.443	0.443	0.339	0.246		0.425	0.349	0.427	0.427	0.321	0.229							
20		0.773	0.585	0.775	0.757	0.665	0.462		0.750	0.561	0.748	0.729	0.632	0.430							
25		0.902	0.686	0.904	0.881	0.829	0.562		0.888	0.647	0.886	0.861	0.801	0.519							
30		0.963	0.744	0.963	0.942	0.921	0.621		0.950	0.694	0.950	0.916	0.896	0.546							

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