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Testing for Persistence Change in Fractionally Integrated Models: An Application to World Inflation Rates*

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Abstract

In this paper we propose an approach to detect persistence changes in fractionally integrated models based on recursive forward and backward estimation of the Breitung and Hassler (2002) test. This procedure generalises to fractionally integrated processes the approaches of Leybourne, Kim, Smith and Newbold (2003) and Leybourne and Taylor (2003), which are ADF and seasonal unit root type tests, respectively, for the conventional intenger value context. Asymptotic results are derived and the performance of the new procedures evaluated in a Monte Carlo exercise. The finite sample size and power performance of the procedures are very encouraging and compare very favourably to available tests, such as those recently proposed by Hassler and Sheithauer (2009) and Sibbertsen and Kruse (2007). We also apply the test statistics introduced to several world inflation rates and find evidence of change in persistence in most series.

Keywords: LM tests, nonstationarity, fractional integration, persistence change, inflation
JEL classification: C20, C22

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1 Introduction

Testing for the presence of unit roots is now routine practice in empirical research given the different statistical and economic implications of classifying a series as stationary or nonstationary. Establishing this distinction is meaningful in that it helps understand the effects of shocks on economic and financial variables. While the impact of shocks will be transitory for stationary series, for nonstationary ones any random shock may have persistent effects. In other words, while a stationary time series will display mean-reverting behaviour, a nonstationary variable will display persistent behaviour, *i.e.*, shocks will have long lasting effects, thus preventing the series from returning to any defined level.

However, in recent years, it has been observed that macroeconomic variables may display both stationary and nonstationary features within a specific period; see, for instance, Halunga, Osborn and Sensier (2009). Indeed, it seems that some series could be switching from I(0) to I(1) behavior, or vice-versa. This has motivated the development of test procedures which look to infer whether a stationary (I(0)) or a nonstationary (I(1)) series has changed its persistence over time to I(1) or I(0), respectively; see, *inter alia,* Kim (2000), Kim, Belaire-Franch and Amador (2002), Busetti and Taylor (2004) and Harvey Leybourne and Taylor (2006).

In recent work Sibbertsen and Kruse (2009), Hassler and Sheithauer (2009) and Hassler and Meller (2009) look at this problem from a long-range dependencies perspective. They move from the integer valued context to the fractional context in order to evaluate whether a time series observed a persistence change from $I(d_0)$ to $I(d_1)$, with $d_0 \neq d_1$, or vice versa.

Hassler and Scheithauer (2009) evaluate the tests proposed by Kim (2000), Kim *et al.* (2002) and Busetti and Taylor (2004) for the null hypothesis of short-memory against a change to nonstationarity, I(1), and show that these tests are also consistent to test for changes from I(0) to I(d), $d > 0$ (long-memory). However, they observe that the estimators proposed for the integer case ($d = 1$) are only reliable if d is close to 1.

Sibbertsen and Kruse (2009) follow Leybourne, Taylor and Kim [LTK] (2007) and adapt their CUSUM of squares-based test statistics, computed from forward and reverse evaluation of time series, to the context of long range dependencies and show that the break point estimator proposed by LTK is consistent under long memory although at a slower rate of convergence (which depends on d). Sibbertsen and Kruse (2009) observe that the LTK procedure suffers from serious size distortions if the DGP has long memory and therefore provide new critical values, appropriate for the I(d) framework, which depend on the memory parameter d.

Hassler and Meller (2009) introduced a test procedure which considers the regression-based Lagrange Multiplier [LM] test of Demetrescu, Kuzin and Hassler (2008). In particular, they include a dummy variable in the test regression with the objective of accounting for a possible break in long-memory. Allowing for a break fraction λ, such that $\lambda \in [\pi, 1-\pi]$, the *supremum* of the sequence of squared t-statistics for the significance of the break parameter is computed and compared to the critical values in Andrews (1993). Through Monte Carlo simulations Hassler and Meller (2009) show that this procedure presents good power particularly when the difference of the orders of integration before and after the break is larger than 0.3.

In this paper we propose a new method to detect persistence changes in fractionally integrated models based on recursive forward and backward estimation of the Breitung and Hassler (2002) test, in the spirit of the approach of Leybourne, Kim, Smith and Newbold [LKSN] (2003). Asymptotic results are derived and the performance of the new procedures evaluated in a Monte Carlo exercise. The finite sample size and power performance of the procedures are very encouraging and compare very favourably with available tests, such as those proposed by Hassler and Sheithauer (2009) and Sibbertsen and Kruse (2009). The performance of the test together with its simplicity of application make it an interesting approach for empirical analysis. We apply the new test statistics to several world inflation rates and find evidence of persistence change in most of the series.

This paper is organized as follows. Section 2 introduces the new procedures. Section 3 discusses the finite sample properties of the test statistics and Section 4 presents an empirical application which investigates persistence change in inflation series. Finally, Section 5 concludes the paper and an appendix collects the proofs.

2 Fractional Persistence Change

Consider data generated from a fractionally integrated process of order d_t ($FI(d_t)$), such that,

$$(1-L)^{d_t} y_t = \varepsilon_t, \tag{1}$$

where $y_t = 0$ for $t \leq 0$, and ε_t satisfies a set of assumptions that will be discussed below. Under the null hypothesis it will be assumed that the fractional integration parameter d_t is constant over the sample, *i.e.*, $d_t = d_0$. However, under the alternative two situations can be considered, i) H_{01} : y_t is $I(d_0)$ changing to $I(d_1)$ at time $\lfloor \tau^* T \rfloor$ $\therefore$ $d_t = d_0$ for $t \leq \lfloor \tau^* T \rfloor$ and $d_t = d_1$ for $t > \lfloor \tau^* T \rfloor$. Here, τ^* is unknown in $\Lambda = [\Lambda_l, \Lambda_u] \subset (0,1)$ and symmetric around 0.5; and ii) H_{10} : y_t is $I(d_1)$ changing to $I(d_0)$ at time $\lfloor \tau^* T \rfloor$.

Remark 2.1: *Owing to nonstationarity, it is customary in the literature related to fractional integration to assume* $y_t I_{(t \leq 0)} = 0$, *either explicitly (e.g., Tanaka, 1999; Demetrescu, Kuzin* *and Hassler [DKH], 2008, and Hassler, Rodrigues and Rubia [HRR], 2009), or indirectly, by requiring* $\varepsilon_t I_{(t\leq 0)} = 0$ *(e.g., Nielsen, 2004, 2005). This restriction ensures that the observable process is well-defined in the mean-square sense regardless of the values of* d*; see Marinucci and Robinson (1999), Tanaka (1999) and Robinson (2005) for further details. It is important to note that the truncation imposed poses no loss of generality for the procedures proposed; see also HRR. However, the assumption on the initial values is not a trivial one, as was shown by Davidson and Hashimzade (2009), and care needs to be taken, particularly in contexts where this difference is likely to be crucial.*

To be more precise regarding the assumptions underlying ε_t in (1) we consider a set of assumptions similar to those of DKH and HRR.

Assumptions:

$\mathcal{A}.1$) *The innovation process* $\{\varepsilon_t, \mathcal{G}_t\}_{-\infty}^{\infty}$, $\mathcal{G}_t = \sigma(\varepsilon_j : j \leq t)$, *forms a martingale difference sequence (MDS) and verifies* $E(\varepsilon_t^2) = \sigma^2 < \infty$, $E(\varepsilon_t^2 | \mathcal{G}_{t-1}) > 0$ *almost surely, with one of the following restrictions holding true:*

$\mathcal{A}.1.1$) $\{\varepsilon_t\}$ *is independent and identically distributed and* $E(|\varepsilon_t^4|^{1+r})$ is *uniformly bounded for some* $r > 0$.

$\mathcal{A}.1.2$) $\{\varepsilon_t\}$ *is strictly stationary and ergodic with*

$$\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}\dots\sum_{l_7=-\infty}^{\infty}|\kappa_\varepsilon(0, l_1, \dots, l_7)| < \infty,$$

where $\kappa_\varepsilon(0, l_1, \dots, l_7)$ *is the eight-order joint cumulant of* $\{\varepsilon_t\}$.

As indicated in HRR, assumption $\mathcal{A}.1.1$ can be weakened by requiring that, conditional on the σ-field of events $\mathcal{G}_t$, moments up to the fourth-order (and suitable cross-products of elements of ε_t) equal the corresponding unconditional moments, so that essentially $\{\varepsilon_t\}$ is only required to behave as an i.i.d process up to the fourth-order moment. The main purpose of $\mathcal{A}.1.2$ is to allow for (unknown) time-varying conditional volatility patterns in $\{\varepsilon_t\}$. For instance, GARCH-type and Stochastic Volatility models are permitted, among other forms of conditional heteroskedasticity, under restrictions that limit the extent of temporal dependence. As in Gonçalves and Kilian (2007), DKH and HRR, this holds by requiring the absolute summability of the eight-order joint cumulants.

In our analysis we will relax Assumption $\mathcal{A}.1$, by also allowing for stationary AR(p) dynamics in the DGP, which may appear jointly with time-varying volatility patterns. Therefore, we also consider as an alternative to assumption $\mathcal{A}.1$ the following:

$\mathcal{A}.2)$ *The innovation process satisfies* $a(L)\varepsilon_t = v_t$, *where* $a(L) = 1 - \sum_j^p a_j L^j$, $p \geq 0$, *such that* $a(z)$ *has all its roots outside the unit circle and* $\{v_t, \mathcal{G}_t\}$, *is a strictly stationary and ergodic MDS satisfying the restrictions in either Assumption* $\mathcal{A}.1.1$ or $\mathcal{A}.1.2$.

For practical purposes, the short-run dynamics may be characterized by a stationary and invertible linear process $\varepsilon_t = \sum_{j=0}^{\infty} b_j v_{t-j}$ such that the AR(p) model, for some large enough $p < \infty$, approaches the underlying AR representation reasonably well. The actual performance of this approximation, when the underlying correlation structure in the short-run component is unknown, is ultimately an empirical question which we shall address in the Monte Carlo section.

2.1 Tests for persistence change

To introduce the persistence change tests, consider data generated from (1) with $d_t = d_0$ and where ε_t satisfies assumption A.1. In this case, for each fixed τ ($\tau \in (\pi, 1-\pi)$) the auxiliary regression is simply,

$$x_t = \phi(\tau)\, x_{t-1}^* + e_t, \qquad t = 2, ..., \lfloor \tau T \rfloor, \tag{2}$$

where $x_t = \triangle^{d_0} y_t$ and $x_{t-1}^* = \sum_{j=1}^{t-1} \frac{x_{t-j}}{j}$; see Breitung and Hassler (2002) for details on the set up of this regression for testing for fractional integration in the time domain (see also Robinson, 1994, for the approach in the frequency domain). Here, we use this test to look for changes in the memory parameter by recursively estimating (2) over the complete sample. In practice, the parameter π that defines the set of values for τ is an arbitrary value, typically π =0.15 or π =0.2.

Remark 2.2*: If our DGP is* $y_t = \mu_t + x_t$ and $(1-L)^d x_t = e_t$, *where* $\mu_t = z_t'\beta$ *is a deterministic kernel (such as a constant or a constant and a time trend), the procedure just presented can still be used but* x_t *has to be replaced by* $\overline{x}_t = x_t - z_t'\widehat{\beta}$, *which when* d *is an integer corresponds to the least-squares residual obtained from the de-meaning or the de-trending regression of* x_t *on* $z_t = 1$ *or* $z_t = (1,t)'$, *respectively, for* $t = 1, ..., \lfloor \tau T \rfloor$. *For example, in the constant case,* $\mu_t = \mu$, $\overline{x}_t = x_t - \overline{x}(\tau)$, *where* $\overline{x}(\tau) = \frac{1}{\lfloor \tau T \rfloor} \sum_{t=1}^{\lfloor \tau T \rfloor} x_t$. *However, when* d *is not an integer, following Robinson (1994) and DKH (p. 184) regress* $x_t = (1-L)^d y_t$ *on the differenced trend function,*

$$(1-L)^d z_t = \sum_{i=0}^{t-1} \xi_i z_{t-i},$$

where $\xi_i = \frac{i-1-d}{i}\xi_{i-1}$ *and* $\xi_0 = 1$. *Similarly as in the integer case, the residuals* $\overline{x}_{1,t} = x_t - \widehat{\beta}'(1-L)^d z_t$ *replace* x_t *in the test regression. According to Robinson (1994), Breitung and Hassler (2002, p.171) and DKH, it can be shown that demeaning or detrending* x_t *prior to* *computing the test statistic does not affect the limiting distribution of the test. That is, under the null hypothesis and replacing x_t by $\overline{x}_{1,t}$, the limit distributions of our tests do not change, i.e., these are invariant to deterministic components.*

Proposition 2.1: *Considering the auxiliary regression in (2) the OLS t- and squared t-statistics to test $\widehat{\phi}(\tau) = 0$, which we denote as $\zeta_f(\tau)$* and *$\zeta_f^2(\tau)$, respectively are,*

$$\zeta_f(\tau) = \frac{\sum_{t=2}^{\lfloor \tau T \rfloor} x_t x_{t-1}^*}{\widehat{\sigma}_e(\tau)\sqrt{\sum_{t=2}^{\lfloor \tau T \rfloor} x_{t-1}^{*2}}} \tag{3}$$

and $\zeta_f^2(\tau) = \left[\zeta_f(\tau)\right]^2$, where $\widehat{\sigma}_e(\tau) = \sqrt{\frac{1}{\lfloor \tau T \rfloor - 2}\sum_{t=2}^{\lfloor \tau T \rfloor} \widehat{e}_t^2}$ and $\widehat{e}_t$ is the least squares residual of the auxiliary regression (2).

Generalizing the results of LTK, it follows that $\zeta_f(\tau)$ is consistent against H_{10} (change from $I(d_1)$ to $I(d_0)$, $d_0 < d_1$) but not against H_{01} (change from $I(d_0)$ to $I(d_1)$, $d_0 < d_1$). Thus, to obtain power against H_{01} one needs to compute the reverse statistic, *i.e.*, $\zeta_r(\tau)$, where x_t is now replaced by the time-reversed series $w_t = x_{T-t+1}$. Thus, considering the remaining $(1-\tau)T$ observations, it follows that the test regression necessary to compute the reverse statistic is,

$$w_t = \phi(\tau) w_{t-1}^* + \widetilde{e}_t, \qquad t = 2, ..., \lfloor (1-\tau) T \rfloor, \tag{4}$$

where $w_t = x_{T-t+1}$ and $w_{t-1}^* = \sum_{j=1}^{t-1} \frac{w_{t-j}}{j} = \sum_{j=1}^{t-1} \frac{x_{T-t+j+1}}{j} = x_{T-t+2}^*$.

Proposition 2.2: *Considering the auxiliary regression in (4), the OLS t- and squared t-statistics to test $\widehat{\phi}(\tau) = 0$ are thus,*

$$\zeta_r(\tau) = \frac{\sum_{t=2}^{\lfloor (1-\tau)T \rfloor} w_t w_{t-1}^*}{\widehat{\sigma}_{\widetilde{e}}(\tau)\sqrt{\sum_{t=2}^{\lfloor (1-\tau)T \rfloor} w_{t-1}^{*2}}} = \frac{\sum_{t=2}^{\lfloor (1-\tau)T \rfloor} x_{T-t+1} x_{T-t+2}^*}{\widehat{\sigma}_{\widetilde{e}}(\tau)\sqrt{\sum_{t=2}^{\lfloor (1-\tau)T \rfloor} x_{T-t+2}^{*2}}} \tag{5}$$

and $\zeta_r^2(\tau) = [\zeta_r(\tau)]^2$, respectively, where $\widehat{\sigma}_{\widetilde{e}}(\tau)\sqrt{\frac{1}{\lfloor (1-\tau)T \rfloor - 2}\sum_{t=2}^{\lfloor (1-\tau)T \rfloor} \widehat{\widetilde{e}}_t^2}$ and $\widehat{\widetilde{e}}_t$ is the least squares residual of the auxiliary regression (4).

If the direction of change is known under the alternative, either ζ_f and ζ_f^2 or ζ_r and ζ_r^2, can be computed. However, if the direction is not known *a priori* as is generally the case, then the use of the $\zeta_{\min} = \min\left\{\zeta_f, \zeta_r\right\}$ and $\zeta_{\max}^2 = \max\left\{\zeta_f^2, \zeta_r^2\right\}$ statistics to achieve higher power is recommended. This follows along similar lines as the statistics proposed in Harvey *et al.* (2006) for the integer I(1) *versus* I(0) or I(0) *versus* I(1) cases.

Given that generally the time of change is not known, the statistics of Propositions 2.1 and 2.2 are not directly usable. Instead the *infimum* and *supremum* of the t and squared t statistics, respectively have to be considered as given in the next proposition.

Proposition 2.3: *Based on the t-* and squared *t*-statistics of Propositions 2.1 and 2.2, *the Infimum and Supremum statistics over* $\tau \in \Lambda^*$, *computed to investigate changes in the memory parameter are respectively,*

$$\zeta_k \equiv \inf_{\tau \in \Lambda^*} \zeta_k(\tau) \ \text{ for } \Lambda^* \equiv [\Lambda_l, \Lambda_u], \tag{6}$$

and

$$\zeta_k^2 \equiv \sup_{\tau \in \Lambda^*} \zeta_k^2(\tau) \ \text{ for } \Lambda^* \equiv [\Lambda_l, \Lambda_u], \tag{7}$$

where $k = r, f$ *and* Λ_l *and* Λ_u *correspond to the lower and upper bounds of* Λ^*, *respectively.*

2.2 Limit Null Distributions

For the purpose of exposition and without loss of generality, under the null hypothesis we consider that $d_0 = 1$. Note that for $d_0 = 0$ the analysis follows along the same lines, and the limit results will also be the same. Furthermore, the results presented will also hold when d_0 is a real value as will be discussed below.

Theorem 2.1 *Considering data generated from (1) with* $d_t = 1$ *(or* $d_t = 0$*) and where* ε_t *satisfies assumption A.1, i.e.* y_t *is* $I(1)$ *(or I(0)) throughout the sample period, it follows as* $T \to \infty$ *that for any fixed, known,* $\tau \in \Lambda^*$, *the statistics provided in Propositions 2.1 and 2.2 have the following limit distributions,*

$$\zeta_f(\tau), \zeta_r(\tau) \xrightarrow{d} N(0,1) \ \text{ and } \zeta_f^2(\tau), \zeta_r^2(\tau) \xrightarrow{d} \chi^2_{(1)}. \tag{8}$$

For proof see appendix.

Furthermore, regarding the statistics of Proposition 2.3, their respective asymptotic distributions are provided in the following theorem.

Theorem 2.2 *Under the same assumptions of Theorem 2.1 and defining* $\Lambda^* \equiv [\Lambda_l, \Lambda_u]$, *it follows as* $T \to \infty$ *that,*

$$\zeta_f, \zeta_r \xrightarrow{d} \inf_{\tau \in \Lambda^*} \Phi_\tau \ \text{ and } \zeta_f^2, \zeta_r^2 \xrightarrow{d} \sup_{\tau \in \Lambda^*} \chi_\tau, \tag{9}$$

where $\Phi_\tau \sim N(0,1)$ *and* $\chi_\tau \sim \chi^2_{(1)}$. *Furthermore,*

$$\zeta_{\min} \xrightarrow{d} \min\left\{\inf_{\tau \in \Lambda^*} \Phi_\tau, \inf_{\tau \in \Lambda^*} \Phi_\tau\right\} \ \text{ and } \zeta^2_{\max} \xrightarrow{d} \max\left\{\sup_{\tau \in \Lambda^*} \chi_\tau, \sup_{\tau \in \Lambda^*} \chi_\tau\right\}. \tag{10}$$

For proof see appendix.

Remark 2.3: *Although the analytical expressions of the limiting distributions of the tests statistics are not straightforwardly determined, these have some noticeable properties. Take, for* *instance, the random variable* $\inf_{\tau \in \Lambda^*} \Phi_\tau$*, which is the infimum of an uncountable number of standard normal random variables. If the random variables were independent and of a finite number (which is commonly done in practice, with* $\lfloor (1-\Lambda_l) T \rfloor - \lfloor \Lambda_l T \rfloor = T^*$ *and* $\Lambda_l = 0.2$*), then the distribution function would be given by* $f_{T^*}(x) = 1 - (1 - \Phi(x))^{T^*}$*, which is the minimum order statistic of a finite number of independent standard normals. Clearly, the simulated critical values in Table 3.1 do not resemble those of* $f(x)$*. A similar argument applies to the case of* $\sup_{\tau \in \Lambda^*} \chi_\tau$ *with chi-squared random variables.*

Thus far our analysis only considered the restricted case that under the null hypothesis d_0 is an integer ($d_0 = 0$ or $d_0 = 1$). However, the results presented are quite general in the sense that they also hold when d_0 is a real value. Hence the following corollary can be stated:

Corollary 2.1 *Assuming that data is generated from* (1) *under assumptions A.1 or A.2 and considering the stationary parameter space* $d_0 < 0.5$, *with* d_0 *known the limit results of Theorems 2.1 and 2.2 hold in this context as well as long as adequately filtered data is used, i.e.,* $x_t = (1-L)^{d_0} y_t$.

In this context, and following Breitung and Hassler (2002), DKH and HRR, the auxiliary regression is set up using $x_t = (1-L)^{d_0} y_t$ and consequently, the previously derived limit results hold under assumptions A.1 or A.2 for ε_t.

However, in general, d_0 is unknown and therefore the previous corollary needs to be adapted in order to cover this empirically relevant case.

Corollary 2.2 *Assuming that data is generated from* (1) *under assumptions A.1 or A.2, but considering* d_0 *unknown, the limit results of Theorems 2.1 and 2.2 will hold in this context as well as long as a* $\sqrt{T}-$ *consistent estimator of d,* $\widehat{d}_T \equiv \widehat{d}$, $\sqrt{T}\left(\widehat{d} - d\right) = O_p(1)$ *is used, such as, for instance, the spectral MLE estimator of Fox and Taqqu (1986). Once a consistent estimator is obtained, the transformed data* $\widehat{x}_t = (1-L)^{\widehat{d}} y_t$ *can be used to set up the test regression ensuring that the limit null distributions presented in Theorems 2.1 and 2.2, hold in this context as well.*

For proof see appendix.

We have until now assumed the null hypothesis to be true. Under the alternative H_{01} (or H_{10}), $\widehat{x}_t$ does not share the same properties of the DGPs $x_{t,0} \equiv \varepsilon_t(d_0)$, for $t \leq \lfloor \tau^* T \rfloor$ and $x_{t,1} \equiv \varepsilon_t(d_1)$, for $t > \lfloor \tau^* T \rfloor$, as these have different behaviors whenever $d_0 \neq d_1$. Thus, to understand the behaviour of the tests under the alternative hypotheses, we provide next the respective power functions, assuming under the null hypothesis that $d_0 = 1$. To show analytically that the tests have power when d_0 and d_1 are both real is not trivial and we refer to the Monte Carlo experiments below.

2.3 Power Functions

To characterize the power functions of the tests introduced in Propositions 2.1, 2.2 and 2.3, we consider the behaviour of the statistics in a local and non-local context. Theorems 2.3 and 2.4 below detail the behaviour of the procedures in both contexts.

Remark 2.4: *To ensure stationarity and invertibility of the fractional process, we need to restrict d to the interval $(-0.5, 0.5)$. For this interval, it is possible to derive the limiting laws of $\sum_{t=2}^{T} x_{t-1}^{*2}$ and $\sum_{t=2}^{T} x_t x_{t-1}^{*}$ which are needed for the power analysis and the limit null distributions (when d_0 is real) of the tests. Nevertheless, we can study the properties of the tests when d does not belong to the $(-0.5, 0.5)$ interval by resorting to Monte Carlo simulations.*

Theorem 2.3 *(Local Power). Consider data generated from (1) under H_{01}, i.e., y_t is $I(1)$ changing to $I(d_{1,T})$, where $d_{1,T} = 1 - \frac{\delta}{\sqrt{T}}, \delta > 0$, at time $\lfloor \tau^* T \rfloor$, with τ^* unknown in $\Lambda = [\Lambda_l, \Lambda_u] \subset (0,1)$, symmetric around 0.5 and with no serial correlation. Then, as $T \to \infty$,*

$$\zeta_f \xrightarrow{d} N(0, \tau^*) + N\left(-\delta\sqrt{(1-\tau^*)\frac{\pi^2/6}{\sigma_\varepsilon^4}}, 1-\tau^*\right) \tag{11}$$

and

$$\zeta_r \xrightarrow{d} N\left(-\delta\sqrt{\frac{\pi^2/6}{\sigma_\varepsilon^4}}, 1\right). \tag{12}$$

For proof see appendix.

Theorem 2.4 *(Non-Local Power). Consider again H_{01} but now y_t is $I(1)$ changing to $I(d_1)$, where $d_1 = d \in (0.5, 1)$ at time $\lfloor \tau^* T \rfloor$, with τ^* unknown in $\Lambda = [\Lambda_l, \Lambda_u] \subset (0,1)$, symmetric around 0.5 and with no serial correlation. As $T \to \infty$,*

$$\frac{1}{\sqrt{T}}\zeta_f \xrightarrow{p} \sqrt{(1-\tau^*)}\frac{\gamma_*(d-1)}{\sigma_\varepsilon\sqrt{\frac{\tau^*}{(1-\tau^*)}\sigma_\varepsilon^2\frac{\pi^2}{6} + \sigma_{x^*}^2(d-1)}} < 0 \tag{13}$$

and

$$\frac{1}{\sqrt{T}}\zeta_r \xrightarrow{p} \sqrt{(1-\tau^*)}\frac{\gamma_*(d-1)}{\sigma_\varepsilon\sigma_{x^*}(d-1)} < 0 \tag{14}$$

where $\gamma_*(d-1)$ and $\sigma_{x^*}(d-1)$ are defined in the appendix.

For proof see appendix.

Regarding the reverse alternative, H_{10}, the following corollaries can be stated.

Corollary 2.3 *(Local Power). Consider H_{10}, where $d_{1,T} = 1 - \frac{\delta}{\sqrt{T}}, \delta > 0$. As $T \to \infty$,*

$$\zeta_f \xrightarrow{d} N\left(-\delta\sqrt{\frac{\pi^2/6}{\sigma_\varepsilon^4}}, 1\right) = O_p(1) \tag{15}$$

and

$$\zeta_r \xrightarrow{d} N(0, \tau^*) + N\left(-\delta\sqrt{\tau^*\frac{\pi^2/6}{\sigma_\varepsilon^4}}, \tau^*\right) = O_p(1). \tag{16}$$

Corollary 2.4 *(Non-Local Power). Consider H_{10}, where $d_1 = d \in (0.5, 1)$. As $T \to \infty$,*

$$\frac{1}{\sqrt{T}}\zeta_f \xrightarrow{p} \sqrt{\tau^*}\frac{\gamma_*(d-1)}{\sigma_\varepsilon \sigma_{x^*}(d-1)} < 0 \tag{17}$$

and

$$\frac{1}{\sqrt{T}}\zeta_r \xrightarrow{p} \sqrt{\tau^*}\frac{\gamma_*(d-1)}{\sigma_\varepsilon\sqrt{\frac{(1-\tau^*)}{\tau^*}\sigma_\varepsilon^2\frac{\pi^2}{6} + \sigma_{x^*}^2(d-1)}} < 0. \tag{18}$$

As can be observed from Theorems 2.3 and 2.4, under the local H_{01} case, where $d_{1,T} = 1 - \frac{\delta}{\sqrt{T}}, \delta > 0$, or the non-local H_{01} case, where $d_1 = d$, the ζ_f and ζ_r, test statistics reach a minimum at $\tau = 1$ and $\tau = \tau^*$, respectively. Thus, it follows that under H_{01} (H_{10}) ζ_r and ζ_r^2 (ζ_f and ζ_f^2) are more powerful tests than ζ_f and ζ_f^2 (ζ_r and ζ_f^2) and the difference in power increases (decreases) with τ^* (see also the Monte Carlo section below). Furthermore, it follows that under H_{01} (in both the local and non-local contexts) the ζ_r (ζ_r^2) test statistics can be used to obtain a consistent estimator of τ^* as $\widehat{\tau} = \underset{\tau\in\Lambda^*}{\arg\min}\zeta_r(\tau)$ whereas under H_{10}, ζ_f is the more powerful test and can therefore be used to obtain a consistent estimator for τ^* as $\widehat{\tau} = \underset{\tau\in\Lambda^*}{\arg\min}\zeta_f(\tau)$.

Corollary 2.5 *Under H_{01} (local and non-local power) it follows from Theorems 2.2 and 2.3 that ζ_f^2, ζ_r^2 are consistent tests with ζ_r^2 having the highest power and useful to obtain a consistent estimate of τ^*. Similarly, under H_{10} (local and non-local power) it follows from Corollaries 2.3 and 2.4 that ζ_f^2, ζ_r^2 are also consistent tests but now ζ_f^2 has the largest power and is useful to obtain a consistent estimate of τ^*.*

Corollary 2.6 *Under H_{01} or H_{10}, where $d_1 = d \in (0.5, 1)$, $\zeta_{\min}$ and $\zeta_{\max}$ are $\sqrt{T}-$consistent tests.*

The proof of Corollary 2.4 follows from Theorems 2.3 and 2.4, noting that $\zeta_{\min} = \min\left\{\zeta_f, \zeta_r\right\}$ and $\zeta_{\max} = \max\left\{\zeta_f^2, \zeta_r^2\right\}$. For an unspecified alternative (union of H_{01} and H_{10}), and assuming that H_{01} is true, then, $\zeta_{\min} = \min\left\{\zeta_f, \zeta_r\right\} = \zeta_r$. If, on the other hand, H_{10} is true, it follows that $\zeta_{\min} = \min\left\{\zeta_f, \zeta_r\right\} = \zeta_f$.

Remark 2.5: *If we reject the null hypothesis with* $\zeta^{\min} = \min\left\{\zeta_f, \zeta_r\right\} = \zeta_r$*, then, the null hypothesis is rejected in favour of* H_{01} *and* $\widehat{\tau} = \underset{\tau\in\Lambda^*,}{\arg\min}\zeta_r(\tau)$*; on the contrary, if* $\zeta^{\min} = \min\left\{\zeta_f, \zeta_r\right\} = \zeta_f$*, support for* H_{10} *is found and* $\widehat{\tau} = \underset{\tau\in\Lambda^*,}{\arg\min}\zeta_f(\tau)$.

2.4 Serial Dependence

Consider the DGP as in (1) with $d_t = d_0$ and where ε_t now satisfies assumption A.2. Clearly, $x_t = \triangle^{d_0} y_t$ will share the same proprieties with ε_t, under the null, as $x_t = \varepsilon_t$ and, by invertibility, $x_t = \sum_{j=1}^{\infty} a_j x_{t-j} + \varepsilon_t$.

For the purpose of testing for persistence change, *i.e.*, changes in d when the errors are autocorrelated we use the augmented LM test (under H_0 of $d_t = d_0$) proposed by DKH; see also HRR. The application of the augmented LM test that we suggest, considers for each fixed τ the following test regressions,

$$x_t = \phi(\tau)\, x_{t-1}^* + \sum_{k=1}^{p} \gamma_k(\tau)\, x_{t-k} + v_t, \qquad t = p+1, ..., \lfloor \tau T \rfloor, \tag{19}$$

and

$$w_t = \phi(\tau)\, w_{t-1}^* + \sum_{k=1}^{p} \vartheta_k(\tau)\, w_{t-k} + u_t, \qquad t = p+1, ..., \lfloor (1-\tau) T \rfloor, \tag{20}$$

where

$$x_t = (1-L)^{d_0} y_t, \;\; w_t = x_{T-t+1}, \;\; x_{t-1}^* = \sum_{j=1}^{t-1} \frac{x_{t-j}}{j} \text{ and } w_{t-1}^* = \sum_{j=1}^{t-1} \frac{w_{t-j}}{j}.$$

Based on successive applications of (19) and (20) we construct the sequences $\{(\zeta_f(\tau), \zeta_r(\tau)),$ $\tau \in [\Lambda_l, \Lambda_u]\}$, where $\zeta_k(\tau)$, $k = f, r$, is the t-ratio associated with $\widehat{\phi}(\tau)$ computed from the above sub-sample regressions. The null hypothesis of a constant d is rejected for large negative values of the statistics computed as in (6) and (7) using the forward and reverse regressions.

Under the null hypothesis, $\phi(\tau) = 0$ for any τ and it is assumed that $p = o\left(T^{1/4}\right)$ as $p \to \infty$ and $T \to \infty$; see DKH (p.181) or HRR, for details. In practice, p is not known. Following DKH, we use an automated, deterministic optimal lag length selection as suggested by Schwertz (1989), *i.e.*,

$$p_K = \left[K\left(\frac{T}{100}\right)^{1/4}\right], \text{ with } K = 4 \text{ or } K = 12,$$

where $[\cdot]$ denotes the largest integer part of a real number. It is important to note that in this case, computation of the t-statistic requires using White standard errors $\widehat{s}\left(\widehat{\phi}(\tau)\right)$ (see DKH, p.182, and HRR, for details).

3 Finite Sample Results

In this section, we address the finite sample properties of the test procedures proposed in this paper. We first provide the finite sample critical values for the statistics introduced and proceed next to the analysis of the empirical size and power performance of the tests.

3.1 Finite Sample Critical Values

For the purpose of computing the necessary critical values for the ζ_f, ζ_f^2, ζ_r, ζ_r^2, $\min(\zeta_f, \zeta_r)$ and $\max(\zeta_f^2, \zeta_r^2)$ statistics, we consider the DGP $(1-L)\, y_t = \varepsilon_t$, where $y_t = 0$ for $t \leq 0$ and $\varepsilon_t \sim nid(0,1)$. Table 3.1 presents critical values for different sample sizes T, $T \in \{100, 250, 500, 1000\}$, which were computed based on 5000 Monte Carlo replications.

⟨Please insert table 3.1 about here⟩

Note that the critical values are valid for test regressions with and without deterministics, given that as previously indicated (see Remark 2.2) the statistics are invariant to these variables. Hence, all results presented in this Section are computed for test regressions in which no deterministic variables where considered. However, experiments with demeaned and detrended variables were also considered, but since the results obtained where qualitatively the same as those reported below we have omitted them.

One immediate consequence of the results in Table 3.1 is that they confirm the results put forward in Theorem 2.2. We observe from this Table that the critical values for the ζ_f and ζ_r, and the ζ_f^2 and ζ_r^2 statistics are in fact the same as put forward in this Theorem.

3.2 Empirical Size and Power

In order to evaluate the finite sample size and power performance of the statistics proposed in this paper, data was generated from the following DGP,

$$(1-L)^{d_1}\, y_t = \varepsilon_t, \quad t = 1, ..., [\tau T] \tag{21}$$

$$(1-L)^{d_2}\, y_t = \varepsilon_t, \quad t = [\tau T]+1, ..., T \tag{22}$$

where $\tau = 0.5$, $d_1 = 0$, $d_2 = 0 + \delta$, $\delta = \{0, 0.1, 0.2, ..., 0.9\}$, $y_t = 0$ for $t \leq 0$ and $\varepsilon_t \sim nid(0,1)$. Hence, the size performance of the tests is evaluated when the data is generated from a white noise process ($d_1 = d_2 = 0$). The results can be found in Tables 3.2 - 3.4.[1] Results on the size and power performance of the tests when d_1 and d_2 are real are provided in Table 3.5.

[1] In order to save space and given that qualitatively the results were the same, we do not report the results for the case when $d_1 = 1$ and $d_2 = 1 - \delta$. However, these results can be obtained from the authors.

⟨Please insert tables 3.2 - 3.5 about here⟩

Table 3.2 presents the empirical size and power performance of the different tests presented in Theorem 2.2. Given the nature of the null hypothesis ($d_1 = d_2 = 0$) and the alternatives considered ($d_1 = 0$, $d_2 = 0 + \delta$), we expect, based on the theoretical results put forward in the text (see Theorems 2.3 and 2.4, and Corolaries 2.3 and 2.4) that the ζ_r and ζ_r^2 statistics will display the largest power. This is in fact confirmed by the results of Table 3.2. We observe from this Table that all tests have empirical size (when $\delta = 0$) very close to the 5% nominal level considered and that as we move away from 0, *i.e.* as δ increases the empirical power of the ζ_r and ζ_r^2 tests increases as well. This behaviour is even more marked for T=250.

Given the importance of allowing for serial correlation, in Tables 3.3 - 3.4 we present the finite sample behaviour of the tests when applied to data generated from a DGP such as (21) - (22) with short-run dynamics in the errors of the type: $(1 - \phi L)\varepsilon_t = (1 + \theta L)u_t$, with $u_t \sim nid(0, 1)$, $(\phi = 0.5, \theta = 0)$, $(\phi = 0, \theta = 0.5)$ and $(\phi = 0, \theta = -0.5)$. In order to decide on the order of augmentation to use, following DKH and as suggested in Section 2.4, we resort to Schwertz's (1989) rule, *i.e.*, $p = int[4(T/100)^{1/4}]$. Table 3.3 presents the results of the tests when the errors display autoregressive (AR) dynamics and Table 3.4, when moving average (MA) dynamics is considered.

The results in Table 3.3 are informative with respect to the impact of the inclusion of unnecessary lags on the procedures' performance. Using the Schwertz rule for T=100, we used 4 lags of the dependent variable to correct for autocorrelation, however, in effect one lag would have been sufficient to account for this short-run dynamics. The implications on power of the use of the additional unnecessary lags in small samples (T=100) is severe. Comparing the results of Table 3.3 with those of Table 3.2 we observe that, for instance, for $d_1 = 0, d_2 = 0.9$ power, for a sample of T=100, in the iid case was around 0.99 for ζ_r and 0.98 for ζ_r^2, but reduces to 0.28 for ζ_r and ζ_r^2 in the context of Table 3.3. It is important to highlight that power considerably improves in larger samples (see results for T=250).

Table 3.4 presents the finite sample behaviour of the tests when the errors follow MA dynamics. One immidiate observation that can be made is that negative MA dynamics ($\theta = -0.5$) has larger implications on the tests' power performance than positive MA dynamics ($\theta = 0.5$). The tests are slightly undersized in both sample sizes considered and power is severely affected particularly when T=100. However, also in this case it is observed that as the sample increases so does the performance of the test.

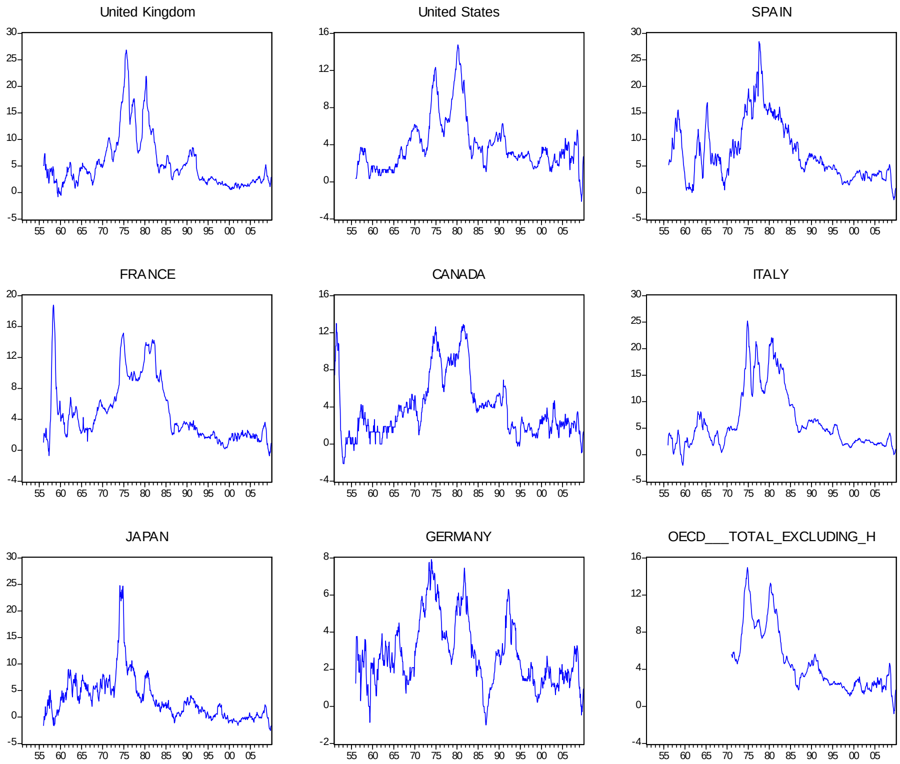
Table 3.5 consideres a different exercise. Instead of imposing a fractional parameter d under the null hypothesis, we first estimated d for the whole sample using the spectral MLE estimator of Fox and Taqqu (1986) and then computed the tests using the estimate, $\hat{d}$. Two cases where considered under the null hypothesis $d_1 = d_2 = 0$ and $d_1 = d_2 = 0.3$. The top panel of Table 3.5 presents the test results when the null hypothesis is $d_1 = d_2 = 0$ and the alternative $d_1 = 0, d_2 = 0 + \delta$. Note from this panel that the results in this case in terms of test performance are switched, note that the best performance is now observed for the ζ_f and ζ_f^2 tests. This is an obvious consequence of the filtering that has been used to set up the test regression in this context. Given that in the Monte Carlo set up we are considering that $d_1 < d_2$,thus as a result of the filtering we obtain d_1^* and d_2^* for the corresponding subsamples, so that now $d_1^* > d_2^*$. Overall we observe that there is a small sample bias in the estimation of d, which naturally translates negatively into the performance of the test. However, considering both cases under analysis and the sizes of the samples used, we conclude that the distortion observed, which is relatively small, is acceptable.

4 Empirical Application

The macro foundations of the reduced-form New Keynesian Phillips curve equations for inflation in developed countries has played a crucial role in showing that inflation exhibits very strong persistence, approaching that of a random-walk process, which is constant over time, specially after WWII; see, *inter alia*, Fuhrer and Moore (1995), Pivetta and Reis (2007) for the US economy and O'Reilly and Whelan (2005) for the Euro area. However, Benatti (2008), building upon distinct world experiences of monetary regimes, casts doubts on the stability of the structural parameters that measure inflation persistence, bringing relevance to the Lucas critique. It is not surprising that a structural change in persistence is observed over time given the shifts in monetary policy regimes occurred since WWII. Hence, the aim of this section is to add some further discussion to this literature by applying our proposed test statistics to several world inflation rates.

A single persistence change in the yearly inflation series was tested for the following economies: the United Kingdom, the United States, Spain, France, Canada, Italy, Japan, Germany and OECD countries (excluding high inflation countries). These are monthly series spanning from January 1951 to December 2009 for Canada (708 observations), from March 1956 to December 2009 for Spain, from January 1971 to December 2009 for the OECD and from January 1956 to December 2009 for the remaining countries. Figure 4.1 presents plots of the time series.

Figure 4.1: Yearly Inflation Rates



We perform the tests under the null hypothesis that the series are I(1), *i.e.* that $d_0 = 1$, and also for I(d), d real, running the regressions of interest with a constant only and with $p_K = \left[K\left(\frac{T}{100}\right)^{1/4}\right], K = 4$, lags to accommodate for serial correlation in the data. For the fractionally integrated case I(d), d real, we considered two approaches for the estimation of $d : \widehat{d}$ using the entire sample; and $\widehat{d}(\tau)$ for the forward and time-reversed subsamples.

For those cases where we find a change in persistence, we obtained the estimated break point, $\widehat{\tau}$, and the two memory parameters, one for each subperiod, either with the forward or the reversed version of the test. Whenever needed, we estimated the long-memory parameter by the spectral maximum likelihood estimator of Fox and Taqqu (1986). In most cases, we obtain similar results using the Geweke and Porter-Hudak (1983) method. The results are provided in Tables 4.1 - 4.3.

Table 4.1: Persistence change test results ($d_0 = 1$)

H_0: y_t is $I(1)$	$\min(\zeta_f, \zeta_r)$	$\max(\zeta_f^2, \zeta_r^2)$	$\widehat{\tau}$ (date)	$\hat{d}_1, \hat{d}_2$
UK	−2.6453*	6.9978*	0.691 (1993:4)	0.7667** → 0.4544**
USA	−2.7286**	7.4457*	0.492 (1982:7)	1# → 0.3085**
Spain	−3.1238**	9.7584**	0.559 (1986:3)	1#→ 0.5898**
France	−2.7626**	7.6323*	0.552 (1985:10)	0.6375** → 0.7287*
Canada	−3.2113**	10.313**	0.693 (1991:10)	1# → 0.4893**
Italy	−2.398*	5.7507	0.324 (1973:6)	1# → 0.6502**
Japan	−1.6834	2.8338	–	–
Germany	−2.1723	4.7191	–	–
OECD	−2.9616**	8.7713**	0.531 (1991:8)	1# → 0.4065**

Notes: ** and * indicate significant at 5% and 10% nominal levels, respectively;
# indicates that the null of d=1 cannot be rejected (value reported under the null).

Only for two out of the nine countries (Japan and Germany) can the hypothesis of constant persistence over time not be rejected (see Table 4.1). For the remaining series we find support for a change in memory based on the reversed version of the test, $\min(\zeta_f, \zeta_r) = \zeta_r$ and $\max(\zeta_f^2, \zeta_r^2) = \zeta_r^2$. According to the estimation results, inflation behaved as a random walk up to a certain point but, more recently, price stability was achieved with a long-memory estimate at the mean and level-reversion range. The estimated break point occurs roughly in the middle of the sample, except for the cases of Italy (1973 oil prices shock), the UK (impact of Britain's exit from the ERM in the second half of 1992) and Canada (1991 recession).

The conclusions which can be taken from the tests do not necessarily match those based on the MLE estimation; results for the cases of the UK and France suggest that $d_1 = 1$ is generally rejected. Evidence of long-memory properties can be found for most of the inflation rates using the entire sample (UK: 0.7684; USA: 0.7062; France: 0.7278; Canada: 0.6211; Japan: 0.6442). This phenomenon is consistent with the results in Hassler and Wolters (1995). The test results for d_0 real are provided in Tables 4.2 and 4.3.

Table 4.2: Persistence change test results (d_0 real and $\widehat{d}$ fixed)

H_0: y_t is $I(d_0)$	$\min(\zeta_f, \zeta_r)$	$\max(\zeta_f^2, \zeta_r^2)$	$\widehat{\tau}$ (date)	$\hat{d}_1, \hat{d}_2$
UK	−3.0765**	9.4652**	0.697 (1993:9)	1.021& → 0.7808**
USA	−2.2669	5.1390	−	−
Spain	−3.2022**	10.2544**	0.612 (1989:2)	0.8332& → 0.6039**
France	−3.2129**	10.3228**	0.462 (1981:1)	1.0494& → 0.7264**
Canada	−3.1708**	10.0541**	0.745 (1994:2)	1.0212& → 0.7783*
Italy	−2.8781**	8.2839**	0.510 (1983:7)	0.9639& → 0.9164*
Japan	−1.8076	3.2675	−	−
Germany	−2.2162	4.9117	−	−
OECD	−1.6341	2.6703	−	−

Notes: ** and * indicate significant at 5% and 10% nominal levels, respectively;

& indicates that the null of d=1 cannot be rejected (MLE point estimate is reported);

Table 4.3: Persistence change test results (d_0 real and $\widehat{d}(\tau)$)

H_0: y_t is $I(d_0)$	$\min(\zeta_f, \zeta_r)$	$\max(\zeta_f^2, \zeta_r^2)$	$\widehat{\tau}$ (date)	$\hat{d}_1$, $\hat{d}_2$
UK	−3.2782**	10.7469**	0.341 (1974:7)	1.1923& → 0.6332**
USA	−2.9464**	8.6816**	0.635 (1990:4)	1.0573& → 0.4481**
Spain	−3.4050**	11.5940**	0.559 (1986:4)	0.7833& → 0.5898**
France	−3.2013**	10.2483**	0.321 (1973:6)	0.5627** → 0.8992&
Canada	−3.2079**	10.2911**	0.691 (1990:12)	0.9251& → 0.6642**
Italy	−3.0310**	9.1870**	0.323 (1973:6)	0.7407& → 0.6600**
Japan	−1.6729	2.7987	−	−
Germany	−2.3118	5.3446	−	−
OECD	−2.9594**	8.7580**	0.695 (1998:2)	0.8648& → 0.2974**

Notes: ** and * indicate significant at 5% and 10% nominal levels, respectively;

& indicates that the null of d=1 cannot be rejected (MLE point estimate is reported);

In general, the results reinforce the previous findings. With the exception of Japan and Germany, countries experimented a decline in inflation rate persistence with no level-reversion up to the 1980's followed by relative price stability during recent years. More specifically, inflation resembles an $I(1)$ process changing to $I(d_2), d_2 < 1$, at time $\lfloor \widehat{\tau} T \rfloor$.

Sibbertsen and Kruse (2009) also applied their long-range dependency tests to the US inflation rate. Using quarterly CPI data from 1953Q1 to 2004Q4, they found 1982Q1 to be the estimated breakpoint and $\widehat{d_2} = 0.246$. Our change point is basically the same (1982:7, eleven quarters after the beginning of Volcker's chairmanship at the Federal Reserve) and we estimate a slightly higher persistence level for the second-half of the sample, $\widehat{d}_2 = 0.308$.

The US inflation rate was also studied by Kim (2000), Kim *et al.* (2002) and Busetti and Taylor (2004), in the integer context, with all finding evidence of a change of persistence. In Busetti and Taylor (2004), the change goes from I(1) to I(0) with an estimated break in the fourth quarter of 1990, which is extremely close to ours when d_0 is real (see Table 4.3). Using the GNP deflator from the second quarter of 1948 to the third quarter of 2000, Kim *et al.* (2002) concluded that inflation rate undergoes changes from stationarity to a unit root around the fourth quarter of 1973. The increase in persistence is due to the estimation of an earlier changepoint. We revisited their partial sums ratio tests using our own updated US monthly sample. The null is strongly rejected for all tests (*mean*, *exp*, *max*, in both directions under the alternative). Following Hassler and Sheithauer (2009), the series changes from $I(1)$ to $I(d)$ in January 1982 (based on LTK's change point estimator) which is very similar to what we and Sibbertsen and Kruse (2009) conclude about the inflation rate during the postwar period.

5 Conclusion

In this paper, we propose regression-based procedures that allow testing for persistence change in fractionally integrated models. The tests can be computed from simple least-squares regressions. Augmented versions of these tests are asymptotically robust against weakly-dependent errors following unknown patterns under quite general conditions, and exhibit good statistical performance in samples of moderate size.

Furthermore, the application of the tests to World inflation rates reveal, with the exception of Japan and Germany, a shift in persistence in the inflation series considered. In particular, the results indicate a change from more persistent to less persistent behaviour, suggesting the possible application of improved monetary policy measures over the latter part of the series.

Hence, the simplicity of application and the good performance in finite samples makes the procedures discussed in this paper a valuable tool when addressing persistence change in a fractional context.

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A Appendix

Proof of Theorem 2.1

Considering the DGP in (1), under the null hypothesis with $d_0 = 1$, assumption A.1 and $\tau \in \Lambda^*$ and fixed. Thus, $x_t = \Delta y_t = \varepsilon_t$ and the statistics in (3) and (5) of Propositions 2.1 and 2.2 are, respectively,

$$\zeta_f(\tau) \overset{H_0}{=} \frac{\sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_t \varepsilon_{t-1}^*}{\widehat{\sigma}_e(\tau)\sqrt{\sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_{t-1}^{*2}}} \quad \text{and} \quad \zeta_r(\tau) \overset{H_0}{=} \frac{\sum_{t=2}^{\lfloor (1-\tau) T \rfloor} \varepsilon_{T-t+1} \varepsilon_{T-t+2}^*}{\widehat{\sigma}_{\widetilde{e}}(\tau)\sqrt{\sum_{t=2}^{\lfloor (1-\tau) T \rfloor} \varepsilon_{T-t+2}^{*2}}},$$

where $\varepsilon_{t-1}^* = \sum_{j=1}^{t-1} \frac{\varepsilon_{t-j}}{j}$ and $\varepsilon_{T-t+2}^* = \sum_{j=1}^{t-1} \frac{\varepsilon_{T-t+j+1}}{j}$.

Since, following Hassler and Breitung (2006, Lemma A (ii) and (iii), p. 1105) and Tanaka (1999),

$$\frac{1}{\lfloor \tau T \rfloor} \sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_{t-1}^{*2} \xrightarrow{p} \sigma_\varepsilon^2 \frac{\pi^2}{6}, \quad \frac{1}{\sqrt{\lfloor \tau T \rfloor}} \sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_t \varepsilon_{t-1}^* \xrightarrow{d} N\left(0, \sigma_\varepsilon^4 \frac{\pi^2}{6}\right);$$

$$\frac{1}{\sqrt{\lfloor (1-\tau) T \rfloor}} \sum_{t=2}^{\lfloor (1-\tau) T \rfloor} \varepsilon_{T-t+1} \varepsilon_{T-t+2}^* \xrightarrow{d} N\left(0, \sigma_\varepsilon^4 \frac{\pi^2}{6}\right);$$

$$\widehat{\sigma}_{\widetilde{e}}^2(\tau) = \frac{1}{\lfloor (1-\tau) T \rfloor - 2} \sum_{t=2}^{\lfloor (1-\tau) T \rfloor} \widehat{\widetilde{e}}_t^2 = \frac{1}{\lfloor (1-\tau) T \rfloor - 2} \sum_{t=2}^{\lfloor (1-\tau) T \rfloor} \widehat{e}_{T-t+1}^2 \xrightarrow{p} \sigma_\varepsilon^2;$$

$$\frac{1}{\lfloor (1-\tau) T \rfloor} \sum_{t=2}^{\lfloor (1-\tau) T \rfloor} \varepsilon_{T-t+2}^{*2} = \frac{1}{\lfloor (1-\tau) T \rfloor} \left(\sum_{t=2}^{T+1} \varepsilon_{t-1}^{*2} - \sum_{t=2}^{\lfloor \tau T \rfloor + 2} \varepsilon_{t-1}^{*2} \right)$$

$$= \frac{T}{\lfloor (1-\tau) T \rfloor} \frac{1}{T} \sum_{t=2}^{T} \varepsilon_{t-1}^{*2} - \frac{\lfloor \tau T \rfloor}{\lfloor (1-\tau) T \rfloor} \frac{1}{\lfloor \tau T \rfloor} \sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_{t-1}^{*2} + o_p(1) \xrightarrow{p} \sigma_\varepsilon^2 \frac{\pi^2}{6}$$

and $\widehat{\sigma}_e^2(\tau) = \frac{1}{\lfloor \tau T \rfloor - 2}\sum_{t=2}^{\lfloor \tau T \rfloor} \widehat{e}_t^2 \xrightarrow{p} \sigma_\varepsilon^2$. These moments have the same asymptotic distribution due to the fact that ε_t is *i.i.d.*. Hence, it follows that, as $T \to \infty$, $\zeta_f(\tau) \overset{a.s}{=} \zeta_r(\tau) \xrightarrow{d} N(0,1)$ and consequently, $\zeta_f^2(\tau) \overset{a.s}{=} \zeta_r^2(\tau) \xrightarrow{d} \chi^2_{(1)}$ for any given $\tau \in \Lambda^*$. ■

Proof of Theorem 2.2

The proof follows from the results in Theorem 2.1. ■

Before providing the proof of Theorem 2.3 consider first the following Lemmatta.

Lemma A.1 *Consider $\varepsilon_t \sim i.i.d.\left(0, \sigma_\varepsilon^2\right)$ and define*

$$\varepsilon_{t-1}^* = \sum_{j=1}^{t-1} \frac{\varepsilon_{t-j}}{j} \text{ and } \varepsilon_{t-1}^{*,\infty} = \sum_{j=1}^{\infty} \frac{\varepsilon_{t-j}}{j} = \sum_{j=0}^{\infty} \psi_j^* \varepsilon_{t-1-j},$$

where $\psi_j^* = \frac{1}{j+1}$ and

$$\varepsilon_{t-2}^{**} = \sum_{j=1}^{t-1} \frac{\varepsilon_{t-j-1}^*}{j} \text{ and } \varepsilon_{t-2}^{**,\infty} = \sum_{j=1}^{\infty} \frac{\varepsilon_{t-1-j}^{*,\infty}}{j} = \sum_{j=0}^{\infty} \psi_j^* \sum_{k=0}^{\infty} \psi_k^* \varepsilon_{t-2-j-k}.$$

Then, as $T \to \infty$,

$$\begin{aligned} \frac{1}{T}\sum_{t=2}^{T}\left(\varepsilon_{t-2}^{**}\right)^2 \xrightarrow{p} E\left(\varepsilon_{t-2}^{**,\infty}\right)^2 &= \sigma_{**}^2, \qquad \frac{1}{\sqrt{T}}\sum_{t=2}^{T}\varepsilon_t\varepsilon_{t-2}^{**} = O_p(1), \\ \frac{1}{T}\sum_{t=2}^{T}\varepsilon_t\varepsilon_{t-2}^{**} \xrightarrow{p} E\left(\varepsilon_t\varepsilon_{t-2}^{**,\infty}\right) &= 0 \quad \text{and} \quad \frac{1}{T}\sum_{t=2}^{T}\varepsilon_{t-1}^{*}\varepsilon_{t-2}^{**} \xrightarrow{p} E\left(\varepsilon_{t-1}^{*,\infty}\varepsilon_{t-2}^{**,\infty}\right) = \gamma_{*,**}, \end{aligned}$$

for some $\sigma_{**}^2 > 0$ and $\gamma_{*,**} \neq 0$.

Proof of Lemma A.1

The process $\varepsilon_{t-1}^{*,\infty}$ is stationary and ergodic since $\varepsilon_t \sim i.i.d.$ and ψ_j^* is square summable, $\sum_{j=0}^{\infty}\psi_j^{*2} = \sum_{j=0}^{\infty}\frac{1}{(j+1)^2} < \infty$. Similarly, $\varepsilon_{t-2}^{**,\infty}$ also satisfies the properties of stationarity and ergodicity (see, for example, DKH, p. 208). Moreover,

$$\varepsilon_{t-1}^* = \varepsilon_{t-1}^{*,\infty} - \sum_{j=t}^{\infty}\psi_j^*\varepsilon_{t-j} = \varepsilon_{t-1}^{*,\infty} - O_p\left(1/\sqrt{t}\right),$$

because

$$E\left(\sum_{j=t}^{\infty}\psi_j^*\varepsilon_{t-j}\right)^2 = \sum_{j=t}^{\infty}\psi_j^{*2}E\left(\varepsilon_{t-j}^2\right) = \sigma_\varepsilon^2\left(\sum_{j=0}^{\infty}\psi_j^{*2} - \sum_{j=0}^{t-1}\psi_j^{*2}\right) = O(1/t)$$

and $\varepsilon_{t-2}^{**} = \varepsilon_{t-2}^{**,\infty} - O_p\left(\ln t/\sqrt{t}\right)$; see DKH. Therefore, as $T \to \infty$,

$$\begin{aligned} \frac{1}{T}\sum_{t=2}^{T}\left(\varepsilon_{t-2}^{**}\right)^2 &= \frac{1}{T}\sum_{t=2}^{T}\left(\varepsilon_{t-2}^{**,\infty}\right)^2 + O_p(1)\frac{1}{T}\sum_{t=2}^{T}\left(\ln t/\sqrt{t}\right)^2 + O_p(1)\,2\frac{1}{T}\sum_{t=2}^{T}\varepsilon_{t-2}^{**,\infty}\left(\ln t/\sqrt{t}\right) \\ &= \frac{1}{T}\sum_{t=2}^{T}\left(\varepsilon_{t-2}^{**,\infty}\right)^2 + o_p(1) \xrightarrow{p} E\left(\varepsilon_{t-2}^{**,\infty}\right)^2 = \sigma_{**}^2; \end{aligned}$$

$$\begin{aligned}\frac{1}{T}\sum_{t=2}^{T}\varepsilon_t\varepsilon_{t-2}^{**} &= \frac{1}{T}\sum_{t=2}^{T}\varepsilon_t\varepsilon_{t-2}^{**,\infty} - \frac{1}{T}\sum_{t=2}^{T}\varepsilon_t O_p\left(\ln t/\sqrt{t}\right) \\ &= \frac{1}{T}\sum_{t=2}^{T}\varepsilon_t\varepsilon_{t-2}^{**,\infty} + o_p(1) \xrightarrow{p} E\left(\varepsilon_t\varepsilon_{t-2}^{**,\infty}\right) = 0; \\ \frac{1}{\sqrt{T}}\sum_{t=2}^{T}\varepsilon_t\varepsilon_{t-2}^{**} &= \frac{1}{\sqrt{T}}\sum_{t=2}^{T}\varepsilon_t\varepsilon_{t-2}^{**,\infty} - \frac{1}{\sqrt{T}}\sum_{t=2}^{T}\varepsilon_t O_p\left(\ln t/\sqrt{t}\right) = O_p(1)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{T}\sum_{t=2}^{T}\varepsilon_{t-1}^{*}\varepsilon_{t-2}^{**} &= \frac{1}{T}\sum_{t=2}^{T}\left(\varepsilon_{t-1}^{*,\infty} - O_p\left(1/\sqrt{t}\right)\right)\left(\varepsilon_{t-2}^{**,\infty} - O_p\left(\ln t/\sqrt{t}\right)\right) \\ &= \frac{1}{T}\sum_{t=2}^{T}\varepsilon_{t-1}^{*,\infty}\varepsilon_{t-2}^{**,\infty} + o_p(1) \xrightarrow{p} E\left(\varepsilon_{t-1}^{*,\infty}\varepsilon_{t-2}^{**,\infty}\right) = \gamma_{*,**},\end{aligned}$$

where the last two results follow from the fact that $\varepsilon_{t-2}^{**,\infty}$ will correlate with $\varepsilon_{t-1}^{*,\infty}$ but not with ε_t due to the *i.i.d.* property of $\{\varepsilon_t\}$. ■

Lemma A.2 *(Local Power) Let $\{y_t\}$ be generated from (1) under $d_T = \frac{\delta}{\sqrt{T}}$ with δ fixed, that is, $x_t \sim I\left(\frac{\delta}{\sqrt{T}}\right)$ with i.i.d. errors, and $x_{t-1}^{*} = \sum_{j=1}^{t-1}\frac{x_{t-j}}{j}$. Then, as $T \to \infty$,*

$$\frac{1}{T}\sum_{t=2}^{T}x_{t-1}^{*2} \xrightarrow{p} \sigma_\varepsilon^2\frac{\pi^2}{6} \qquad \text{and} \qquad \frac{1}{\sqrt{T}}\sum_{t=2}^{T}x_t x_{t-1}^{*} \xrightarrow{d} N\left(\frac{\pi^2}{6}\delta, \sigma_\varepsilon^4\frac{\pi^2}{6}\right).$$

Proof of Lemma A.2

The second result is a direct application of Theorem 3.1 in Tanaka (1999). To show the first result, note that $x_t = (1-L)^{-d_T}\varepsilon_t$, where ε_t satisfies assumption A.1, can be decomposed as

$$x_t = \varepsilon_t + \frac{\delta}{\sqrt{T}}\sum_{k=1}^{t-1}\frac{\varepsilon_{t-k}}{k} + O_p(1/T) = \varepsilon_t + \frac{\delta}{\sqrt{T}}\varepsilon_{t-1}^{*} + o_p(1),$$

following Tanaka (1999, p. 579), and therefore,

$$x_{t-1}^{*} = \sum_{j=1}^{t-1}\frac{y_{t-j}}{j} = \varepsilon_{t-1}^{*} + \frac{\delta}{\sqrt{T}}\varepsilon_{t-2}^{**} + o_p(1).$$

Then, following Hassler and Breitung (2006, Lemma A) and Lemma A.1 we observe that,

$$\frac{1}{T}\sum_{t=2}^{T}x_{t-1}^{*2} = \frac{1}{T}\sum_{t=2}^{T}\varepsilon_{t-1}^{*2} + \frac{\delta^2}{T}\frac{1}{T}\sum_{t=2}^{T}\varepsilon_{t-2}^{**2} + 2\frac{\delta}{\sqrt{T}}\frac{1}{T}\sum_{t=2}^{T}\varepsilon_{t-1}^{*}\varepsilon_{t-2}^{**} + o_p(1) \xrightarrow{p} \sigma_\varepsilon^2\frac{\pi^2}{6}.$$

■

Lemma A.3 *(Global Power) Let* $\{y_t\}$ *be generated by model (1) with* $d_t = d \in (-0.5, 0.5)$ *(non-local alternative), that is,* $x_t \sim I(d)$ *with i.i.d. errors and* $x^*_{t-1} = \sum_{j=1}^{t-1} \frac{x_{t-j}}{j}$*. Then, as* $T \to \infty$,

$$\frac{1}{T}\sum_{t=2}^{T} x^{*2}_{t-1} \xrightarrow{p} \sigma^2_{y^*}(d) \qquad \text{and} \qquad \frac{1}{T}\sum_{t=2}^{T} x_t x^*_{t-1} \xrightarrow{p} \gamma_*(d),$$

where

$$\sigma^2_{y^*}(d) = \sigma^2_{\varepsilon}\sum_{j=0}^{\infty}\varphi^{*2}_{j,d} \qquad \text{and} \qquad \gamma_*(d) = \sigma^2_{\varepsilon}\sum_{j=1}^{\infty}\varphi_{j,d}\varphi^*_{j-1,d}$$

with

$$\begin{aligned} \varphi_{j,d} &= \frac{\Gamma(j+d)}{\Gamma(d)\Gamma(j+1)} = \frac{1}{j!}\prod_{k=0}^{j-1}(d+k) \approx \frac{1}{\Gamma(d)} j^{d-1} \text{ for large } j, \\ \varphi^*_{j,d} &= \sum_{k=0}^{j}\frac{1}{(k+1)!}\varphi_{j-k,d} \text{ and } \Gamma(\cdot) \text{ is the Gamma function.} \end{aligned}$$

Moreover, if $d \in (0, 0.5)$ *then* $\gamma_*(d) > 0$*; whereas if* $d \in (-0.5, 0)$ *then* $\gamma_*(d) < 0$.

Proof of Lemma A.3:

Here, the DGP is $x_t = (1-L)^{-d}\varepsilon_t$, where ε_t satisfies assumption A.1, and can no longer be decomposed as

$$x_t = \varepsilon_t + d\sum_{k=1}^{t-1}\frac{\varepsilon_{t-k}}{k} + O_p(1/T) = \varepsilon_t + d\varepsilon^*_{t-1} + o_p(1),$$

since

$$\begin{aligned} x_t &= \left(1 + dL + \frac{d(d+1)L^2}{2!} + \frac{d(d+1)(d+2)L^3}{3!} + ...\right)\varepsilon_t \\ &= \varepsilon_t + d\varepsilon_{t-1} + \frac{(d+1)}{1!}d\frac{\varepsilon_{t-2}}{2} + \frac{(d+1)(d+2)}{2!}d\frac{\varepsilon_{t-3}}{3} + ... \\ &= \varepsilon_t + d\varepsilon^*_{t-1} \\ &\quad + \left(\frac{(d+1)}{1!} - 1\right)d\frac{\varepsilon_{t-2}}{2} + ... + \left(\frac{(d+1)(d+2)...(d+t-2)}{(t-2)!} - 1\right)d\frac{\varepsilon_1}{t-1} \\ &\quad + \frac{(d+1)(d+2)...(d+t-1)}{(t-1)!}d\frac{\varepsilon_0}{t} + ... \end{aligned}$$

Even if one assumes that $\varepsilon_t = 0, t \leq 0$,

$$\begin{aligned} x_t &= \varepsilon_t + d\varepsilon^*_{t-1} + d\sum_{l=2}^{t-1}\left(\frac{\Pi_{i=1}^{l-1}(d+i)}{(l-1)!} - 1\right)\frac{\varepsilon_{t-l}}{l} + o_p(1) \\ &= \varepsilon_t + d\varepsilon^*_{t-1} + d\left(\varepsilon_{t-1} + \varepsilon^{*,d}_{t-1}\right) + o_p(1), \end{aligned}$$

where

$$\varepsilon^{*,d}_{t-1} = \sum_{j=1}^{t-1}\psi_{j,d}\frac{\varepsilon_{t-j}}{j}, \qquad \text{with} \qquad \psi_{j,d} = \frac{\Pi_{i=1}^{j-1}(d+i)}{(j-1)!} - 1.$$

Under this assumption,

$$\begin{aligned} x^*_{t-1} &= \sum_{j=1}^{t-1} \frac{x_{t-j}}{j} = \varepsilon^*_{t-1} + d\varepsilon^{**}_{t-2} + d\sum_{j=1}^{t-1} \frac{1}{j}\left(\varepsilon_{t-j-1} + \varepsilon^{*,d}_{t-j-1}\right) + o_p(1) \\ &= \varepsilon^*_{t-1} + d\varepsilon^{**}_{t-2} + d\left(\varepsilon^*_{t-2} + \varepsilon^{**,d}_{t-2}\right) + o_p(1), \end{aligned}$$

where

$$\varepsilon^{**,d}_{t-2} = \sum_{j=1}^{t-1} \frac{\varepsilon^{*,d}_{t-j-1}}{j},$$

since

$$\sum_{j=1}^{t-1} \frac{1}{j}\varepsilon_{t-j-1} = \sum_{j=1}^{t-2} \frac{1}{j}\varepsilon_{t-j-1} + \frac{1}{t-1}\varepsilon_0 = \varepsilon^*_{t-2} + o_p(1).$$

It would then remain to be shown that $\left\{\varepsilon^{*,d,\infty}_{t-1}\right\}$, where $\varepsilon^{*,d,\infty}_{t-1} = \sum_{j=1}^{\infty} \psi_{j,d} \frac{\varepsilon_{t-j}}{j}$, with $\psi_{j,d} = \frac{\Pi_{i=1}^{j-1}(d+i)}{(j-1)!} - 1$, is stationary and ergodic, which, by Lemma A.1, would imply

$$\frac{1}{T}\sum_{t=2}^{T} x^{*2}_{t-1} \xrightarrow{p} \sigma^2_{y^*}(d) \quad \text{and} \quad \frac{1}{T}\sum_{t=2}^{T} x_t x^*_{t-1} \xrightarrow{p} \gamma_*(d),$$

for some $\sigma^2_{y^*}(d)$ and $\gamma_*(d)$ that depend on d, where the last result follows from the fact that y_t will correlate with x^*_{t-1}. To show that $\psi^*_{j,d} = \frac{1}{j}\left(\frac{\Pi_{i=1}^{j-1}(d+i)}{(j-1)!} - 1\right)$ is square summable (or absolute summable) for some interval of d can be a tedious job.

Thus, to prove the result in this Lemma and to obtain closed form expressions for $\sigma^2_{y^*}(d)$ and $\gamma_*(d)$ we pursue an alternative approach. Let $d < 0.5$. Then, $\{x_t\}$ is stationary and has the infinite order MA representation $x_t = \sum_{j=0}^{\infty} \varphi_{j,d}\varepsilon_{t-j}$, where $\varphi_{j,d} = \frac{\Gamma(j+d)}{\Gamma(d)\Gamma(j+1)} = \frac{1}{j!}\prod_{k=0}^{j-1}(d+k) \approx \frac{1}{\Gamma(d)} j^{d-1}$ for large j, and $\Gamma(\cdot)$ is the Gamma function (see, for example, Baillie, 1996). Then, from DKH and Lemma 2.1 it follows that,

$$x^{*,\infty}_{t-1} = \sum_{j=1}^{\infty} \frac{x_{t-j}}{j} = \sum_{j=0}^{\infty} \varphi^*_{j,d}\varepsilon_{t-1-j},$$

where $\varphi^*_{j,d} = \sum_{k=0}^{j} \frac{1}{(k+1)!}\varphi_{j-k,d} = \sum_{k=0}^{j} \frac{1}{(k+1)!} \frac{\Gamma(j-k+d)}{\Gamma(d)\Gamma(j-k+1)}$ is stationary and ergodic ($\left\{\varphi^*_{j,d}\right\}_j$ is square summable). Because $x^*_{t-1} = x^{*,\infty}_{t-1} - O_p\left(1/\sqrt{t}\right)$ it follows that,

$$\frac{1}{T}\sum_{t=2}^{T} x^{*2}_{t-1} = \frac{1}{T}\sum_{t=2}^{T} x^{*2,\infty}_{t-1} + o_p(1) \xrightarrow{p} \sigma^2_{\varepsilon}\sum_{j=0}^{\infty}\varphi^{*2}_{j,d} \equiv \sigma^2_{y^*}(d) > 0$$

see DKH (p. 193). Moreover,

$$\begin{aligned} \frac{1}{T}\sum_{t=2}^{T} x_t x^*_{t-1} &= \frac{1}{T}\sum_{t=2}^{T} x_t x^{*,\infty}_{t-1} + o_p(1) \\ &= \frac{1}{T}\sum_{t=2}^{T}\left(\sum_{j=1}^{\infty}\varphi_{j,d}\varepsilon_{t-j}\right)\left(\sum_{k=0}^{\infty}\varphi^*_{k,d}\varepsilon_{t-1-k}\right) + o_p(1) \\ &\xrightarrow{p} \sigma^2_{\varepsilon}\sum_{i=1}^{\infty}\varphi_{i,d}\varphi^*_{i-1,d} \equiv \gamma_*(d) \end{aligned}$$

by the $i.i.d.$ assumption of $\{\varepsilon_t\}$. Clearly, if $0 < d < 0.5$ then $\gamma_*(d) > 0$. This statement can be proven given any of the following equalities:

$$\begin{aligned}
&\sum_{i=1}^{\infty} \varphi_{i,d}\varphi_{i-1,d}^{*} \\
&= \sum_{i=1}^{\infty}\sum_{k=0}^{i-1} \frac{1}{(k+1)!}\varphi_{i,d}\varphi_{i-1-k,d} = \sum_{i=1}^{\infty}\sum_{k=0}^{i-1} \frac{1}{(k+1)!}\varphi_{i-1-k,d}^{2}\prod_{j=0}^{k}\left(\frac{i+d-1-j}{i-j}\right) \\
&= \sum_{i=0}^{\infty}\varphi_{i,d}^{2}\left[\sum_{k=1}^{\infty}\frac{1}{k!}\prod_{j=0}^{k-1}\left(\frac{d+i+j}{i+j+1}\right)\right] \\
&= \sum_{k=1}^{\infty}\frac{1}{k!}\prod_{j=0}^{k-1}\left(\frac{d+j}{j+1}\right) + \sum_{i=1}^{\infty}\varphi_{i,d}^{2}\left[\sum_{k=1}^{\infty}\frac{1}{k!}\prod_{j=0}^{k-1}\left(\frac{d+i+j}{i+j+1}\right)\right].
\end{aligned}$$

On the other hand, if $-0.5 < d < 0$,

$$\sum_{k=1}^{\infty}\frac{1}{k!}\prod_{j=0}^{k-1}\left(\frac{d+j}{j+1}\right) = d\sum_{k=1}^{\infty}\frac{1}{k!}\prod_{j=1}^{k-1}\left(\frac{d+j}{j+1}\right) < 0$$

whereas

$$\sum_{i=1}^{\infty}\varphi_{i,d}^{2}\left[\sum_{k=1}^{\infty}\frac{1}{k!}\prod_{j=0}^{k-1}\left(\frac{d+i+j}{i+j+1}\right)\right] > 0,$$

not being clear which of the two dominates. It is known that for the "antipersistent" fractional white noise process with $d \in (-0.5, 0)$, all autocovariances, γ_l, $l \geq 1$, are negative. Then, if $-0.5 < d < 0$, we have $\gamma_*(d) < 0$ because

$$\sigma_\varepsilon^2\sum_{i=1}^{\infty}\varphi_{i,d}\varphi_{i-1,d}^{*} = \sigma_\varepsilon^2\sum_{i=1}^{\infty}\sum_{k=0}^{i-1}\frac{1}{(k+1)!}\varphi_{i,d}\varphi_{i-1-k,d} = \sum_{l\geq 1}\frac{1}{l!}\gamma_l < 0.$$

■

Proof of Theorem 2.3

Assume H_{01}, where $d_{1,T} = 1 - \frac{\delta}{\sqrt{T}}, \delta > 0$. Fix $\tau \in \Lambda^*$ and consider the case where $\tau \leq \tau^*$. From Theorem 2.1, and regardless of whether local or global alternatives are considered, $\zeta_f(\tau) = O_p(1)$ as $\zeta_f(\tau) \xrightarrow{d} N(0,1)$. Now, consider the case where $\tau > \tau^*$, so that,

$$\zeta_f(\tau) = \frac{\sum_{t=2}^{\lfloor \tau^* T\rfloor} x_t x_{t-1}^* + \sum_{t=\lfloor \tau^* T\rfloor+1}^{\lfloor \tau T\rfloor} x_t x_{t-1}^*}{\widehat{\sigma}_e(\tau)\sqrt{\sum_{t=2}^{\lfloor \tau^* T\rfloor} x_{t-1}^{*2} + \sum_{t=\lfloor \tau^* T\rfloor+1}^{\lfloor \tau T\rfloor} x_{t-1}^{*2}}},$$

where $\frac{1}{\lfloor \tau^* T\rfloor}\sum_{t=2}^{\lfloor \tau^* T\rfloor} x_{t-1}^{*2} \xrightarrow{p} \sigma_\varepsilon^2\frac{\pi^2}{6}$ and $\frac{1}{\sqrt{\lfloor \tau^* T\rfloor}}\sum_{t=2}^{\lfloor \tau^* T\rfloor} x_t x_{t-1}^* \xrightarrow{d} N\left(0, \sigma_\varepsilon^4\frac{\pi^2}{6}\right)$.On the other hand, for the time period $t = \lfloor \tau^* T\rfloor + 1, ..., \lfloor \tau T\rfloor$, $\{y_t\}$ follows model (1) with $d_t = d_{1,T} = 1 - \frac{\delta}{\sqrt{T}}, \delta > 0$ and $x_t = (1-L)y_t = (1-L)^{-d_{1,T}}\varepsilon_t \sim I(d_{1,T} - 1)$. Then, $x_t \sim I\left(-\frac{\delta}{\sqrt{T}}\right)$, and by Lemma A.2,

$$\frac{1}{\sqrt{\lfloor(\tau-\tau^*)T\rfloor}}\sum_{t=\lfloor \tau^* T\rfloor+1}^{\lfloor \tau T\rfloor} x_t x_{t-1}^* \xrightarrow{d} N\left(-\frac{\pi^2}{6}\delta, \sigma_\varepsilon^4\frac{\pi^2}{6}\right) \text{ and } \frac{1}{\lfloor(\tau-\tau^*)T\rfloor}\sum_{t=\lfloor \tau^* T\rfloor+1}^{\lfloor \tau T\rfloor} x_{t-1}^{*2} \xrightarrow{p} \sigma_\varepsilon^2\frac{\pi^2}{6}.$$

Hence, when $\tau > \tau^*$ and the alternative is local,

$$\begin{aligned}\zeta_f(\tau) &= \frac{\mathcal{A}_f(\tau,\tau^*)}{\mathcal{B}_f(\tau,\tau^*)} \\ &\overset{d}{\to} \frac{N\left(0,\sigma_\varepsilon^4\frac{\pi^2}{6}\right)+\sqrt{\frac{\tau-\tau^*}{\tau^*}}N\left(-\frac{\pi^2}{6}\delta,\sigma_\varepsilon^4\frac{\pi^2}{6}\right)}{\sigma_\varepsilon\sqrt{\sigma_\varepsilon^2\frac{\pi^2}{6}+\left(\frac{\tau-\tau^*}{\tau^*}\right)\sigma_\varepsilon^2\frac{\pi^2}{6}}} \\ &\equiv N\left(0,\frac{\tau^*}{\tau}\right)+N\left(-\delta\sqrt{\left(\frac{\tau-\tau^*}{\tau}\right)\frac{\pi^2/6}{\sigma_\varepsilon^4}},\frac{\tau-\tau^*}{\tau}\right),\end{aligned}$$

where

$$\mathcal{A}_f(\tau,\tau^*) = \frac{1}{\sqrt{\lfloor\tau^*T\rfloor}}\sum_{t=2}^{\lfloor\tau^*T\rfloor}x_tx_{t-1}^* + \frac{\sqrt{\lfloor(\tau-\tau^*)T\rfloor}}{\sqrt{\lfloor\tau^*T\rfloor}}\frac{1}{\sqrt{\lfloor(\tau-\tau^*)T\rfloor}}\sum_{t=\lfloor\tau^*T\rfloor+1}^{\lfloor\tau T\rfloor}x_tx_{t-1}^*$$

and

$$\mathcal{B}_f(\tau,\tau^*) = \widehat{\sigma}_e(\tau)\sqrt{\frac{1}{\lfloor\tau^*T\rfloor}\sum_{t=2}^{\lfloor\tau^*T\rfloor}x_{t-1}^{*2}+\frac{\lfloor(\tau-\tau^*)T\rfloor}{\lfloor\tau^*T\rfloor}\frac{1}{\lfloor(\tau-\tau^*)T\rfloor}\sum_{t=\lfloor\tau^*T\rfloor+1}^{\lfloor\tau T\rfloor}x_{t-1}^{*2}}.$$

Thus, $\zeta_f(\tau)$ is a normal random variable with expectation $-\delta\sqrt{\left(\frac{\tau-\tau^*}{\tau}\right)\frac{\pi^2/6}{\sigma_\varepsilon^4}} < 0$ and variance larger than 1 (note that $\sum_{t=2}^{\lfloor\tau^*T\rfloor}x_tx_{t-1}^*$ and $\sum_{t=\lfloor\tau^*T\rfloor+1}^{\lfloor\tau T\rfloor}x_tx_{t-1}^*$ have a positive covariance that equals a multiple of σ_ε^2 because both processes depend on $\varepsilon_s, s \leq \lfloor\tau^*T\rfloor$.) Note that when $\tau = \tau^*$ or $\delta = 0$ we have the null distribution $N(0,1)$. By the CMT,

$$\begin{aligned}\zeta_f &= \inf_{\tau\in\Lambda^*,}\zeta_f(\tau) \overset{d}{\to} \inf_{\tau\in\Lambda^*,}\begin{cases} N(0,1) & if \quad \tau\leq\tau^* \\ N\left(0,\frac{\tau^*}{\tau}\right)+N\left(-\delta\sqrt{\left(\frac{\tau-\tau^*}{\tau}\right)\frac{\pi^2/6}{\sigma_\varepsilon^4}},\frac{\tau-\tau^*}{\tau}\right) & if \quad \tau>\tau^*\end{cases} \\ &\overset{a.s.}{=} N(0,\tau^*)+N\left(-\delta\sqrt{(1-\tau^*)\frac{\pi^2/6}{\sigma_\varepsilon^4}},1-\tau^*\right)\end{aligned}$$

because $-\delta\sqrt{\left(\frac{\tau-\tau^*}{\tau}\right)\frac{\pi^2/6}{\sigma_\varepsilon^4}}$ is monotonically decreasing in τ (the *infimum* is attained at $\tau = 1$). The further the departure from the null hypothesis (δ larger) and/or the earlier the break occurs (τ^* smaller), the more likely it is to reject the null as the distribution shifts to the left of zero. This proves the existence of (local) power of the ζ_f statistic.

Regarding the $\zeta_r(\tau)$ statistic, consider that $\tau > \tau^*$, where $\zeta_r(\tau)$ is computed for $I(d_{1,T})$ data, and $d_{1,T} = 1 - \frac{\delta}{\sqrt{T}}, \delta > 0$. Then, by Lemma A.2,

$$\zeta_r(\tau) = \frac{\frac{1}{\sqrt{\lfloor(1-\tau)T\rfloor}}\sum_{t=2}^{\lfloor(1-\tau)T\rfloor}x_{T-t+1}x_{T-t+2}^*}{\widehat{\sigma}_{\widetilde{\varepsilon}}(\tau)\sqrt{\frac{1}{\lfloor(1-\tau)T\rfloor}\sum_{t=2}^{\lfloor(1-\tau)T\rfloor}x_{T-t+2}^{*2}}} \overset{d}{\to} \frac{N\left(-\frac{\pi^2}{6}\delta,\sigma_\varepsilon^4\frac{\pi^2}{6}\right)}{\sigma_\varepsilon\sqrt{\sigma_\varepsilon^2\frac{\pi^2}{6}}} \overset{a.s.}{=} N\left(-\delta\sqrt{\frac{\pi^2/6}{\sigma_\varepsilon^4}},1\right),$$

which does not depend on τ.

In the case of $\tau \leq \tau^*$,

$$\begin{aligned}
\zeta_r(\tau) &= \frac{\mathcal{A}_r(\tau,\tau^*)}{\mathcal{B}_r(\tau,\tau^*)} \\
&\xrightarrow{d} \frac{N\left(-\frac{\pi^2}{6}\delta, \sigma_\varepsilon^4 \frac{\pi^2}{6}\right) + \sqrt{\frac{\tau^*-\tau}{1-\tau^*}} N\left(0, \sigma_\varepsilon^4 \frac{\pi^2}{6}\right)}{\sigma_\varepsilon \sqrt{\sigma_\varepsilon^2 \frac{\pi^2}{6} + \left(\frac{\tau^*-\tau}{1-\tau^*}\right) \sigma_\varepsilon^2 \frac{\pi^2}{6}}} \\
&\overset{a.s.}{\equiv} N\left(-\delta\sqrt{\left(\frac{1-\tau^*}{1-\tau}\right)\frac{\pi^2/6}{\sigma_\varepsilon^4}}, \frac{1-\tau^*}{1-\tau}\right) + N\left(0, \frac{\tau^*-\tau}{1-\tau}\right),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A}_r(\tau,\tau^*) &= \frac{1}{\sqrt{\lfloor (1-\tau^*)T \rfloor}} \sum_{t=2}^{\lfloor (1-\tau^*)T \rfloor} x_{T-t+1} x^*_{T-t+2} \\
&+ \frac{\sqrt{\lfloor (\tau^*-\tau)T \rfloor}}{\sqrt{\lfloor (1-\tau^*)T \rfloor}} \frac{1}{\sqrt{\lfloor (\tau^*-\tau)T \rfloor}} \sum_{t=\lfloor (1-\tau^*)T \rfloor+1}^{\lfloor (1-\tau)T \rfloor} x_{T-t+1} x^*_{T-t+2}
\end{aligned}$$

and

$$\mathcal{B}_r(\tau,\tau^*) = \widehat{\sigma}_{\widetilde{\varepsilon}}(\tau) \sqrt{\frac{1}{\lfloor (1-\tau^*)T \rfloor} \sum_{t=2}^{\lfloor (1-\tau^*)T \rfloor} x^{*2}_{T-t+2} + \frac{\lfloor (\tau^*-\tau)T \rfloor}{\lfloor (1-\tau^*)T \rfloor} \frac{1}{\lfloor (\tau^*-\tau)T \rfloor} \sum_{t=\lfloor (1-\tau^*)T \rfloor+1}^{\lfloor (1-\tau)T \rfloor} x^{*2}_{T-t+2}}.$$

Thus, $\zeta_r(\tau)$ is also a normal random variable but with expectation $-\delta\sqrt{\left(\frac{1-\tau^*}{1-\tau}\right)\frac{\pi^2/6}{\sigma_\varepsilon^4}} < 0$ and variance larger than 1. By the CMT,

$$\zeta_r \xrightarrow{d} \inf_{\tau \in \Lambda^*,} \begin{cases} N\left(-\delta\sqrt{\left(\frac{1-\tau^*}{1-\tau}\right)\frac{\pi^2/6}{\sigma_\varepsilon^4}}, \frac{1-\tau^*}{1-\tau}\right) + N\left(0, \frac{\tau^*-\tau}{1-\tau}\right) & if \quad \tau \leq \tau^* \\ N\left(-\delta\sqrt{\frac{\pi^2/6}{\sigma_\varepsilon^4}}, 1\right) & if \quad \tau > \tau^* \end{cases} \equiv N\left(-\delta\sqrt{\frac{\pi^2/6}{\sigma_\varepsilon^4}}, 1\right),$$

because $-\delta\sqrt{\left(\frac{1-\tau^*}{1-\tau}\right)\frac{\pi^2/6}{\sigma_\varepsilon^4}} > -\delta\sqrt{\frac{\pi^2/6}{\sigma_\varepsilon^4}}$ is monotonically decreasing in τ, for $\tau \leq \tau^*$ (the infimum is attained at $\tau = \tau^*$). This makes proof that the statistic ζ_r also has (local) power.

■

Proof of Theorem 2.4:

Assume H_{01}, where $d_1 = d$. Fix $\tau \in \Lambda^*$ and consider the case $\tau \leq \tau^*, \zeta_f(\tau) = O_p(1)$, from Theorem 2.1. Now, consider the case where $\tau > \tau^*$.

Then, by Lemma A.3,

$$\begin{aligned}
\frac{1}{\sqrt{T}}\zeta_f(\tau) &= \frac{\frac{1}{\sqrt{T}\sqrt{\lfloor (\tau-\tau^*)T \rfloor}} \sum_{t=2}^{\lfloor \tau^* T \rfloor} x_t x^*_{t-1} + \frac{\sqrt{\lfloor (\tau-\tau^*)T \rfloor}}{\sqrt{T}} \frac{1}{\lfloor (\tau-\tau^*)T \rfloor} \sum_{t=\lfloor \tau^* T \rfloor+1}^{\lfloor \tau T \rfloor} x_t x^*_{t-1}}{\widehat{\sigma}_e(\tau) \sqrt{\frac{\lfloor \tau^* T \rfloor}{\lfloor (\tau-\tau^*)T \rfloor} \frac{1}{\lfloor \tau^* T \rfloor} \sum_{t=2}^{\lfloor \tau^* T \rfloor} x^{*2}_{t-1} + \frac{1}{\lfloor (\tau-\tau^*)T \rfloor} \sum_{t=\lfloor \tau^* T \rfloor+1}^{\lfloor \tau T \rfloor} x^{*2}_{t-1}}} \\
&\xrightarrow{p} \frac{\sqrt{(\tau-\tau^*)}\gamma_*(d-1)}{\sigma_\varepsilon \sqrt{\frac{\tau^*}{(\tau-\tau^*)}\sigma_\varepsilon^2 \frac{\pi^2}{6} + \sigma^2_{x^*}(d-1)}}.
\end{aligned}$$

By the CMT,

$$\frac{1}{\sqrt{T}}\zeta_f = \inf_{\tau\in\Lambda^*,}\frac{1}{\sqrt{T}}\zeta_f(\tau) \xrightarrow{d} \inf_{\tau\in\Lambda^*,}\begin{cases} N(0,1) & if \quad \tau\leq\tau^* \\ \frac{\sqrt{(\tau-\tau^*)}\gamma_*(d-1)}{\sigma_\varepsilon\sqrt{\frac{\tau^*}{(\tau-\tau^*)}\sigma_\varepsilon^2\frac{\pi^2}{6}+\sigma_{x^*}^2(d-1)}} & if \quad \tau>\tau^* \end{cases}.$$

Then, the *infimum* is reached at $\tau = 1$ and

$$\frac{1}{\sqrt{T}}\zeta_f \xrightarrow{p} \frac{\sqrt{(1-\tau^*)}\gamma_*(d-1)}{\sigma_\varepsilon\sqrt{\frac{\tau^*}{(1-\tau^*)}\sigma_\varepsilon^2\frac{\pi^2}{6}+\sigma_{x^*}^2(d-1)}} < 0 \text{ (see Lemma A.3)},$$

which, essentially, depends on τ^* (power decreases with τ^*) and d (power decreases with d - departure from unity). Hence, ζ_f is $\sqrt{T}$ consistent against $H_{01} : \zeta_f \to -\infty$, as $T \to \infty$, under H_{01} (global).

Regarding $\zeta_r(\tau)$, for $\tau > \tau^*$, by Lemma A.3 $\zeta_r(\tau) = O_p(1)$ and when $\tau \leq \tau^*$,

$$\begin{aligned}\frac{1}{\sqrt{T}}\zeta_r(\tau) &= \frac{\mathcal{A}_r^*(\tau,\tau^*)}{\mathcal{B}_r^*(\tau,\tau^*)} \\ &\xrightarrow{p} \frac{\sqrt{(1-\tau^*)}\gamma_*(d-1)}{\sigma_\varepsilon\sqrt{\sigma_{x^*}^2(d-1)+\left(\frac{\tau^*-\tau}{1-\tau^*}\right)\sigma_\varepsilon^2\frac{\pi^2}{6}}},\end{aligned}$$

where

$$\begin{aligned}\mathcal{A}_r^*(\tau,\tau^*) &= \frac{\sqrt{\lfloor(1-\tau^*)T\rfloor}}{\sqrt{T}}\frac{1}{\lfloor(1-\tau^*)T\rfloor}\sum_{t=2}^{\lfloor(1-\tau^*)T\rfloor} x_{T-t+1}x_{T-t+2}^* \\ &+\frac{1}{\sqrt{T}}\frac{\sqrt{\lfloor(\tau^*-\tau)T\rfloor}}{\sqrt{\lfloor(1-\tau^*)T\rfloor}}\frac{1}{\sqrt{\lfloor(\tau^*-\tau)T\rfloor}}\sum_{t=\lfloor(1-\tau^*)T\rfloor+1}^{\lfloor(1-\tau)T\rfloor} x_{T-t+1}x_{T-t+2}^*\end{aligned}$$

and

$$\mathcal{B}_r^*(\tau,\tau^*) = \widehat{\sigma}_{\widetilde{\varepsilon}}(\tau)\sqrt{\frac{1}{\lfloor(1-\tau^*)T\rfloor}\sum_{t=2}^{\lfloor(1-\tau^*)T\rfloor} x_{T-t+2}^{*2} + \frac{\lfloor(\tau^*-\tau)T\rfloor}{\lfloor(1-\tau^*)T\rfloor}\frac{1}{\lfloor(\tau^*-\tau)T\rfloor}\sum_{t=\lfloor(1-\tau^*)T\rfloor+1}^{\lfloor(1-\tau)T\rfloor} x_{T-t+2}^{*2}}.$$

Therefore,

$$\frac{1}{\sqrt{T}}\zeta_r = \inf_{\tau\in\Lambda^*,}\frac{1}{\sqrt{T}}\zeta_r(\tau) \xrightarrow{p} \inf_{\tau\in\Lambda^*,}\begin{cases} \sqrt{(1-\tau^*)}\frac{\gamma_*(d-1)}{\sigma_\varepsilon\sqrt{\sigma_{x^*}^2(d-1)+\left(\frac{\tau^*-\tau}{1-\tau^*}\right)\sigma_\varepsilon^2\frac{\pi^2}{6}}} & if \quad \tau\leq\tau^* \\ \sqrt{(1-\tau)}\frac{\gamma_*(d-1)}{\sigma_\varepsilon\sqrt{\sigma_{x^*}^2(d-1)}} & if \quad \tau>\tau^* \end{cases}.$$

Clearly, the infimum is attained at $\tau = \tau^*$ and

$$\frac{1}{\sqrt{T}}\zeta_r \xrightarrow{p} \sqrt{(1-\tau^*)}\frac{\gamma_*(d-1)}{\sigma_\varepsilon\sigma_{x^*}(d-1)} < 0 \text{ (see Lemma A.3)},$$

which, essentially, also depends on τ^* (power decreases with τ^*) and d (power decreases with d). Hence, ζ_r is also $\sqrt{T}$ consistent against $H_{01} : \zeta_r \to -\infty$, as $T \to \infty$, under H_{01} (global).

■

Proof of Corolary 2.2

Let $\widehat{d}_T \equiv \widehat{d}$ be any $\sqrt{T}-$ consistent estimator of d for a $FI(d)$ model: $\sqrt{T}\left(\widehat{d}-d\right) = O_p(1)$ (Geweke and Porter-Hudak (1983), among several others) and define the estimated process $\widehat{x}_t = (1-L)^{\widehat{d}}\, y_t$ to be used in the testing regression, where $\widehat{d}$ follows from using the entire sample, $t = 1, ..., T$. Then in this case the null limit distributions are as previously presented in Theorems 2.1 and 2.2, under assumptions A.1 or A.2 for ε_t, if, for any fixed τ, $\frac{1}{\lfloor \tau T \rfloor}\sum_{t=2}^{\lfloor \tau T \rfloor} \widehat{\varepsilon}_{t-1}^{*2} = \frac{1}{\lfloor \tau T \rfloor}\sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_{t-1}^{*2} + o_p(1)$ and $\frac{1}{\sqrt{\lfloor \tau T \rfloor}}\sum_{t=2}^{\lfloor \tau T \rfloor} \widehat{\varepsilon}_t\widehat{\varepsilon}_{t-1}^{*} = \frac{1}{\sqrt{\lfloor \tau T \rfloor}}\sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_t\varepsilon_{t-1}^{*} + o_p(1)$. A sketch of the proof goes as follows (a more detailed argument can be made along the lines of the proof of Proposition 3 in Breitung and Hassler, 2006, p.1108). Under the null hypothesis, $\widehat{x}_t = (1-L)^{\widehat{d}}\, y_t \equiv \widehat{\varepsilon}_t$, for all t, where $\widehat{d} \xrightarrow{p} d_0$, as $T \to \infty$. For a particular consistent estimator of d, let $\widehat{d} = d_0 + \xi_T$, where $\xi_T = o_p(1)$, as $T \to \infty$, for some process ξ_T. Then, under the null hypothesis it follows that,

$$
\begin{aligned}
\widehat{x}_t &\equiv \widehat{\varepsilon}_t = \sum_{j=0}^{\infty} \pi_j\left(\widehat{d}\right) L^j y_t = \sum_{j=0}^{\infty} (-1)^j \frac{\widehat{d}\left(\widehat{d}-1\right)\left(\widehat{d}-2\right)...\left(\widehat{d}-j+1\right)}{j!} L^j y_t \\
&= \sum_{j=0}^{\infty} (-1)^j \frac{(d_0+\xi_T)(d_0+\xi_T-1)(d_0+\xi_T-2)...(d_0+\xi_T-j+1)}{j!} L^j y_t \\
&= \sum_{j=0}^{\infty} \left[(-1)^j \frac{d_0(d_0-1)(d_0-2)...(d_0-j+1)}{j!} + \eta_T\right] L^j y_t \\
&= \left(\sum_{j=0}^{\infty} \left[(-1)^j \frac{d_0(d_0-1)(d_0-2)...(d_0-j+1)}{j!}\right] L^j + \eta_T\right) y_t \\
&= \left((1-L)^{d_0} + \eta_T\right) y_t = \varepsilon_t + y_t\eta_T,
\end{aligned}
$$

for some $\eta_T = o_p(1)$, and $\widehat{\varepsilon}_{t-1}^{*} = \sum_{j=1}^{t-1} \frac{\widehat{\varepsilon}_{t-j}}{j} = \varepsilon_{t-1}^{*} + \eta_T \sum_{j=1}^{t-1} \frac{y_{t-j}}{j} = \varepsilon_{t-1}^{*} + \eta_T y_{t-1}^{*}$. For any fixed τ,

$$
\begin{aligned}
\frac{1}{\lfloor \tau T \rfloor}\sum_{t=2}^{\lfloor \tau T \rfloor} \widehat{\varepsilon}_{t-1}^{*2} &= \frac{1}{\lfloor \tau T \rfloor}\sum_{t=2}^{\lfloor \tau T \rfloor} \left(\varepsilon_{t-1}^{*} + \eta_T y_{t-1}^{*}\right)^2 \\
&= \frac{1}{\lfloor \tau T \rfloor}\sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_{t-1}^{*2} + \eta_T^2 \frac{1}{\lfloor \tau T \rfloor}\sum_{t=2}^{\lfloor \tau T \rfloor} y_{t-1}^{*2} + 2\eta_T \frac{1}{\lfloor \tau T \rfloor}\sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_{t-1}^{*} y_{t-1}^{*}.
\end{aligned}
$$

Due to stationarity and ergodicity of y_t, $\frac{1}{\lfloor \tau T \rfloor}\sum_{t=2}^{\lfloor \tau T \rfloor} y_{t-1}^{*2} \xrightarrow{p} Var\left(y_{t-1}^{**}\right) = \sigma_{y^{**}}^2 < \infty$; see DKH (p.208) and Hassler and Breitung (2006, p.1106). By the same token,

$$\begin{aligned}
\frac{1}{\lfloor \tau T \rfloor} \sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_{t-1}^{*} y_{t-1}^{*} &= \frac{1}{\lfloor \tau T \rfloor} \sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_{t-1}^{*} \sum_{j=1}^{t-1} \frac{y_{t-j}}{j} \\
&= \frac{1}{\lfloor \tau T \rfloor} \sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_{t-1}^{*} \left(\sum_{k=0}^{\infty} \pi_k \left(-d_0\right) \sum_{j=1}^{t-1} \frac{\varepsilon_{t-j-k}}{j} \right) \\
&= \frac{1}{\lfloor \tau T \rfloor} \sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_{t-1}^{*} \left(\sum_{k=0}^{\infty} \pi_k \left(-d_0\right) \varepsilon_{t-1-k}^{*} \right) \\
&= \sum_{k=0}^{\infty} \pi_k \left(-d_0\right) \left(\frac{1}{\lfloor \tau T \rfloor} \sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_{t-1}^{*} \varepsilon_{t-1-k}^{*} \right) \xrightarrow{p} \sum_{k=0}^{\infty} \pi_k \left(-d_0\right) \gamma_{\varepsilon} = \gamma_{\varepsilon,\infty}.
\end{aligned}$$

Hence, $\frac{1}{\lfloor \tau T \rfloor} \sum_{t=2}^{\lfloor \tau T \rfloor} \widehat{\varepsilon}_{t-1}^{*2} = \frac{1}{\lfloor \tau T \rfloor} \sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_{t-1}^{*2} + o_p\left(1\right)$, as $T \to \infty$.

With respect to the second result,

$$\begin{aligned}
\frac{1}{\sqrt{\lfloor \tau T \rfloor}} \sum_{t=2}^{\lfloor \tau T \rfloor} \widehat{\varepsilon}_t \widehat{\varepsilon}_{t-1}^{*} &= \frac{1}{\sqrt{\lfloor \tau T \rfloor}} \sum_{t=2}^{\lfloor \tau T \rfloor} \left(\left(\varepsilon_t + y_t \eta_T\right) \left(\varepsilon_{t-1}^{*} + \eta_T y_{t-1}^{*}\right) \right) \\
= \frac{1}{\sqrt{\lfloor \tau T \rfloor}} \sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_t \varepsilon_{t-1}^{*} &+ \eta_T \frac{1}{\sqrt{\lfloor \tau T \rfloor}} \sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_t y_{t-1}^{*} \\
&+ \eta_T \frac{1}{\sqrt{\lfloor \tau T \rfloor}} \sum_{t=2}^{\lfloor \tau T \rfloor} y_t \varepsilon_{t-1}^{*} + \eta_T^2 \frac{1}{\sqrt{\lfloor \tau T \rfloor}} \sum_{t=2}^{\lfloor \tau T \rfloor} y_t y_{t-1}^{*}.
\end{aligned}$$

Let us first consider $\frac{1}{\sqrt{\lfloor \tau T \rfloor}} \sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_t y_{t-1}^{*}$. In this case,

$$\begin{aligned}
\frac{1}{\sqrt{\lfloor \tau T \rfloor}} \sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_t y_{t-1}^{*} &= \frac{1}{\sqrt{\lfloor \tau T \rfloor}} \sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_t \left(\sum_{k=0}^{\infty} \pi_k \left(-d_0\right) \varepsilon_{t-1-k}^{*} \right) \\
&= \sum_{k=0}^{\infty} \pi_k \left(-d_0\right) \left(\frac{1}{\sqrt{\lfloor \tau T \rfloor}} \sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_t \varepsilon_{t-1-k}^{*} \right) = O_p\left(1\right);
\end{aligned}$$

see also Hassler and Breitung (2006, p.1105). Similarly,

$$\begin{aligned}
\frac{1}{\sqrt{\lfloor \tau T \rfloor}} \sum_{t=2}^{\lfloor \tau T \rfloor} y_t \varepsilon_{t-1}^{*} &= \frac{1}{\sqrt{\lfloor \tau T \rfloor}} \sum_{t=2}^{\lfloor \tau T \rfloor} \left(\sum_{k=0}^{\infty} \pi_k \left(-d_0\right) \varepsilon_{t-k} \right) \varepsilon_{t-1}^{*} \\
&= \sum_{k=0}^{\infty} \pi_k \left(-d_0\right) \left(\frac{1}{\sqrt{\lfloor \tau T \rfloor}} \sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_{t-k} \varepsilon_{t-1}^{*} \right) = O_p\left(1\right).
\end{aligned}$$

Finally,

$$\begin{aligned}
\frac{1}{\sqrt{\lfloor \tau T \rfloor}} \sum_{t=2}^{\lfloor \tau T \rfloor} y_t y_{t-1}^{*} &= \frac{1}{\sqrt{\lfloor \tau T \rfloor}} \sum_{t=2}^{\lfloor \tau T \rfloor} \left(\sum_{k=0}^{\infty} \pi_k \left(-d_0\right) \varepsilon_{t-k} \right) \left(\sum_{i=0}^{\infty} \pi_i \left(-d_0\right) \varepsilon_{t-1-i}^{*} \right) \\
&= \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \pi_k \left(-d_0\right) \pi_i \left(-d_0\right) \left(\frac{1}{\sqrt{\lfloor \tau T \rfloor}} \sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_{t-k} \varepsilon_{t-1-i}^{*} \right) = O_p\left(1\right).
\end{aligned}$$

Therefore, $\frac{1}{\sqrt{\lfloor \tau T \rfloor}} \sum_{t=2}^{\lfloor \tau T \rfloor} \widehat{\varepsilon}_t \widehat{\varepsilon}_{t-1}^* = \frac{1}{\sqrt{\lfloor \tau T \rfloor}} \sum_{t=2}^{\lfloor \tau T \rfloor} \varepsilon_t \varepsilon_{t-1}^* + o_p(1)$.

Table 3.1: Finite sample critical values

T		0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
100	ζ_f	-2.709	-2.481	-2.289	-2.023	0.251	0.621	0.949	1.313
	ζ_f^2	0.288	0.402	0.525	0.693	5.081	6.155	7.140	8.396
	ζ_r	-2.821	-2.499	-2.302	-2.043	0.320	0.639	0.976	1.322
	ζ_r^2	0.284	0.395	0.515	0.691	5.055	6.150	7.224	8.733
	$\min(\zeta_f, \zeta_r)$	-2.922	-2.684	-2.480	-2.276	-0.446	-0.204	0.067	0.364
	$\max(\zeta_f^2, \zeta_r^2)$	0.615	0.787	0.977	1.266	6.056	7.087	8.200	9.431
250	ζ_f	-2.956	-2.601	-2.336	-2.029	0.265	0.611	0.892	1.252
	ζ_f^2	0.344	0.437	0.547	0.731	5.242	6.571	8.073	9.440
	ζ_r	-2.878	-2.567	-2.344	-2.055	0.288	0.639	0.934	1.303
	ζ_r^2	0.311	0.420	0.551	0.738	5.393	6.570	7.814	9.394
	$\min(\zeta_f, \zeta_r)$	-3.074	-2.841	-2.562	-2.319	-0.425	-0.153	0.096	0.345
	$\max(\zeta_f^2, \zeta_r^2)$	0.606	0.793	1.022	1.312	6.409	7.814	9.035	10.439
500	ζ_f	-2.994	-2.682	-2.416	-2.068	0.296	0.628	0.957	1.316
	ζ_f^2	0.358	0.465	0.602	0.791	5.450	6.875	8.276	10.075
	ζ_r	-3.014	-2.680	-2.407	-2.063	0.325	0.707	0.990	1.314
	ζ_r^2	0.360	0.453	0.573	0.759	5.521	6.943	8.360	10.042
	$\min(\zeta_f, \zeta_r)$	-3.268	-2.920	-2.655	-2.369	-0.386	-0.117	0.112	0.448
	$\max(\zeta_f^2, \zeta_r^2)$	0.647	0.825	1.006	1.298	6.781	8.250	9.560	11.536
1000	ζ_f	-3.000	-2.712	-2.388	-2.045	0.305	0.617	0.885	1.256
	ζ_f^2	0.372	0.479	0.595	0.780	5.508	6.963	8.374	10.177
	ζ_r	-2.982	-2.678	-2.390	-2.076	0.321	0.638	0.905	1.286
	ζ_r^2	0.368	0.467	0.586	0.778	5.610	7.006	8.305	10.007
	$\min(\zeta_f, \zeta_r)$	-3.223	-2.927	-2.671	-2.368	-0.394	-0.115	0.149	0.447
	$\max(\zeta_f^2, \zeta_r^2)$	0.637	0.805	0.999	1.319	6.825	8.254	9.466	11.607

Table 3.2: Empirical rejection frequencies

DGP: $(1-L)^{d_1} y_t = \varepsilon_t, \quad t = 1, ..., [\tau T]$; $(1-L)^{d_2} y_t = \varepsilon_t, \quad t = [\tau T]+1, ..., T$, $\tau = 0.5$, $d_1 = 0$, $d_2 \in \{0, 0.1, ..., 0.9\}$ and $\varepsilon_t \sim nid(0,1)$.

T=100							T=250						
d_2	ζ_f	ζ_f^2	ζ_r	ζ_r^2	$\min(\zeta_f,\zeta_r)$	$\max(\zeta_f^2,\zeta_r^2)$	d_2	ζ_f	ζ_f^2	ζ_r	ζ_r^2	$\min(\zeta_f,\zeta_r)$	$\max(\zeta_f^2,\zeta_r^2)$
0.0	0.0520	0.0522	0.0466	0.0502	0.0518	0.0518	0.0	0.0514	0.0482	0.0530	0.0524	0.0534	0.0526
0.1	0.0572	0.0552	0.1258	0.0856	0.1068	0.0780	0.1	0.0744	0.0640	0.2782	0.1922	0.2196	0.1432
0.2	0.0742	0.0616	0.2978	0.2048	0.2306	0.1476	0.2	0.1282	0.0972	0.706	0.5902	0.6068	0.4706
0.3	0.0948	0.0706	0.5012	0.3900	0.4090	0.2972	0.3	0.2016	0.1518	0.9492	0.9004	0.9032	0.8254
0.4	0.1234	0.0936	0.7210	0.6010	0.6134	0.4908	0.4	0.2954	0.2362	0.9934	0.9872	0.9878	0.9732
0.5	0.1640	0.1284	0.8468	0.7538	0.7606	0.6446	0.5	0.4064	0.3248	0.9998	0.9996	0.9996	0.9976
0.6	0.2082	0.1626	0.9320	0.8790	0.8842	0.7968	0.6	0.5082	0.4356	1	1	1	0.9998
0.7	0.2362	0.1810	0.9656	0.9266	0.9294	0.8686	0.7	0.6040	0.5292	1	1	1	1
0.8	0.2892	0.2302	0.9842	0.9644	0.9662	0.9238	0.8	0.7010	0.6302	1	1	1	1
0.9	0.3456	0.2732	0.9916	0.9794	0.9804	0.9544	0.9	0.7610	0.6988	1	1	1	1

Table 3.3: Empirical rejection frequencies - AR short-run dynamics

DGP: $(1-L)^{d_1} y_t = \varepsilon_t, \quad t = 1, ..., [\tau T]$; $(1-L)^{d_2} y_t = \varepsilon_t, \quad t = [\tau T]+1, ..., T$, $\tau = 0.5$, $d_1 = 0$, $d_2 \in \{0, 0.1, ..., 0.9\}$, $(1-0.5L)\varepsilon_t = u_t$ and $u_t \sim nid(0,1)$.

T=100							T=250						
d_2	ζ_f	ζ_f^2	ζ_r	ζ_r^2	$\min(\zeta_f,\zeta_r)$	$\max(\zeta_f^2,\zeta_r^2)$	d_2	ζ_f	ζ_f^2	ζ_r	ζ_r^2	$\min(\zeta_f,\zeta_r)$	$\max(\zeta_f^2,\zeta_r^2)$
0.0	0.0276	0.0208	0.0304	0.0208	0.0258	0.0198	0.0	0.0286	0.0230	0.0316	0.0240	0.0306	0.0200
0.1	0.0280	0.0236	0.0444	0.0328	0.0384	0.0274	0.1	0.0310	0.0222	0.0656	0.0408	0.0492	0.0298
0.2	0.0330	0.0228	0.0582	0.0440	0.0494	0.0340	0.2	0.0356	0.0216	0.1138	0.0720	0.0786	0.0436
0.3	0.0306	0.0246	0.0826	0.0534	0.0594	0.0346	0.3	0.0372	0.0228	0.1864	0.1234	0.1274	0.0778
0.4	0.0276	0.0214	0.1134	0.0764	0.0790	0.0494	0.4	0.0350	0.0252	0.2894	0.2078	0.2030	0.1306
0.5	0.0366	0.0274	0.1526	0.1048	0.1058	0.0716	0.5	0.0386	0.0244	0.4018	0.2926	0.2818	0.1862
0.6	0.0392	0.0284	0.1906	0.1358	0.1366	0.0934	0.6	0.0400	0.0278	0.5068	0.4178	0.3870	0.2802
0.7	0.0346	0.0248	0.2132	0.1608	0.1506	0.1000	0.7	0.0442	0.0304	0.6064	0.5176	0.4804	0.3758
0.8	0.0386	0.0254	0.2492	0.1984	0.1838	0.1300	0.8	0.0408	0.0338	0.6866	0.6062	0.5562	0.4614
0.9	0.0416	0.0282	0.2754	0.2294	0.2090	0.1564	0.9	0.0472	0.0318	0.7654	0.7044	0.6472	0.5620

Note: The number of lags used in the augmented test regression is selected as $p = int[4(T/100)^{1/4}]$; see Schwertz (1989).

Table 3.4: Empirical rejection frequencies - MA short-run dynamics

a) DGP: $(1-L)^{d_1} y_t = \varepsilon_t$, $t = 1, ..., [\tau T]$; $(1-L)^{d_2} y_t = \varepsilon_t$, $t = [\tau T]+1, ..., T$, $\tau = 0.5$, $d_1 = 0$, $d_2 \in \{0, 0.1, ..., 0.9\}$, $\varepsilon_t = (1-0.5L)u_t$ and $u_t \sim nid(0,1)$.

T=100							T=250						
d_2	ζ_f	ζ_f^2	ζ_r	ζ_r^2	$\min(\zeta_f,\zeta_r)$	$\max(\zeta_f^2,\zeta_r^2)$	d_2	ζ_f	ζ_f^2	ζ_r	ζ_r^2	$\min(\zeta_f,\zeta_r)$	$\max(\zeta_f^2,\zeta_r^2)$
0.0	0.0352	0.0250	0.0342	0.0264	0.0384	0.0236	0.0	0.0396	0.0278	0.0410	0.0280	0.0426	0.0250
0.1	0.0354	0.0226	0.0414	0.0252	0.0396	0.0240	0.1	0.0424	0.0246	0.0674	0.0350	0.0522	0.0242
0.2	0.0352	0.0228	0.0490	0.0288	0.0440	0.0232	0.2	0.0442	0.0250	0.1072	0.0568	0.0738	0.0346
0.3	0.0322	0.0190	0.0646	0.0354	0.0472	0.0244	0.3	0.0416	0.0264	0.1722	0.0998	0.1126	0.0572
0.4	0.0290	0.0202	0.0730	0.0422	0.0536	0.0306	0.4	0.0430	0.0242	0.2564	0.1582	0.1652	0.0966
0.5	0.0368	0.0244	0.0978	0.0566	0.0730	0.0390	0.5	0.0466	0.0276	0.3270	0.2092	0.2192	0.1316
0.6	0.0382	0.0224	0.1042	0.0630	0.0750	0.0466	0.6	0.0466	0.0282	0.3808	0.2586	0.2620	0.1576
0.7	0.0370	0.0270	0.1146	0.0746	0.0854	0.0510	0.7	0.0474	0.0306	0.4318	0.3026	0.3078	0.1866
0.8	0.0366	0.0222	0.1152	0.0694	0.0768	0.0464	0.8	0.0466	0.0274	0.4468	0.3268	0.3274	0.2020
0.9	0.0334	0.0196	0.1086	0.0674	0.0756	0.0438	0.9	0.0516	0.0352	0.4688	0.3414	0.3386	0.2266

b) DGP: $(1-L)^{d_1} y_t = \varepsilon_t,\ t = 1, ..., [\tau T]$; $(1-L)^{d_2} y_t = \varepsilon_t,\ t = [\tau T]+1, ..., T$, $\tau = 0.5$, $d_1 = 0$, $d_2 \in \{0, 0.1, ..., 0.9\}$, $\varepsilon_t = (1+0.5L)u_t$ and $u_t \sim nid(0,1)$.

T=100							T=250						
d_2	ζ_f	ζ_f^2	ζ_r	ζ_r^2	$\min(\zeta_f,\zeta_r)$	$\max(\zeta_f^2,\zeta_r^2)$	d_2	ζ_f	ζ_f^2	ζ_r	ζ_r^2	$\min(\zeta_f,\zeta_r)$	$\max(\zeta_f^2,\zeta_r^2)$
0.0	0.0284	0.0248	0.0276	0.0232	0.0270	0.0222	0.0	0.0372	0.0276	0.0388	0.0280	0.0382	0.0256
0.1	0.0278	0.0220	0.0416	0.0320	0.0350	0.0230	0.1	0.0374	0.0272	0.0732	0.0442	0.0580	0.0340
0.2	0.0284	0.0242	0.0524	0.0358	0.0416	0.0266	0.2	0.0420	0.0276	0.1260	0.0786	0.0914	0.0482
0.3	0.0300	0.0236	0.0788	0.0502	0.0528	0.0340	0.3	0.0444	0.0268	0.2116	0.1414	0.1486	0.0852
0.4	0.0286	0.0192	0.0998	0.0678	0.0706	0.0434	0.4	0.0454	0.0286	0.3206	0.2294	0.2288	0.1470
0.5	0.0332	0.0276	0.1398	0.0994	0.1018	0.0638	0.5	0.0448	0.0288	0.4382	0.3226	0.3184	0.2090
0.6	0.0364	0.0284	0.1794	0.1282	0.1252	0.0918	0.6	0.0470	0.0328	0.5496	0.4406	0.4198	0.3108
0.7	0.0302	0.0204	0.2050	0.1534	0.1412	0.0984	0.7	0.0552	0.0374	0.6478	0.5454	0.5210	0.4038
0.8	0.0374	0.0284	0.2368	0.1838	0.1712	0.1244	0.8	0.0494	0.0368	0.7268	0.6358	0.6006	0.4972
0.9	0.0416	0.0292	0.2664	0.2158	0.1918	0.1454	0.9	0.0546	0.0368	0.8038	0.7394	0.6944	0.5920

Note: The number of lags used in the augmented test regression is selected as $p = int[4(T/100)^{1/4}]$; see Schwertz (1989).

Table 3.5: Empirical rejection frequencies (d is estimated using spectral MLE)

a) DGP: $(1-L)^{d_1} y_t = \varepsilon_t,\ t = 1, ..., [\tau T]$; $(1-L)^{d_2} y_t = \varepsilon_t,\ t = [\tau T]+1, ..., T$, $\tau = 0.5$, $d_1 = 0$, $d_2 \in \{0, 0.1, ..., 0.9\}$, $\varepsilon_t \sim nid(0,1)$.

T=100								T=250							
$\widehat{d}$	d_2	ζ_f	ζ_f^2	ζ_r	ζ_r^2	$\min(\zeta_f,\zeta_r)$	$\max(\zeta_f^2,\zeta_r^2)$	$\widehat{d}$	d_2	ζ_f	ζ_f^2	ζ_r	ζ_r^2	$\min(\zeta_f,\zeta_r)$	$\max(\zeta_f^2,\zeta_r^2)$
-0.0525	0.0	0.0198	0.0576	0.0188	0.0672	0.0196	0.0678	-0.0213	0.0	0.0158	0.0304	0.0138	0.0318	0.0144	0.0282
0.0071	0.1	0.0440	0.0548	0.0090	0.0820	0.0312	0.0724	0.0338	0.1	0.0616	0.0400	0.0056	0.0660	0.0360	0.0532
0.0774	0.2	0.0998	0.0692	0.0050	0.0870	0.0616	0.0862	0.0987	0.2	0.2312	0.1508	0.0024	0.1406	0.1510	0.1482
0.1542	0.3	0.2024	0.1418	0.0040	0.1050	0.1384	0.1446	0.1703	0.3	0.5362	0.4062	0.0010	0.2982	0.4070	0.3644
0.2475	0.4	0.3820	0.2804	0.0034	0.1182	0.2786	0.2528	0.2509	0.4	0.8190	0.7244	0.0018	0.4880	0.7254	0.6684
0.3443	0.5	0.5696	0.4602	0.0080	0.1258	0.4618	0.4054	0.3429	0.5	0.9630	0.9252	0.0002	0.6426	0.9256	0.8936
0.4553	0.6	0.7424	0.6462	0.0094	0.1312	0.6468	0.5696	0.4402	0.6	0.9962	0.9880	0.0008	0.7054	0.9882	0.9788
0.5696	0.7	0.8612	0.7794	0.0140	0.1264	0.7796	0.7136	0.5421	0.7	0.9994	0.9982	0.0030	0.6782	0.9982	0.9960
0.6872	0.8	0.9412	0.8888	0.0218	0.1232	0.8894	0.8332	0.6486	0.8	0.9998	0.9998	0.0060	0.5998	0.9998	0.9996
0.7923	0.9	0.9678	0.9402	0.0236	0.1138	0.9406	0.8964	0.7539	0.9	1.0000	1.0000	0.0042	0.5260	1.0000	1.0000

b) DGP: $(1-L)^{d_1} y_t = \varepsilon_t,\ t = 1, ..., [\tau T]$; $(1-L)^{d_2} y_t = \varepsilon_t,\ t = [\tau T]+1, ..., T$, $\tau = 0.5$, $d_1 = 0.3$, $d_2 \in \{0, 0.1, ..., 0.9\}$, $\varepsilon_t \sim nid(0,1)$.

T=100								T=250							
$\widehat{d}$	d_2	ζ_f	ζ_f^2	ζ_r	ζ_r^2	$\min(\zeta_f,\zeta_r)$	$\max(\zeta_f^2,\zeta_r^2)$	$\widehat{d}$	d_2	ζ_f	ζ_f^2	ζ_r	ζ_r^2	$\min(\zeta_f,\zeta_r)$	$\max(\zeta_f^2,\zeta_r^2)$
0.1527	0.0	0.0028	0.1144	0.1834	0.1322	0.1268	0.1378	0.1694	0.0	0.0008	0.3080	0.4910	0.3724	0.3736	0.3518
0.1875	0.1	0.0056	0.0748	0.1010	0.0682	0.0634	0.0788	0.2019	0.1	0.0022	0.1380	0.2202	0.1422	0.1434	0.1358
0.2318	0.2	0.0090	0.0382	0.0508	0.0372	0.0348	0.0382	0.2418	0.2	0.0062	0.0498	0.0680	0.0408	0.0418	0.0426
0.2834	0.3	0.0256	0.0318	0.0258	0.0250	0.0266	0.0280	0.2900	0.3	0.0156	0.0252	0.0168	0.0166	0.0158	0.0184
0.3516	0.4	0.0646	0.0456	0.0162	0.0270	0.0434	0.0364	0.3493	0.4	0.0844	0.0550	0.0086	0.0308	0.0556	0.0378
0.4281	0.5	0.1664	0.1110	0.0132	0.0290	0.1142	0.0810	0.4230	0.5	0.3108	0.2118	0.0022	0.0634	0.2130	0.1506
0.5221	0.6	0.3240	0.2372	0.0144	0.0324	0.2418	0.1764	0.5066	0.6	0.6586	0.5486	0.0038	0.1284	0.5496	0.4400
0.6255	0.7	0.5104	0.4076	0.0198	0.0446	0.4126	0.3218	0.6004	0.7	0.8964	0.8304	0.0086	0.1830	0.8308	0.7520
0.7315	0.8	0.7064	0.5968	0.0310	0.0552	0.6012	0.4994	0.6994	0.8	0.9816	0.9606	0.0114	0.2474	0.9608	0.9332
0.8279	0.9	0.8260	0.7380	0.0258	0.0578	0.7408	0.6426	0.7971	0.9	0.9974	0.9954	0.0098	0.2748	0.9954	0.9888

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