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The Flexible Fourier Form and Local GLS De-trended Unit Root Tests*

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Abstract

In two recent papers Enders and Lee (2008) and Becker *et al.* (2006) provide Lagrange multiplier and OLS de-trended unit root tests, and stationarity tests, respectively, which incorporate a Fourier approximation element in the deterministic component. Such an approach can prove useful in providing robustness against a variety of breaks in the deterministic trend function of unknown form and number. In this paper, we generalise the unit root testing procedure based on local GLS de-trending proposed by Elliott, Rothenberg and Stock (1996) to allow for a Fourier approximation to the unknown deterministic component in the same way. We show that although the resulting unit root tests possess good finite sample size and power properties, their limit null distributions are undefined.

Keywords: Local GLS de-trending; flexible Fourier approximation; trend breaks; unit root.
JEL classifications: C20, C22

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1 Introduction

The difficulties inherent in testing for a unit root in a time series which is subject to structural breaks in its deterministic trend function are well documented in the econometric time series literature. Since the seminal work of Perron (1989), a large literature has developed around providing unit root test procedures which account for such breaks; see Perron (2006) for a recent review. Initial research considered the presence of at most one break in the data generation process [DGP] while more recent research has focused on the possibility of multiple possible breaks in the level and/or the trend; see, in particular, the local generalised least squares [GLS] de-trended unit root tests of Carrion-i-Silvestre, Kim and Perron (2009). The performance of extant unit root tests depends crucially on the estimated break location(s) and on the assumed maximum number of breaks; see, *inter alia,* Enders and Lee (2008), Becker, Enders and Hurn (2004) and Becker, Enders and Lee (2006).

It has been observed by, among others, Gallant (1981), Davies (1987), Becker, Enders and Hurn (2004) and Harvey, Leybourne and Xiao (2008) that a Fourier approximation can, to any desired degree of accuracy, capture the behaviour of a deterministic trend function of unknown form, even if the function itself is aperiodic. This result has been recently employed by Enders and Lee (2008) who generalise the Schmidt and Phillips (1992) and Schmidt and Lee (1991) Lagrange multiplier [LM] type unit root test (which employ first difference [FD] de-trending) together with the ordinary least squares [OLS] de-trended Dickey-Fuller [DF] (1979) unit root tests through the introduction of a Fourier approximation to the deterministic trend component. They show the resulting tests to be robust against a large variety of possible break mechanisms in the deterministic trend function. Becker, Enders and Lee (2006) provide stationarity tests using the same framework and show that the resulting Kwiatkowski, Phillips, Schmidt and Shin (1992) [KPSS] type tests display good size and power properties in the presence of a variety of structural break designs. An empirically attractive feature of these procedures is that there is no need to assume either that the potential break dates or the number of breaks are known to the practitioner, *a priori.* The simplicity with which this approximation can be implemented is also an important advantage of this approach relative to existing methods which require numerically involved searching procedures and are in any case operationally infeasible if the practitioner wishes to allow for more than two putative breaks; see Carrion-i-Silvestre, Kim and Perron (2009).

Our objective in this paper is to apply the flexible Fourier form to the local GLS unit root testing procedure of Elliott, Rothenberg and Stock [ERS] (1996) and to compare this with the corresponding DF and LM based unit root tests of Enders and Lee (2008). It is known that local GLS de-trending can yield unit root tests which are considerably more powerful than their OLS and FD de-trended counterparts; see, in particular, ERS for the constant and linear trend cases, Perron and Rodríguez (2003) for the case of a single break in level/trend, and Carrion-i-Silvestre, Kim and Perron (2009) for the case of multiple level/trend breaks. In this paper we demonstrate that these power gains carry over, at least in finite samples, when using local GLS de-trended unit roots based around the flexible Fourier form.

The paper is organised as follows. Section 2 introduces the local GLS de-trended unit root tests which incorporate the flexible Fourier form and briefly outlines the corresponding LM and OLS tests of Enders and Lee (2008). In section 3 we provide finite sample critical values for the GLS de-trended tests and compare the finite sample size and power properties of these tests with the corresponding OLS de-trended and LM tests. Large sample properties of the local GLS de-trended tests are discussed in section 4. Section 5 concludes. Proofs are contained in a mathematical appendix. In what follows we use the notation '$x := y$' ('$x =: y$') to indicate that x is defined by y (y is defined by x), and '$\Rightarrow$' to denote weak convergence.

2 The Model and Unit Roots Tests

2.1 The Flexible Fourier Model

Consider data generated according to following DGP:

$$y_t = \alpha_0 + \alpha_1 t + \alpha_2 \sin\left(\frac{2\pi\kappa t}{T}\right) + \alpha_3 \cos\left(\frac{2\pi\kappa t}{T}\right) + x_t, \; t = 1, ..., T \quad (1)$$

$$x_t = \phi x_{t-1} + u_t \quad (2)$$

where it is assumed, for the present, that $u_t \sim iid(0, \sigma^2)$ and that the starting value, x_0, is an $O_p(1)$ random variable. The Fourier frequency, κ, is taken to be a fixed value. Our interest in this paper lies in testing the unit null hypothesis, $H_0 : \phi = 1$, against the stationary alternative, $H_1 : |\phi| < 1$, in (1).

Remark 2.1: The deterministic kernel considered in (1) includes a linear time trend, but we may also consider the case where only a constant and the two Fourier terms are considered; i.e., the case where $\alpha_1 = 0$ in (1). This will be referred to as the constant case in what follows, while the more general case where $\alpha_1 \neq 0$ will be termed the linear trend case.

Remark 2.2: The model in (1)-(2) contains a single Fourier frequency. We focus our attention on this model, based on the observations made in Enders and Lee (2008) and Becker, Enders and Lee (2006) that a single Fourier frequency can mimic a large variety of breaks in the deterministic trend function. Enders and Lee (2008) note that, for any desired level of accuracy, a more general Fourier expansion of the form $f_{t,n}(\kappa) := \alpha_0 + \sum_{i=1}^{\ell} \alpha_i \sin\left(\frac{2\pi\kappa_i t}{T}\right) + \sum_{i=1}^{\ell} \beta_i \cos\left(\frac{2\pi\kappa_i t}{T}\right)$, where $1 \leq \ell < T/2$, and with $\kappa_1 < \kappa_2 < \cdots < \kappa_\ell$, could be considered. However, Enders and Lee (2008) argue against the use of many Fourier frequency components because it can lead to problems of over-fitting.

Remark 2.3: Equation (1) can be re-written as,

$$y_t = z_t'\boldsymbol{\alpha} + f_t(\kappa)'\boldsymbol{\varphi} + x_t,$$

where $z_t := (1, t)'$ and $\boldsymbol{\alpha} := (\alpha_0, \alpha_1)'$ (or, in the constant case $z_t := 1$ and $\boldsymbol{\alpha} := \alpha_0$), $f_t(\kappa) := (\sin\left(\frac{2\pi\kappa t}{T}\right), \cos\left(\frac{2\pi\kappa t}{T}\right))'$, and $\boldsymbol{\varphi} := (\varphi_1, \varphi_2)'$, or in vector notation as

$$\mathbf{y} = \mathbf{Z}\boldsymbol{\alpha} + \mathbf{f}(\kappa)\boldsymbol{\varphi} + \mathbf{x}$$

where $\mathbf{Z} := (z_1', ..., z_T')'$ and $\mathbf{f}(\kappa) := (f_1(\kappa)', ..., f_T(\kappa)')'$ are $T \times 2$ matrices ($\mathbf{Z}$ is a $T \times 1$ vector in the constant case) and $\mathbf{y}$ and $\mathbf{x}$ are $T \times 1$ vectors.

2.2 GLS De-trended Unit Root Tests

In this section, we extend the local GLS de-trending approach of ERS to the problem of testing for a unit root within the context of (1)-(2). This is achieved through a two-step procedure.

In the first step we estimate the OLS regression of

$$\mathbf{y}_{\bar{c}} := \left(y_1, y_2 - (1 + \frac{\bar{c}}{T}) y_1, ..., y_T - (1 + \frac{\bar{c}}{T}) y_{T-1} \right)' \tag{3}$$

onto

$$\mathbf{V}_{\bar{c}} := \left[v_1, v_2 - (1 + \frac{\bar{c}}{T}) v_1, ..., v_T - (1 + \frac{\bar{c}}{T}) v_{T-1} \right]' \tag{4}$$

where $v_t := (z_t', \ f_t(\kappa)')'$, to obtain an estimate of the parameter vector $\boldsymbol{\theta} := (\boldsymbol{\alpha}', \boldsymbol{\varphi}')'$. Denote this estimate by $\widehat{\boldsymbol{\theta}}_{\bar{c}} := (\widehat{\boldsymbol{\alpha}}_{\bar{c}}', \widehat{\boldsymbol{\varphi}}_{\bar{c}}')'$. The value of the local GLS de-trending parameter, $\bar{c}$, depends on the form of the deterministic component in (1); ERS suggest using $\bar{c} = -7$ for the constant case and $\bar{c} = -13.5$ for the linear trend case.

In the second step we run the DF-type unit root test regression on the local GLS de-trended series, $y_t^{\bar{c}} := y_t - z_t'\widehat{\boldsymbol{\alpha}}_{\bar{c}} - f_t(\kappa)'\widehat{\boldsymbol{\varphi}}_{\bar{c}}$, $t = 1, ..., T$; that is, compute the t-statistic for $\phi = 0$ in the regression equation

$$\Delta y_t^{\bar{c}} = \phi y_{t-1}^{\bar{c}} + u_t. \tag{5}$$

We denote the resulting statistic as $t_\phi^{ERS_f^\zeta}$, $\zeta = \mu, \tau$, where μ indicates that the statistic is computed for the constant case, $z_t = 1$, and τ that the statistic is computed for the linear trend case, $z_t = (1, t)'$. In what follows, where generic statements are being made which apply in both the constant and linear trend cases, we will omit the superscript ζ.

Remark 2.4: Enders and Lee (2008) extend the LM type unit root tests of Schmidt and Phillips (1992) and Schmidt and Lee (1991) to this context. In the first step of this testing procedure, the parameters of the deterministic variables (constant, time trend and Fourier terms) are estimated under the null hypothesis, i.e.,

$$\Delta y_t = \Delta z_t \delta_1 + \Delta f_t(\kappa)'\boldsymbol{\delta} + \Delta x_t$$

where $\Delta z_t := 1$, $\boldsymbol{\delta} := (\delta_2, \delta_3)'$ and $\Delta f_t(\kappa) := \left(\Delta \sin\left(\frac{2\pi\kappa t}{T}\right), \Delta \cos\left(\frac{2\pi\kappa t}{T}\right)\right)'$. The estimated coefficients, $\tilde{\delta}_j$, $j = 1, ..., 3$, from this regression are then used to construct the FD de-trended series:

$$y_t^{LM} := y_t - z_t'\widetilde{\boldsymbol{\alpha}} - f_t(\kappa)'\widetilde{\boldsymbol{\delta}}, \qquad t = 2, ..., T$$

where $\widetilde{\boldsymbol{\alpha}} := (\widetilde{\psi}, \widetilde{\delta}_1)$, $\widetilde{\psi} := y_1 - \widetilde{\delta}_1 - \widetilde{\delta}_2 \sin\left(\frac{2\pi\kappa}{T}\right) - \widetilde{\delta}_3 \cos\left(\frac{2\pi\kappa}{T}\right)$, and $\widetilde{\boldsymbol{\delta}} := (\tilde{\delta}_2, \tilde{\delta}_3)'$. The second step then involves estimating the auxiliary regression

$$\Delta y_t = \Delta z_t \vartheta_1 + \Delta f_t(\kappa)'\boldsymbol{\vartheta} + \phi y_{t-1}^{LM} + u_t \tag{6}$$

to obtain the regression t-statistic for $\phi = 0$ in (6), $t_\phi^{LM_f}$ say.

Remark 2.5: Enders and Lee (2008) also consider OLS de-trended DF-type statistics for testing H_0 again H_1 in (1)-(2). In this case, the appropriate DF-type regression is given by

$$\Delta y_t = v_t'\boldsymbol{\omega} + \phi y_{t-1} + v_t \tag{7}$$

where $v_t := (z_t', f_t'(\kappa))'$. The OLS de-trended DF-type statistic is then given by the regression t-statistic for $\phi = 0$ in (7), say $t_\phi^{DF_f^\zeta}$, where the nomenclature $\zeta = \mu, \tau$ has the same meaning as outlined for $t_\phi^{ERS_f^\zeta}$ above. Notice that this procedure is asymptotically equivalent to the two-step procedure where H_0 is tested using the regression t-statistic for $\phi = 0$ in

$$\Delta\breve{y}_t = \phi\breve{y}_{t-1} + \breve{v}_t \quad (8)$$

where $\breve{y}_t := y_t - z_t'\breve{\boldsymbol{\alpha}} - f_t(\kappa)'\breve{\boldsymbol{\varphi}}$, are the OLS de-trended data from regressing y_t onto v_t ($\breve{\boldsymbol{\alpha}}$ and $\breve{\boldsymbol{\varphi}}$ being the resulting OLS estimates of $\boldsymbol{\alpha}$ and $\boldsymbol{\varphi}$ respectively), $t = 1, ..., T$.

Remark 2.6: It is straightforward to show that all of the three unit root statistics discussed above, namely $t_\phi^{ERS_f}$, $t_\phi^{LM_f}$ and $t_\phi^{DF_f}$ are exact invariant with respect to the parameters characterising the deterministic trend function in (1)-(2). The three statistics differ purely in the manner in which this invariance is achieved; i.e., through the de-trending method they employ.

Remark 2.7: We have assumed thus far that u_t in (2) is serially uncorrelated. Short run dynamics in the u_t process can be handled in the usual way by augmenting test regressions (5), (6) and (7), with sufficient lags of the dependent variable to correct for the serial correlation present; see, *inter alia*, ERS, Chang and Park (2002) and Ng and Perron (2001).

3 Finite Sample Simulations

In this section we provide finite sample critical values for the unit root tests outlined in the previous section, together with an investigation of their relative finite sample size and power properties.

3.1 Finite Sample Critical Values

Table 1 below, presents a selection of finite sample critical values for the $t_\phi^{ERS_f}$, $t_\phi^{DF_f}$ and $t_\phi^{LM_f}$ unit root tests from section 2. The critical values provided are valid for the constant (μ) and linear trend (τ) cases. For the ERS type test statistics we followed Elliott et al.'s (1996) suggestion and set the local GLS de-trending parameter to $\bar{c} = -7$ in the constant case and $\bar{c} = -13.5$ in the linear trend case. The reported critical values were computed by Monte Carlo from the random walk process $x_t = x_{t-1} + u_t$, with $u_t \sim NIID(0, 1)$. Without loss of generality, we set $x_0 = 0$, the three tests all being exact similar with respect to x_0. The test regressions used for each procedure were those described in the previous section; i.e., (5), (6) and (7) for $t_\phi^{ERS_f}$, $t_\phi^{LM_f}$ and $t_\phi^{DF_f}$, respectively. Critical values are reported for $\kappa \in (1, 2, 3, 4, 5)$ and $T \in (100, 200, 1000)$. All of the simulations reported in this paper were programmed in Gauss 9.0 using 10000 Monte Carlo replications.

Table 1 about here

Although these critical values are generated assuming a known value of the Fourier frequency parameter, κ, they can also be used as an approximation to the finite sample critical values in cases where the value of κ is unknown but has been estimated. As Becker *et al.*

(2006,p.390) argue[1] "In most instances with highly persistent macroeconomic data, using the value $k = 1$ or $k = 2$ should be sufficient to capture the important breaks in the data. However, there are circumstances where the researcher may want to select some frequency other than $k = 1$ or $k = 2$. Hence ... we consider is to select k for using a completely data-driven method." To that end, Davies (1987) shows that a consistent estimate of κ can be obtained by minimising the residual sum of squares resulting from estimating a sequence of regressions of the form given in (1) over a suitable grid of values of κ.

An interesting feature that can be observed in the results in Table 1 is that as the Fourier frequency parameter κ increases, so the critical values of the unit root tests which include the Fourier regressors appear to converge, other things being equal, towards the critical values for the unit roots tests that omit the Fourier terms (i.e., the unit root tests of DF, ERS and Schmidt and Phillips, 1992). This result can be attributed to the asymptotic orthogonality that exists between the elements of the frequency zero deterministic regressors in z_t and the Fourier terms in $f_t(\kappa)$ in cases where $\kappa = \lambda T$, $0 < \lambda < 0.5$, such that the Fourier terms are located at the harmonic frequency pair $(2\pi\lambda, 2\pi - 2\pi\lambda)$ which is bounded away from zero and therefore have no impact on the distribution of the unit root tests in the limit[2]. This is, of course, a purely finite sample effect because $\frac{\kappa}{T} \to 0$ in (1), as $T \to \infty$.

3.2 Finite Sample Size and Power of the Tests

3.2.1 Conventional Unit Roots Tests

Before looking at the finite sample size and power properties of the $t_\phi^{ERS_f}$, $t_\phi^{LM_f}$ and $t_\phi^{DF_f}$ unit root tests from section 2, we first investigate the implications for the corresponding conventional unit root tests of ERS, Schmidt and Phillips (1992) and DF, computed using a deterministic kernel which includes a constant only or a constant and a time trend, but which do not take account of the Fourier terms in (1). With an obvious notation we denote these tests by t_ϕ^{ERS}, t_ϕ^{LM} and t_ϕ^{DF}, respectively. To that end, we generate data from the DGP

$$y_t = \alpha_1 \sin\left(\frac{2\pi\kappa t}{T}\right) + \alpha_2 \cos\left(\frac{2\pi\kappa t}{T}\right) + x_t \quad (9)$$

$$x_t = \phi x_{t-1} + u_t, \quad u_t \sim NIID(0,1), \quad t = 1, ..., T \quad (10)$$

with $x_0 \sim N(0,1)$, independent of u_t. The autoregressive parameter is defined as $\phi := 1 + \frac{c}{T}$: Table 2 reports results for $c = 0$ which corresponds to the null hypothesis, H_0, while Table 3 reports corresponding results for $c = -15$ which corresponds to the alternative hypothesis, H_1. The other parameters are varied according to $\kappa \in (1, 2, 3, 4, 5)$, $\alpha_1 \in (0, 3)$ and $\alpha_2 \in (0, 5)$. This corresponds to the simulation design used in Enders and Lee (2008). We report results for samples of length $T = 100$ and $T = 200$

Tables 2 – 3 about here

The results in Table 2 demonstrate that under the unit root null hypothesis all of the conventional tests become under-sized, in many cases very severely so, in the presence of

[1] k in the notation of Becker *et al.* (2006) is equivalent to κ in our notation.

[2] This does not, however, imply that the inclusion of the Fourier terms in the test regression is unnecessary in these cases. As will be seen in the next section, where conventional unit root tests are evaluated in this context the omission of these Fourier terms has severe implications for the finite sample properties of the tests.

neglected Fourier terms[3]. In general the under-sizing is marginally worse, other things being equal, for t_ϕ^{ERS} and t_ϕ^{LM} than for t_ϕ^{DF}, with the degree of under-sizing seen in all three tests becoming increasingly severe as α_1 and/or α_2 become larger. Other things being equal, the size distortions are worse the greater is κ and the smaller is the sample size. Notice that in small samples as κ increases so the Fourier terms present in the DGP move further away from the zero frequency and, hence, the impact of these neglected deterministic terms (i.e. the lack of similarity of the test statistics) becomes increasingly pronounced.

Regarding the empirical power of the procedures, we observe that when no Fourier terms are present in the DGP the ERS test presents the best power performance followed by the LM type test. The DF is the test with the lowest power of the three. Where (neglected) Fourier terms are present in the DGP we see from the results in Table 3 that all of the conventional tests show catastrophic losses in power relative to the case where no Fourier terms are present. Indeed in the majority of reported cases all three tests display rejection frequencies below the nominal 5% level.

3.2.2 Tests with κ Known

In order to evaluate the finite sample power properties of the $t_\phi^{ERS_f}$, $t_\phi^{DF_f}$ and $t_\phi^{LM_f}$ unit root tests we generate data from (9)-(10), again with $x_0 \sim N(0,1)$, independent of u_t, and the autoregressive parameter set as $\rho := 1 + \frac{c}{T}$ but now for $c \in (-5, -10, -15, -20)$. Given the exact invariance of $t_\phi^{ERS_f}$, $t_\phi^{DF_f}$ and $t_\phi^{LM_f}$ to α_1 and α_2 when κ is known we may set $\alpha_1 = \alpha_2 = 0$, without loss of generality.

Table 4 about here

Table 4 presents the finite sample power results for the three tests. It is clear from the results in Table 4 that the local GLS de-trended unit root test, $t_\phi^{ERS_f^\tau}$, proposed in this paper enjoys significant power gains over both the OLS and FD de-trended tests, $t_\phi^{DF_f^\tau}$ and $t_\phi^{LM_f}$ respectively. It is clear that the OLS de-trended test consistently displays the lowest power among the three tests while the local GLS de-trended test consistently displays the highest power among the three tests.

3.2.3 Tests with κ Unknown

Following the discussion in section 3.2.1, we now turn to an evaluation of the finite sample size and power properties of the $t_\phi^{ERS_f^\tau}$, $t_\phi^{DF_f^\tau}$ and $t_\phi^{LM_f}$ tests in the case where κ is taken to be unknown and is estimated from the data. Data are again generated from (9)-(10) for $\rho := 1 + \frac{c}{T}$ with $c \in (-5, -10, -15, -20)$. Because the tests which are based on an estimate of κ are no longer exact invariant to parameters of the Fourier terms (they are, however, asymptotically invariant to these parameters) we generated the Fourier terms for $\kappa \in (1, 2, 3, 4, 5)$ with, in each case, $\alpha_1 \in (0, 3)$ and $\alpha_2 \in (0, 5)$.

In order to make the tests operational we must first estimate the true but unknown Fourier frequency parameter, κ. This is done using the approach of Davies (1987). Following Becker

[3]Corresponding experiments with a constant only were also computed but gave qualitatively similar results to those presented for the constant and linear trend case in Table 2 and are therefore omitted. These results can be obtained from the authors on request.

et at. (2006, p.390) we estimate the regression equation

$$y_t = \alpha_0 + \alpha_1 t + \alpha_2 \sin\left(\frac{2\pi k t}{T}\right) + \alpha_3 \cos\left(\frac{2\pi k t}{T}\right) + x_t. \tag{11}$$

for each integer value of k in the interval $1 \leq k \leq k^{max}$. The estimated value, $\widehat{\kappa}$, is then given by the value of k which minimises the residual sum of squares across these estimated regression equations. Following the arguments given in Becker *et al.* (2006, p. 390) we set the maximum frequency at $k^{max} = 5$. The small sample behaviour of this estimator is explored in detail in section 3 of Becker *et al.* (2006) and is shown to perform well in practice. The $t_\phi^{ERS_f^\tau}$, $t_\phi^{DF_f^\tau}$ and $t_\phi^{LM_f}$ tests are then calculated as before taking $\widehat{\kappa}$ as if it were the true value of κ.

Tables 5 – 6 about here

Tables 5 ($T = 100$) and Table 6 ($T = 200$) present the empirical rejection frequencies of the resulting $t_\phi^{ERS_f^\tau}$, $t_\phi^{DF_f^\tau}$ and $t_\phi^{LM_f}$ tests for the unknown κ case. Although some small size distortions are observed, when $\alpha_1 \neq 0$ and/or $\alpha_2 \neq 0$, the results are qualitatively very similar to those reported in Table 3 for the known κ case, suggesting that the estimation procedure for κ works well in practice, at least from the perspective of maintaining the size and power properties of the resulting unit root tests relative to the known κ case.

4 Asymptotic Results

In this section, we show that the local GLS de-trended unit root test statistic, $t_\phi^{ERS_f}$, from (5) is asymptotically infeasible. This is established by showing in Theorem 1 that the first stage local GLS de-trending regression of $\mathbf{y}_{\bar{c}}$ onto $\mathbf{V}_{\bar{c}}$ is undefined in the limit due to the asymptotic singularity of the associated (scaled) Gram matrix, $\frac{1}{T}\mathbf{V}_{\bar{c}}'\mathbf{V}_{\bar{c}}$.

Theorem 1 *Let $\{y_t\}$ be generated according to (1)-(2) under the conditions stated in section 2. Then under $H_0 : \rho = 1$, and as $T \to \infty$*

$$\begin{aligned} \sqrt{T}\left(\widehat{\boldsymbol{\theta}}_{\bar{c}} - \boldsymbol{\theta}\right) &= \left(\frac{1}{T}\mathbf{V}_{\bar{c}}'\mathbf{V}_{\bar{c}}\right)^{-1} \frac{1}{T^{1/2}}\mathbf{V}_{\bar{c}}'\mathbf{x}_{\bar{c}} \\ &\Rightarrow \sigma \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1+\bar{c}+\frac{\bar{c}^2}{3} \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ 0 \\ x_1 \\ \mathfrak{W} \end{bmatrix} \end{aligned} \tag{12}$$

where $\mathbf{x}_{\bar{c}} := \left(x_1, x_2 - (1+\frac{\bar{c}}{T})x_1, ..., x_T - (1+\frac{\bar{c}}{T})x_{T-1}\right)'$, $\mathfrak{W} :=(1-\bar{c})W(1) - 2\bar{c}\int_0^1 W(r)dr - \bar{c}^2\int_0^1 rW(r)dr$, with $W(r)$ a standard Brownian motion, and where $\bar{c}$ is the value of the local GLS de-trending parameter used.

Remark 4.1: Note that (12) corresponds to the limit when $z_t = (1,t)'$ is used in de-trending the data. For the constant only case, $z_t = 1$, it is straightforward to show, using results from the proof of Theorem 1, that for this case the limit reduces to $\sigma \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ 0 \\ x_1 \end{bmatrix}$.

Remark 4.2: It is immediately seen from (12) that $\frac{1}{T}\mathbf{V}_{\bar{c}}'\mathbf{V}_{\bar{c}}$ is asymptotically rank deficient, converging to a (non-random) matrix with rank two (for the constant only case of Remark 4.1 the Gram matrix is asymptotically rank deficient with rank one). Consequently, the statistic $t_{\phi}^{ERS_f^{\zeta}}$, $\zeta = \mu, \tau$, of (5) is asymptotically undefined under the unit root null hypothesis.

Remark 4.3: If, instead of using local GLS de-trending, the parameters of the deterministics are estimated by OLS de-trending from regressing y_t onto v_t, where v_t is as defined in section 2 (as is done in the DF-type test of Enders and Lee, 2008, or with the KPSS test of Becker *et al.* 2006 - see Remark 2.5), the asymptotic singularity problem observed in Theorem 4.1 is not encountered, since here the corresponding quantity $\frac{1}{T}\mathbf{V}'\mathbf{V}$, where $\mathbf{V} := [v_1, v_2, ..., v_T]'$ is non-singular in the limit in both the constant and linear trend cases; see Becker *et al.* (2006) for further details.

Remark 4.4: The results in Theorem 1 show that the Fourier regressors $\sin\left(\frac{2\pi\kappa t}{T}\right)$ and $\cos\left(\frac{2\pi\kappa t}{T}\right)$ are asymptotically collinear with the constant term when subjected to the local GLS transformation in (4). An alternative to the full local GLS de-trending approach outlined in section 2 might then be to apply the local GLS de-trending stage in the first step only to the elements of z_t, and to then include the Fourier terms directly in the second step regression. That is, to compute the t-statistic for $\phi = 0$, t_{ϕ}^{HYBRID} say, in the regression equation

$$\Delta\bar{y}_t = v_t'\boldsymbol{\omega} + \phi\bar{y}_{t-1} + \bar{v}_t$$

where $\bar{y}_t := y_t - z_t'\bar{\boldsymbol{\alpha}}$ with $\bar{\boldsymbol{\alpha}}$ the estimated parameter vector from regressing $\mathbf{y}_{\bar{c}}$ onto $\mathbf{z}_{\bar{c}} := \left[z_1, z_2 - (1+\frac{\bar{c}}{T})z_1, ..., z_T - (1+\frac{\bar{c}}{T})z_{T-1}\right]'$. Although t_{ϕ}^{HYBRID} can be shown (available from the authors on request) to have a pivotal and well-defined limiting null distribution its exact distribution, unlike the $t_{\phi}^{ERS_f^{\zeta}}$, $t_{\phi}^{DF_f^{\zeta}}$ and $t_{\phi}^{LM_f}$ tests, depends on the nuisance parameters characterising the Fourier terms (arising from the fact that the first stage local GLS regression is mis-specified). In unreported Monte Carlo simulations we found t_{ϕ}^{HYBRID} to behave very poorly, both in terms of size and power in small samples.

5 Conclusions

In this paper, we generalise the Dickey-Fuller-type unit root testing procedure based on local GLS de-trending proposed by Elliott, Rothenberg and Stock (1996) to incorporate a Fourier approximation to the unknown deterministic component in the same was as is done for the corresponding OLS and FD de-trended Dickey-Fuller-type unit root tests of Enders and Lee (2008). We show that although the resulting unit root tests possess good finite sample size and power properties when compared to the OLS and FD de-trended tests of Enders and Lee (2008), their limit null distributions are undefined.

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Appendix

Proof of Theorem 1

The scaled local GLS estimator of $\boldsymbol{\alpha}$ can be written as:

$$\sqrt{T}\left(\widehat{\boldsymbol{\theta}}_{\bar{c}}-\boldsymbol{\theta}\right)=\left(\frac{1}{T}\mathbf{V}_{\bar{c}}'\mathbf{V}_{\bar{c}}\right)^{-1}\frac{1}{T^{1/2}}\mathbf{V}_{\bar{c}}'\mathbf{x}_{\bar{c}}. \tag{A.1}$$

The columns of $\mathbf{V}_{\bar{c}}:=(\mathbf{V}_{1,\bar{c}},\mathbf{V}_{2,\bar{c}},\mathbf{V}_{3,\bar{c}}\mathbf{V}_{4,\bar{c}})$ are given by:

$$\begin{aligned}
\mathbf{V}_{1,\bar{c}} &:= (1+cT^{-1})\mathbf{e}_1-\frac{c}{T}\mathbf{1},\\
\mathbf{V}_{2,\bar{c}} &:= \Delta\sin-\frac{\bar{c}}{T}\sin_{-1},\\
\mathbf{V}_{3,\bar{c}} &:= \Delta\cos-\frac{\bar{c}}{T}\cos_{-1}+(1+\frac{\bar{c}}{T})\mathbf{e}_1,\\
\mathbf{V}_{4,\bar{c}} &:= \mathbf{1}+\frac{c}{T}\boldsymbol{\tau},
\end{aligned}$$

where $\mathbf{1}$ is a $T\times 1$ vector of ones, $\mathbf{e}_1$ is a $T\times 1$ vector with first element equal to one and all others equal to zero and $\boldsymbol{\tau}$ is a $T\times 1$ vector such that $\boldsymbol{\tau}:=(0,1,...,T-1)'$, and

$$\begin{aligned}
\Delta\sin &:= \left(\Delta\sin\left(\frac{2\pi\kappa\cdot 1}{T}\right),\Delta\sin\left(\frac{2\pi\kappa\cdot 2}{T}\right),...,\Delta\sin\left(\frac{2\pi\kappa\cdot T}{T}\right)\right)'\\
\sin_{-1} &:= \left(\sin\left(\frac{2\pi\kappa\cdot 0}{T}\right),\sin\left(\frac{2\pi\kappa\cdot 1}{T}\right),...,\sin\left(\frac{2\pi\kappa\cdot T-1}{T}\right)\right)'\\
\Delta\cos &:= \left(\Delta\cos\left(\frac{2\pi\kappa\cdot 1}{T}\right),\Delta\cos\left(\frac{2\pi\kappa\cdot 2}{T}\right),...,\Delta\cos\left(\frac{2\pi\kappa\cdot T}{T}\right)\right)'\\
\cos_{-1} &:= \left(\cos\left(\frac{2\pi\kappa\cdot 0}{T}\right),\cos\left(\frac{2\pi\kappa\cdot 1}{T}\right),...,\cos\left(\frac{2\pi\kappa\cdot T-1}{T}\right)\right)'.
\end{aligned}$$

The following Lemma details the large sample behaviour of the scaled products involved in (A.1). The joint convergence results in (A.2)-(A.6) of Lemma A.1, together with applications of the continuous mapping theorem, are sufficient to establish the stated result in (12).

Lemma A.1 *Let the conditions of Theorem 1 hold. Then, as $T\to\infty$,*

$$\frac{1}{T}\overline{\mathbf{V}}_{1,\bar{c}}'\overline{\mathbf{V}}_{1,\bar{c}} \to 1;\ \frac{1}{T}\overline{\mathbf{V}}_{2,\bar{c}}'\overline{\mathbf{V}}_{2,\bar{c}}\to 0;\ \frac{1}{T}\overline{\mathbf{V}}_{3,\bar{c}}'\overline{\mathbf{V}}_{3,\bar{c}}\to 1; \tag{A.2}$$

$$\frac{1}{T}\overline{\mathbf{V}}_{4,\bar{c}}'\overline{\mathbf{V}}_{4,\bar{c}} \to 1+\bar{c}+\frac{\bar{c}^2}{3};\ \frac{1}{T}\overline{\mathbf{V}}_{1,\bar{c}}'\overline{\mathbf{V}}_{2,\bar{c}}\to 0;\ \frac{1}{T}\overline{\mathbf{V}}_{1,\bar{c}}'\overline{\mathbf{V}}_{3,\bar{c}}\to 1; \tag{A.3}$$

$$\frac{1}{T}\overline{\mathbf{V}}_{1,\bar{c}}'\overline{\mathbf{V}}_{4,\bar{c}} \to 0;\ \frac{1}{T}\overline{\mathbf{V}}_{4,\bar{c}}'\overline{\mathbf{V}}_{2,\bar{c}}\to 0;\ \frac{1}{T}\overline{\mathbf{V}}_{4,\bar{c}}'\overline{\mathbf{V}}_{3,\bar{c}}\to 0 \tag{A.4}$$

and

$$\frac{1}{T^{1/2}}\overline{\mathbf{V}}_{1,\bar{c}}'\mathbf{x}_{\bar{c}} \Rightarrow \sigma x_1;\ \frac{1}{T^{1/2}}\overline{\mathbf{V}}_{2,\bar{c}}'\mathbf{x}_{\bar{c}}\Rightarrow 0;\ \frac{1}{T^{1/2}}\overline{\mathbf{V}}_{3,\bar{c}}'\mathbf{x}_{\bar{c}}\Rightarrow \sigma x_1; \tag{A.5}$$

$$\frac{1}{T^{1/2}}\overline{\mathbf{V}}_{4,\bar{c}}'\mathbf{x}_{\bar{c}} \Rightarrow \sigma\left((1-\bar{c})W(1)-2\bar{c}\int_0^1 W(r)dr-\bar{c}^2\int_0^1 rW(r)dr\right) \tag{A.6}$$

where $\overline{\mathbf{V}}_{\bar{c}}:=(\overline{\mathbf{V}}_{1,\bar{c}},\overline{\mathbf{V}}_{2,\bar{c}},\overline{\mathbf{V}}_{3,\bar{c}}\overline{\mathbf{V}}_{4,\bar{c}})$, is such that $\overline{\mathbf{V}}_{\bar{c}}:=N_T\mathbf{V}_{\bar{c}}$ with $N_T=diag(T^{1/2},T^{1/2},T^{1/2},1)$.

Proof of Lemma A.1

Using the scaling matrix $\Lambda := diag\left\{T^{-1/2}, T^{-1/2}, T^{-1/2}, T^{-1/2}\right\}$ we obtain the following limit results for the main diagonal elements of $\Lambda\overline{\mathbf{V}}_{\bar{c}}'\overline{\mathbf{V}}_{\bar{c}}\Lambda$.

$$\begin{aligned}
\frac{1}{T}\overline{\mathbf{V}}_{1,\bar{c}}'\overline{\mathbf{V}}_{1,\bar{c}} &= \left((1+cT^{-1})\mathbf{e}_1 - \frac{c}{T}\mathbf{1}\right)'\left((1+cT^{-1})\mathbf{e}_1 - \frac{c}{T}\mathbf{1}\right) \\
&= (1+cT^{-1})^2\mathbf{e}_1'\mathbf{e}_1 - \frac{2c}{T}(1+cT^{-1})\mathbf{1}'\mathbf{e}_1 + \left(\frac{c}{T}\right)^2\mathbf{1}'\mathbf{1} \\
&= 1 - \left(\frac{c}{T}\right)^2 + \frac{c^2}{T} = 1 + o(1).
\end{aligned}$$

$$\begin{aligned}
\frac{1}{T}\overline{\mathbf{V}}_{2,\bar{c}}'\overline{\mathbf{V}}_{2,\bar{c}} &= \left(\Delta\sin - \frac{c}{T}\sin_{-1}\right)'\left(\Delta\sin - \frac{c}{T}\sin_{-1}\right) \\
&= \sum_{t=1}^{T}\Delta\sin^2\left(\frac{2\pi kt}{T}\right) - \frac{2c}{T}\sum_{t=1}^{T}\left(\Delta\sin\left(\frac{2\pi kt}{T}\right)\right)\left(\sin\left(\frac{2\pi k\,(t-1)}{T}\right)\right) \\
&\quad + \left(\frac{c}{T}\right)^2\sum_{t=1}^{T}\sin^2\left(\frac{2\pi k\,(t-1)}{T}\right) = o(1),
\end{aligned}$$

where we have used the result from Enders and Lee (2008) that $\Delta\sin\left(\frac{2\pi kt}{T}\right) \equiv \left(\frac{2\pi k}{T}\right)\cos\left(\frac{2\pi kt}{T}\right)$.

$$\begin{aligned}
\frac{1}{T}\overline{\mathbf{V}}_{3,\bar{c}}'\overline{\mathbf{V}}_{3,\bar{c}} &= \left(\Delta\cos - \frac{c}{T}\cos_{-1} + (1+\frac{c}{T})\mathbf{e}_1\right)'\left(\Delta\cos - \frac{c}{T}\cos_{-1} + (1+\frac{c}{T})\mathbf{e}_1\right) \\
&= \sum_{t=1}^{T}\Delta\cos^2\left(\frac{2\pi kt}{T}\right) - \frac{2c}{T}\sum_{t=1}^{T}\left(\Delta\cos\left(\frac{2\pi kt}{T}\right)\right)\left(\cos\left(\frac{2\pi k\,(t-1)}{T}\right)\right) \\
&\quad + \left(\frac{c}{T}\right)^2\sum_{t=1}^{T}\cos^2\left(\frac{2\pi k\,(t-1)}{T}\right) + 2(1+\frac{c}{T})\mathbf{e}_1\Delta\cos - \frac{2c}{T}(1+\frac{c}{T})\mathbf{e}_1\cos_{-1} \\
&\quad + (1+\frac{c}{T})^2 = 1 + o(1),
\end{aligned}$$

using the result from Enders and Lee (2008) that $\Delta\cos\left(\frac{2\pi kt}{T}\right) \equiv -\left(\frac{2\pi k}{T}\right)\sin\left(\frac{2\pi kt}{T}\right)$.

$$\begin{aligned}
\frac{1}{T}\overline{\mathbf{V}}_{4,\bar{c}}'\overline{\mathbf{V}}_{4,\bar{c}} &= \frac{1}{T}\left(\mathbf{1} + \frac{c}{T}\boldsymbol{\tau}\right)'\left(\mathbf{1} + \frac{c}{T}\boldsymbol{\tau}\right) \\
&= \frac{1}{T}\left(\mathbf{1}'\mathbf{1} + 2\frac{c}{T}\boldsymbol{\tau}'\mathbf{1} + \left(\frac{c}{T}\right)^2\boldsymbol{\tau}'\boldsymbol{\tau}\right) \\
&= 1 + c + \frac{c^2}{3} + o(1).
\end{aligned}$$

Turning to the off diagonal elements of the symmetric matrix $\Lambda\overline{\mathbf{V}}_{\bar{c}}'\overline{\mathbf{V}}_{\bar{c}}\Lambda$, we have that:

$$
\begin{aligned}
\frac{1}{T}\overline{\mathbf{V}}'_{1,\bar{c}}\overline{\mathbf{V}}_{2,\bar{c}} &= \left((1+cT^{-1})\mathbf{e}_1 - \frac{c}{T}\mathbf{1}\right)'\left(\Delta\sin - \frac{c}{T}\sin_{-1}\right) \\
&= (1+cT^{-1})\Delta\sin\left(\frac{2\pi k}{T}\right) - \frac{c}{T}\sum_{t=1}^{T}\Delta\sin\left(\frac{2\pi kt}{T}\right) \\
&\quad -(1+cT^{-1})\sin\left(\frac{2\pi k0}{T}\right) + \left(\frac{c}{T}\right)^2\sum_{t=1}^{T}\sin\left(\frac{2\pi k\,(t-1)}{T}\right) \\
&= \sin\left(\frac{2\pi k}{T}\right) + o(1) = o(1)
\end{aligned}
$$

where we have used the identity $\Delta\sin\left(\frac{2\pi k}{T}\right) \equiv \sin\left(\frac{2\pi k}{T}\right) - \sin\left(\frac{2\pi k0}{T}\right)$.

$$
\begin{aligned}
\frac{1}{T}\overline{\mathbf{V}}'_{1,\bar{c}}\overline{\mathbf{V}}_{3,\bar{c}} &= \left((1+cT^{-1})\mathbf{e}_1 - \frac{c}{T}\mathbf{1}\right)'\left(\Delta\cos - \frac{c}{T}\cos_{-1} + (1+cT^{-1})\mathbf{e}_1\right) \\
&= (1+cT^{-1})\Delta\cos\left(\frac{2\pi k}{T}\right) - \frac{c}{T}\sum_{t=1}^{T}\Delta\cos\left(\frac{2\pi kt}{T}\right) - (1+cT^{-1})\frac{c}{T}\cos\left(\frac{2\pi k0}{T}\right) \\
&\quad + \left(\frac{c}{T}\right)^2\sum_{t=1}^{T}\cos\left(\frac{2\pi k\,(t-1)}{T}\right) + (1+cT^{-1})^2 - \frac{c}{T}(1+cT^{-1}) \\
&= (1+cT^{-1})\left(\cos\left(\frac{2\pi k}{T}\right) - 1\right) + \frac{2c\pi k}{T^2}\sum_{t=1}^{T}\sin\left(\frac{2\pi kt}{T}\right) \\
&\quad -2(1+cT^{-1})\frac{c}{T} + \left(\frac{c}{T}\right)^2\sum_{t=1}^{T}\cos\left(\frac{2\pi k\,(t-1)}{T}\right) + (1+cT^{-1})^2 \\
&= \cos\left(\frac{2\pi k}{T}\right) + o(1) = 1 + o(1)
\end{aligned}
$$

where we have used the identity $\Delta\cos\left(\frac{2\pi k}{T}\right) \equiv \cos\left(\frac{2\pi k}{T}\right) - \cos\left(\frac{2\pi k0}{T}\right)$.

$$
\begin{aligned}
\frac{1}{T}\overline{\mathbf{V}}'_{1,\bar{c}}\overline{\mathbf{V}}_{4,\bar{c}} &= \frac{1}{T^{1/2}}\left((1+cT^{-1})\mathbf{e}_1 - \frac{c}{T}\mathbf{1}\right)'\left(\mathbf{1} + \frac{c}{T}\boldsymbol{\tau}\right) \\
&= \frac{1}{T^{1/2}}(1+cT^{-1}) - \frac{c}{T^{1/2}} - \frac{c^2}{2T^{1/2}} = o(1).
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{T}\overline{\mathbf{V}}'_{2,\bar{c}}\overline{\mathbf{V}}_{3,\bar{c}} &= \left(\Delta\sin - \frac{c}{T}\sin_{-1}\right)'\left(\Delta\cos - \frac{c}{T}\cos_{-1} + (1+cT^{-1})\mathbf{e}_1\right) \\
&= \Delta\sin'\Delta\cos - \frac{c}{T}\sin'_{-1}\Delta\cos - \frac{c}{T}\Delta\sin'\cos_{-1} + \left(\frac{c}{T}\right)^2\sin'_{-1}\cos_{-1} \\
&\quad + \Delta\sin'(1+cT^{-1})\mathbf{e}_1 - \frac{c}{T}\sin'_{-1}(1+cT^{-1})\mathbf{e}_1 \\
&= 0 - \frac{c}{T}\sin'_{-1}\Delta\cos - \frac{c}{T}\Delta\sin'\cos_{-1} + 0 + \sin\left(\frac{2\pi k}{T}\right)(1+cT^{-1}) - 0 \\
&= -\frac{c}{T}\sum_{t=1}^{T}\sin\left(\frac{2\pi k\,(t)}{T} + \frac{2\pi k\,(t-1)}{T}\right) + \sin\left(\frac{2\pi k}{T}\right)(1+cT^{-1}) \\
&= \sin\left(\frac{2\pi k}{T}\right) + o(1) = o(1).
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{T}\overline{\mathbf{V}}'_{2,\bar{c}}\overline{\mathbf{V}}_{4,\bar{c}} &= \frac{1}{T^{1/2}}\left(\Delta\sin - \frac{c}{T}\sin_{-1}\right)'\left(\mathbf{1} + \frac{c}{T}\boldsymbol{\tau}\right) \\
&= \frac{1}{T^{1/2}}\left(\Delta\sin'\mathbf{1} - \frac{c}{T}\sin'_{-1}\mathbf{1} + \Delta\sin'\frac{c}{T}\boldsymbol{\tau} - \frac{c}{T}\sin'_{-1}\frac{c}{T}\boldsymbol{\tau}\right) \\
&= \frac{1}{T^{1/2}}\left(0 - 0 + \frac{c}{T}\Delta\sin'\boldsymbol{\tau} - \left(\frac{c}{T}\right)^2\sin'_{-1}\boldsymbol{\tau}\right) \\
&= \frac{1}{T^{1/2}}\left(\frac{c}{T}\sum_{t=1}^{T}(t-1)\Delta\sin\left(\frac{2\pi k\,(t)}{T}\right) - \left(\frac{c}{T}\right)^2\sum_{t=1}^{T}(t-1)\sin\left(\frac{2\pi k\,(t-1)}{T}\right)\right) \\
&= \frac{2c\pi k}{T^{5/2}}\sum_{t=1}^{T}(t-1)\cos\left(\frac{2\pi kt}{T}\right) - \frac{c^2}{T^{5/2}}\sum_{t=1}^{T}(t-1)\sin\left(\frac{2\pi k\,(t-1)}{T}\right) = o(1).
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{T}\overline{\mathbf{V}}'_{3,\bar{c}}\overline{\mathbf{V}}_{4,\bar{c}} &= \frac{1}{T^{1/2}}\left(\Delta\cos - \frac{c}{T}\cos_{-1} + (1+cT^{-1})\mathbf{e}_1\right)'\left(\mathbf{1} + \frac{c}{T}\boldsymbol{\tau}\right) \\
&= \frac{1}{T^{1/2}}\left((1+cT^{-1}) + \frac{c}{T}\sum_{t=1}^{T}(t-1)\Delta\cos\left(\frac{2\pi kt}{T}\right) - \left(\frac{c}{T}\right)^2\sum_{t=1}^{T}(t-1)\cos\left(\frac{2\pi k\,(t-1)}{T}\right)\right) \\
&= \frac{1}{T^{1/2}}(1+cT^{-1}) - \frac{2c\pi k}{T^{5/2}}\sum_{t=1}^{T}(t-1)\sin\left(\frac{2\pi kt}{T}\right) - \frac{c^2}{T^{5/2}}\sum_{t=1}^{T}(t-1)\cos\left(\frac{2\pi k\,(t-1)}{T}\right) = o(1).
\end{aligned}
$$

Turning finally to the numerator, $\mathbf{V}'_{\bar{c}}\mathbf{x}_{\bar{c}}$, in (A.1), noting that $\mathbf{x}_{\bar{c}} = \left[x_1, \Delta x_2 - \frac{\bar{c}}{T}x_1, ..., \Delta x_T - \frac{\bar{c}}{T}x_{T-1}\right]$, we observe that:

$$
\begin{aligned}
\frac{1}{\sqrt{T}}\overline{\mathbf{V}}'_{1,\bar{c}}\mathbf{x}_{\bar{c}} &= \left((1+cT^{-1})\mathbf{e}_1 - \frac{\bar{c}}{T}\mathbf{1}\right)'\mathbf{x}_{\bar{c}} \\
&= (1+\bar{c}T^{-1})x_1 - \frac{\bar{c}}{T}x_1 - \frac{\bar{c}}{T}\sum_{t=2}^{T}\Delta x_t + \left(\frac{\bar{c}}{T}\right)^2\sum_{t=2}^{T}x_{t-1} \\
&= x_1 + o_p(1). \qquad \text{(A.7)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\sqrt{T}}\overline{\mathbf{V}}_{2,\bar{c}}'\mathbf{x}_{\bar{c}} &= \left(\Delta\sin - \frac{c}{T}\sin_{-1}\right)'\mathbf{x}_{\bar{c}} \\
&= \Delta\sin\left(\frac{2\pi k}{T}\right)x_1 + \sum_{t=2}^{T}\Delta\sin\left(\frac{2\pi kt}{T}\right)\left(\Delta x_t - \frac{\bar{c}}{T}x_{t-1}\right) \\
&\quad -\frac{c}{T}\sum_{t=2}^{T}\sin\left(\frac{2\pi k\,(t-1)}{T}\right)\left(\Delta x_t - \frac{\bar{c}}{T}x_{t-1}\right) \\
&= \Delta\sin\left(\frac{2\pi k}{T}\right)x_1 + \frac{2\pi k}{T}\sum_{t=2}^{T}\cos\left(\frac{2\pi kt}{T}\right)u_t - \left(\frac{\bar{c}}{T}\right)^2\frac{2\pi k}{T}\sum_{t=2}^{T}\cos\left(\frac{2\pi kt}{T}\right)x_{t-1} \\
&\quad -\frac{c}{T}\sum_{t=2}^{T}\sin\left(\frac{2\pi k\,(t-1)}{T}\right)u_t + \left(\frac{\bar{c}}{T}\right)^2\sum_{t=2}^{T}\sin\left(\frac{2\pi k\,(t-1)}{T}\right)x_{t-1} \\
&= \sin\left(\frac{2\pi k}{T}\right)x_1 + o_p\left(1\right) = o_p\left(1\right). \qquad \text{(A.8)}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{\sqrt{T}}\overline{\mathbf{V}}_{3,\bar{c}}'\mathbf{x}_{\bar{c}} &= \left(\Delta\cos - \frac{c}{T}\cos_{-1} + (1+cT^{-1})\mathbf{e}_1\right)'\mathbf{x}_{\bar{c}} \\
&= \Delta\cos\left(\frac{2\pi k}{T}\right)x_1 + \sum_{t=2}^{T}\Delta\cos\left(\frac{2\pi kt}{T}\right)\left(\Delta x_t - \frac{\bar{c}}{T}x_{t-1}\right) \\
&\quad -\frac{c}{T}\sum_{t=2}^{T}\cos\left(\frac{2\pi k\,(t-1)}{T}\right)\left(\Delta x_t - \frac{\bar{c}}{T}x_{t-1}\right) + (1+cT^{-1})x_1 \\
&= x_1 + o_p(1). \qquad \text{(A.9)}
\end{aligned}
$$

Remark A.1: We have used results from Bierens (1994, Lemma 9.6.3) in establishing (A.8) and (A.9). These results state that,

$$
\sum_{t=2}^{T}F\left(\frac{t}{T}\right)u_t = F(1)S(1) - \int_0^1 f(r)S_T(r)dr
$$

where $f(r) = F'(r)$. Consequently, for $F\left(\frac{t}{T}\right) = \sin\left(\frac{2\pi kt}{T}\right)$ it follows that $f(\frac{t}{T}) = (2\pi k)\cos\left(\frac{2\pi kt}{T}\right)$ and if $F\left(\frac{t}{T}\right) = \cos\left(\frac{2\pi kt}{T}\right)$ then $f(\frac{t}{T}) = -\left(2\pi k\right)\sin\left(\frac{2\pi kt}{T}\right)$. As a result,

$$
\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\cos\left(\frac{2\pi kt}{T}\right)u_t = \sigma\left(W(1) + 2\pi k\int_0^1\sin\left(2\pi kr\right)W(r)dr\right)
$$

and

$$
\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\sin\left(\frac{2\pi kt}{T}\right)u_t = \sigma\left(\sin\left(2\pi k\right)W(1) - 2\pi k\int_0^1\cos\left(2\pi kr\right)W(r)dr\right). \qquad \square
$$

Finally, to establish the result in (A.6) observe that

$$
\begin{aligned}
\frac{1}{\sqrt{T}}\overline{\mathbf{V}}'_{4,\bar{c}}\mathbf{x}_{\bar{c}} &= \frac{1}{\sqrt{T}}\left(\mathbf{1}+\frac{c}{T}\boldsymbol{\tau}\right)'\mathbf{x}_{\bar{c}} \\
&= \frac{1}{\sqrt{T}}\left(x_1+\sum_{t=2}^{T}\Delta x_t-\frac{\bar{c}}{T}\sum_{t=2}^{T}x_{t-1}+\frac{c}{T}\sum_{t=2}^{T}(t-1)\Delta x_t-\frac{\bar{c}^2}{T^2}\sum_{t=2}^{T}(t-1)x_{t-1}\right)
\end{aligned}
$$

and, hence,

$$
\begin{aligned}
\frac{1}{\sqrt{T}}\overline{\mathbf{V}}'_{4,\bar{c}}\mathbf{x}_{\bar{c}} &= \frac{1}{T^{1/2}}\sum_{t=2}^{T}\Delta x_t-\frac{\bar{c}}{T^{3/2}}\sum_{t=2}^{T}x_{t-1}+\frac{c}{T^{3/2}}\sum_{t=2}^{T}(t-1)\Delta x_t-\frac{\bar{c}^2}{T^{5/2}}\sum_{t=2}^{T}(t-1)x_{t-1}+o_p(1) \\
&\Rightarrow \sigma\left(W(1)-\bar{c}\int_0^1 W(r)dr+\bar{c}\int_0^1 rdW(r)-\bar{c}^2\int_0^1 rW(r)dr\right)=:\sigma\mathfrak{W}.
\end{aligned}
$$

Theorem 2 *Considering the DGP in (1)-(2), under the null hypothesis, $H_0 : \rho = 1$, and assuming that κ is fixed, it follows as $T \to \infty$ that,*

$$\sqrt{T}\left(\widehat{\boldsymbol{\alpha}}_{\overline{c}} - \boldsymbol{\alpha}\right) = \left(\frac{1}{T}\mathbf{Z}'_{\overline{c}}\mathbf{Z}_{\overline{c}}\right)^{-1} \frac{1}{T^{1/2}}\mathbf{Z}'_{\overline{c}}\left(\mathbf{f}_{\overline{c}}(\kappa)\varphi + \mathbf{x}_{\overline{c}}\right) \tag{A.10}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 + c + \frac{c^2}{3} \end{bmatrix}^{-1} \begin{bmatrix} \sigma\left(x_1 + \varphi_2\right) \\ \sigma\mathfrak{W} \end{bmatrix} \equiv \sigma \begin{bmatrix} \boldsymbol{\Xi}_1 \\ \boldsymbol{\Xi}_2 \end{bmatrix} \tag{A.11}$$

where $\mathfrak{W} = (1-\overline{c})W(1) - 2\overline{c}\int_0^1 W(r)dr - \overline{c}^2\int_0^1 rW(r)dr$, $\boldsymbol{\Xi}_1 = x_1 + \varphi_2$ *and* $\boldsymbol{\Xi}_2 = \frac{\mathfrak{W}}{1+c+\frac{c^2}{3}}$.

Note that if $\varphi_2 = 0$, (12) corresponds to the results obtained by Elliott *et al.* (1996).

Theorem 3 *Considering the DGP in (1)-(2), under the null hypothesis, $H_0 : \rho = 1$, and assuming that κ is fixed, it follows as $T \to \infty$ that the limits of the scalled parameter estimates, $\widehat{\boldsymbol{\delta}} = (\widehat{\varphi}_1, \widehat{\varphi}_2, \widehat{\phi})$, obtained from test regression (5) i.e.,*

$$\Lambda^{-1}\widehat{\boldsymbol{\delta}} = \left(\Lambda V_t' V_t \Lambda\right)^{-1} \Lambda V_t' \Delta y_t^{\overline{c}}$$

where $V_t = (f_t(\kappa)', y_{t-1}^{\overline{c}})'$ *and* $\Lambda = diag\left\{\frac{1}{\sqrt{T}}, \frac{1}{\sqrt{T}}, \frac{1}{T}\right\}$, *will be the following,*

$$\Lambda^{-1}\left(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right) \Rightarrow \begin{bmatrix} \int_0^1 \sin^2\left(2\pi\kappa r\right) dr & 0 & \mathfrak{w}_{13} \\ 0 & \int_0^1 \cos^2\left(2\pi\kappa r\right) dr & \mathfrak{w}_{23} \\ \mathfrak{w}_{31} & \mathfrak{w}_{32} & \mathfrak{w}_{33} \end{bmatrix}^{-1} \begin{bmatrix} \mathfrak{D}_1 \\ \mathfrak{D}_2 \\ \mathfrak{D}_3 \end{bmatrix}$$

where $\mathfrak{w}_{13} = \sigma\int_0^1 \sin\left(2\pi\kappa r\right) W(r)dr - \sigma\boldsymbol{\Xi}_2\int_0^1 r\sin\left(2\pi\kappa r\right) dr$, $\mathfrak{w}_{23} = \sigma\int_0^1 \cos\left(2\pi\kappa r\right) W(r)dr - \sigma\boldsymbol{\Xi}_2\int_0^1 r\cos\left(2\pi\kappa r\right) dr$, $\mathfrak{w}_{33} = \sigma^2\int_0^1 W(r)^2 dr - 2\sigma^2\boldsymbol{\Xi}_2\int_0^1 rW(r)dr + \frac{\sigma^2}{3}\boldsymbol{\Xi}_2^2$, $\mathfrak{D}_1 = \sigma\Big(\sin\left(2\pi\kappa\right) W(1) - 2\pi\kappa\int_0^1 \cos(2$
$\mathfrak{D}_2 = \sigma\left(W(1) + 2\pi\kappa\int_0^1 \sin\left(2\pi\kappa r\right) W(r)dr\right)$ *and* $\mathfrak{D}_3 = \sigma^2\Big[\left(\int_0^1 W(r)dW(r) - \boldsymbol{\Xi}_2\int_0^1 rdW(r)\right) - \boldsymbol{\Xi}_2\Big(\int_0^1 W(r)dr$

As can be observed from Theorem 4.2, the limit distributions of the test statistics will only depend on k, the frequency used in the Fourier approximation. Note that although in the first step the limit of $\widehat{\alpha}_{1\overline{c}}$ is a function of x_1 and the unknown coefficient φ_2 the limit distribution of the estimators computed in the second step are free of this nuisance parameter.

Proof of Theorem 2

The proof of theroem 3.2 follows along similar lines as the proof of Theorem 3.1. Hence, consider first the limit results for the parameter estimates of the deterministic component (just a constant or a constant and a time trend) estimated in the first step. We consider the more general quasi-differenced (QD) deterministic kernel which includes a constant and a time trend,

$$\mathbf{Z}_{\overline{c}} = \left(\mathbf{Z}_{1,\overline{c}}, \mathbf{Z}_{2,\overline{c}}\right) \tag{A.12}$$

where

$$\mathbf{Z}_{1,\overline{c}} = (1 + \frac{\overline{c}}{T})\mathbf{e}_1 - \frac{\overline{c}}{T}\mathbf{1},$$
$$\mathbf{Z}_{2,\overline{c}} = \mathbf{1} + \frac{\overline{c}}{T}\boldsymbol{\tau},$$

$\mathbf{1}$ is a $T \times 1$ vector of ones, $\mathbf{e}_1$ is a $T \times 1$ vector with first element equal to one and all others equal to zero and $\boldsymbol{\tau}$ is a $T \times 1$ vector such that $\boldsymbol{\tau} =(0,1,...,T-1)'$. Similarly as in the proof of Theorem 3.1, define the diagonal matrix $N_T = diag\left\{T^{1/2}, 1\right\}$ and consider $\overline{\mathbf{Z}}_{\overline{c}} = \mathbf{Z}_{\overline{c}} N_T$. Consequently, the following Lemma can be stated.

Lemma A.2 *Considering the DGP (1)-(2) under the null hypothesis, $H_0 : \rho = 1$, and assumption A (k fixed), it follows as $T \to \infty$ that,*

$$\frac{1}{T}\overline{\mathbf{Z}}'_{1,\overline{c}}\overline{\mathbf{Z}}_{1,\overline{c}} \to 1 \tag{A.13}$$
$$\frac{1}{T}\overline{\mathbf{Z}}'_{2,\overline{c}}\overline{\mathbf{Z}}_{2,\overline{c}} \to 1+\overline{c}+\frac{\overline{c}^2}{3} \tag{A.14}$$
$$\frac{1}{T}\overline{\mathbf{Z}}'_{1,\overline{c}}\overline{\mathbf{Z}}_{2,\overline{c}} \to 0 \tag{A.15}$$

and

$$\frac{1}{T^{1/2}}\overline{\mathbf{Z}}'_{1,\overline{c}}\mathbf{x}_{\overline{c}} \Rightarrow x_1 \tag{A.16}$$
$$\frac{1}{T^{1/2}}\overline{\mathbf{Z}}'_{2,\overline{c}}\mathbf{x}_{\overline{c}} \Rightarrow \sigma\left((1-\overline{c})W(1) - 2\overline{c}\int_0^1 W(r)dr - \overline{c}^2\int_0^1 rW(r)dr\right) \tag{A.17}$$
$$\frac{1}{T^{1/2}}\overline{\mathbf{Z}}'_{1,\overline{c}}\mathbf{Z}_{\sin,\overline{c}} \to 0;\ \frac{1}{T^{1/2}}\overline{\mathbf{Z}}'_{1,\overline{c}}\mathbf{Z}_{\cos,\overline{c}} \to 1;\ \frac{1}{T^{1/2}}\overline{\mathbf{Z}}'_{2,\overline{c}}\mathbf{Z}_{\sin,\overline{c}} \to 0 \tag{A.18}$$
$$\frac{1}{T^{1/2}}\overline{\mathbf{Z}}'_{2,\overline{c}}\overline{\mathbf{Z}}_{\cos,\overline{c}} \to 0. \tag{A.19}$$

As in the case of Lemma A.1, also these results are useful to characterize the limit results when only a constant is considered in (A.12). Thus, under joint convergence, the results in (A.13) - (A.19) provide the necessary limits to obtain the asymptotic results for the first step estimators.

Proof of Lemma A.2.

Consider first the limit results for the denominator of (A.10) i.e., the results in (A.13) - (A.15). Scaling the elements of $\overline{\mathbf{Z}}'_{\overline{c}}\overline{\mathbf{Z}}_{\overline{c}}$ by $\Lambda = diag\left\{T^{-1/2}, T^{-1/2}\right\}$ we observe that the limits of $\frac{1}{T}\overline{\mathbf{Z}}'_{1,\overline{c}}\overline{\mathbf{Z}}_{1,\overline{c}}$, $\frac{1}{T}\overline{\mathbf{Z}}'_{2,\overline{c}}\overline{\mathbf{Z}}_{2,\overline{c}}$, and $\frac{1}{T}\overline{\mathbf{Z}}'_{1,\overline{c}}\overline{\mathbf{Z}}_{2,\overline{c}}$ are equivalent to those of $\frac{1}{T}\overline{\mathbf{V}}'_{1,\overline{c}}\overline{\mathbf{V}}_{1,\overline{c}}$, $\frac{1}{T}\overline{\mathbf{V}}'_{4,\overline{c}}\overline{\mathbf{V}}_{4,\overline{c}}$ and $\frac{1}{T}\overline{\mathbf{V}}'_{1,\overline{c}}\overline{\mathbf{V}}_{4,\overline{c}}$ provided in (A.7), (A.7) and (A.7), respectively.

Regarding the numerator, i.e. the results in (A.16)-(A.19), consider $\frac{1}{T^{1/2}}\overline{\mathbf{Z}}'_{\overline{c}}\left(\mathbf{f}_{\overline{c}}(\kappa)\varphi + \mathbf{x}_{\overline{c}}\right) = \frac{1}{T^{1/2}}\overline{\mathbf{Z}}'_{\overline{c}}\mathbf{f}_{\overline{c}}(\kappa)\boldsymbol{\varphi} + \frac{1}{T^{1/2}}\overline{\mathbf{Z}}'_{\overline{c}}\mathbf{x}_{\overline{c}}$. For proof of the results we analyse these two terms separately.

Consider first $\frac{1}{T^{1/2}}\overline{\mathbf{Z}}'_{\overline{c}}\mathbf{x}_{\overline{c}}$. Since $\mathbf{x}_{\overline{c}} = \left[x_1, \Delta x_2 - \frac{\overline{c}}{T}x_1, ..., \Delta x_T - \frac{\overline{c}}{T}x_{T-1}\right] = (1+\frac{\overline{c}}{T})x_0\mathbf{e}_1 + \Delta\mathbf{x} - \frac{\overline{c}}{T}\mathbf{x}_{-1}$, it follows that the results for $\frac{1}{\sqrt{T}}\overline{\mathbf{Z}}'_{1,\overline{c}}\mathbf{x}_{\overline{c}}$, and $\frac{1}{\sqrt{T}}\overline{\mathbf{Z}}'_{2,\overline{c}}\mathbf{x}_{\overline{c}}$ correspond to those of $\frac{1}{\sqrt{T}}\overline{\mathbf{V}}'_{1,\overline{c}}\mathbf{x}_{\overline{c}}$ and $\frac{1}{\sqrt{T}}\overline{\mathbf{V}}'_{4,\overline{c}}\mathbf{x}_{\overline{c}}$ presented in (A.7) and (**??**).

Regarding $\frac{1}{T^{1/2}}\overline{\mathbf{Z}}'_{\overline{c}}\mathbf{f}_{\overline{c}}(\kappa)\boldsymbol{\varphi}$, note that

$$\frac{1}{T^{1/2}}\overline{\mathbf{Z}}'_{\overline{c}}\mathbf{f}_{\overline{c}}(\kappa)\boldsymbol{\varphi} = \frac{1}{T^{1/2}}\begin{bmatrix} \overline{\mathbf{Z}}'_{1,\overline{c}}\mathbf{Z}_{\sin,\overline{c}} & \overline{\mathbf{Z}}'_{1,\overline{c}}\mathbf{Z}_{\cos,\overline{c}} \\ \overline{\mathbf{Z}}'_{2,\overline{c}}\mathbf{Z}_{\sin,\overline{c}} & \overline{\mathbf{Z}}'_{2,\overline{c}}\mathbf{Z}_{\cos,\overline{c}} \end{bmatrix}\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$$

where $\mathbf{f}_{\overline{c}}(\kappa) = (\mathbf{Z}_{\sin,\overline{c}}, \mathbf{Z}_{\cos,\overline{c}})$,

$$\begin{aligned}
\mathbf{Z}_{\sin,\overline{c}} &= \Delta \sin - \frac{\overline{c}}{T} \sin_{-1}, \\
\mathbf{Z}_{\cos,\overline{c}} &= \Delta \cos - \frac{\overline{c}}{T} \cos_{-1} + (1 + \frac{\overline{c}}{T})\mathbf{e}_1,
\end{aligned}$$

and the vectors $\boldsymbol{\Delta} \sin$, $\sin_{-1}$, $\Delta \cos$ and $\cos_{-1}$ are as previously defined in (A.2) - (A.2), respectively.

Hence, the elements of $\frac{1}{T^{1/2}} \overline{\mathbf{Z}}'_{\overline{c}} f_{\overline{c}}(\kappa) \boldsymbol{\varphi}$ will have the following limit results:

$$\begin{aligned}
\frac{1}{T^{1/2}} \overline{\mathbf{Z}}'_{1,\overline{c}} \mathbf{Z}_{\sin,\overline{c}} &= \left((1 + \frac{\overline{c}}{T})\mathbf{e}_1 - \frac{\overline{c}}{T}\mathbf{1} \right)' \left(\Delta \sin - \frac{\overline{c}}{T} \sin_{-1} \right) \\
&= (1 + \frac{\overline{c}}{T}) \sin \left(\frac{2\pi\kappa}{T} \right) - \frac{2\overline{c}\pi\kappa}{T^2} \sum_{t=1}^{T} \cos \left(\frac{2\pi\kappa t}{T} \right) + \left(\frac{\overline{c}}{T} \right)^2 \sum_{t=1}^{T} \sin \left(\frac{2\pi\kappa\,(t-1)}{T} \right) \\
&= \sin \left(\frac{2\pi\kappa}{T} \right) + o(1) = o(1). \qquad \text{(A.20)}
\end{aligned}$$

Note that $\sin\left(\frac{2\pi\kappa}{T}\right) \to \frac{2\pi\kappa}{T} \to 0$ as $T \to \infty$. Thus, for fixed κ we observe that as $T \to \infty$, $\frac{1}{T^{1/2}} \overline{\mathbf{Z}}'_{1,\overline{c}} \mathbf{Z}_{\sin,\overline{c}} \to 0$.

Furthermore,

$$\begin{aligned}
\frac{1}{T^{1/2}} \overline{\mathbf{Z}}'_{1,\overline{c}} \mathbf{Z}_{\cos,\overline{c}} &= \left((1 + \frac{\overline{c}}{T})\mathbf{e}_1 - \frac{\overline{c}}{T}\mathbf{1} \right)' \left(\Delta \cos - \frac{\overline{c}}{T} \cos_{-1} + (1 + \frac{\overline{c}}{T})\mathbf{e}_1 \right) \\
&= (1 + \frac{\overline{c}}{T}) \Delta \cos \left(\frac{2\pi\kappa}{T} \right) - \frac{\overline{c}}{T} \sum_{t=1}^{T} \Delta \cos \left(\frac{2\pi\kappa t}{T} \right) - (1 + \frac{\overline{c}}{T}) \frac{\overline{c}}{T} \cos \left(\frac{2\pi\kappa 0}{T} \right) \\
&\quad + \left(\frac{\overline{c}}{T} \right)^2 \sum_{t=1}^{T} \cos \left(\frac{2\pi\kappa\,(t-1)}{T} \right) + (1 + \frac{\overline{c}}{T})^2 - \frac{\overline{c}}{T}(1 + \frac{\overline{c}}{T}) \\
&= (1 + \frac{\overline{c}}{T}) \left(\cos \left(\frac{2\pi\kappa}{T} \right) - \cos \left(\frac{2\pi\kappa 0}{T} \right) \right) + \frac{\overline{c}}{T} \sum_{t=1}^{T} \frac{2\pi\kappa}{T} \sin \left(\frac{2\pi\kappa t}{T} \right) \\
&\quad -2(1 + \frac{\overline{c}}{T}) \frac{\overline{c}}{T} + \left(\frac{\overline{c}}{T} \right)^2 \sum_{t=1}^{T} \cos \left(\frac{2\pi\kappa\,(t-1)}{T} \right) + (1 + \frac{\overline{c}}{T})^2 \\
&= \cos \left(\frac{2\pi\kappa}{T} \right) + o(1) = 1 + o(1). \qquad \text{(A.21)}
\end{aligned}$$

Thus, for fixed κ we observe that as $T \to \infty$, $\frac{1}{T^{1/2}} \overline{\mathbf{Z}}'_{1,\overline{c}} \mathbf{Z}_{\cos,\overline{c}} \to 1$.

Moreover,

$$\begin{aligned}
\frac{1}{T^{1/2}}\overline{\mathbf{Z}}'_{2,\bar{c}}\mathbf{Z}_{\sin,\bar{c}} &= \frac{1}{T^{1/2}}\left(\Delta\sin - \frac{\bar{c}}{T}\sin_{-1}\right)'\left(\mathbf{1}+\frac{\bar{c}}{T}\boldsymbol{\tau}\right) \\
&= \frac{1}{T^{1/2}}\left(\frac{\bar{c}}{T}\sum_{t=1}^{T}(t-1)\Delta\sin\left(\frac{2\pi\kappa\,(t)}{T}\right) - \left(\frac{\bar{c}}{T}\right)^2\sum_{t=1}^{T}(t-1)\sin\left(\frac{2\pi\kappa\,(t-1)}{T}\right)\right) \\
&= \frac{2\bar{c}\pi\kappa}{T^{5/2}}\sum_{t=1}^{T}(t-1)\cos\left(\frac{2\pi\kappa t}{T}\right) - \frac{\bar{c}^2}{T^{5/2}}\sum_{t=1}^{T}(t-1)\sin\left(\frac{2\pi\kappa\,(t-1)}{T}\right) \\
&= o(1); \qquad\qquad (A.22)
\end{aligned}$$

Therefore, $\frac{1}{T^{1/2}}\overline{\mathbf{Z}}'_{2,\bar{c}}\mathbf{Z}_{\sin,\bar{c}} \to 0$. Finally,

$$\begin{aligned}
\frac{1}{T^{1/2}}\overline{\mathbf{Z}}'_{2,\bar{c}}\overline{\mathbf{Z}}_{\cos,\bar{c}} &= \frac{1}{T^{1/2}}\left(\Delta\cos - \frac{\bar{c}}{T}\cos_{-1} + (1+\frac{\bar{c}}{T})\mathbf{e}_1\right)'\left(\mathbf{1}+\frac{\bar{c}}{T}\boldsymbol{\tau}\right) \\
&= \frac{1}{T^{1/2}}\left((1+\frac{\bar{c}}{T}) + \frac{\bar{c}}{T}\sum_{t=1}^{T}(t-1)\Delta\cos\left(\frac{2\pi\kappa t}{T}\right)\right. \\
&\quad \left. - \left(\frac{\bar{c}}{T}\right)^2\sum_{t=1}^{T}(t-1)\cos\left(\frac{2\pi\kappa\,(t-1)}{T}\right)\right) \\
&= \frac{1}{T^{1/2}}(1+\frac{\bar{c}}{T}) - \frac{2\bar{c}\pi\kappa}{T^{5/2}}\sum_{t=1}^{T}(t-1)\sin\left(\frac{2\pi\kappa t}{T}\right) \\
&\quad - \frac{\bar{c}^2}{T^{5/2}}\sum_{t=1}^{T}(t-1)\cos\left(\frac{2\pi\kappa\,(t-1)}{T}\right) \\
&= o(1) \qquad\qquad (A.23)
\end{aligned}$$

and $\frac{1}{T^{1/2}}\overline{\mathbf{Z}}'_{2,\bar{c}}\overline{\mathbf{Z}}_{\cos,\bar{c}} \to 0$. Hence, the results in (A.20) - (A.23) complete the proof of Lemma A.2.

Proof of Theorem 3

In order to derive the limit results of the second step estimators the following test regression with no augmentation is used,

$$\Delta y_t^{\bar{c}} = f_t(\kappa)'\boldsymbol{\varphi} + \phi y_{t-1}^{\bar{c}} + u_t.$$

Considering $\boldsymbol{\delta} = (\varphi_1, \varphi_2, \phi)$ and $V_t = (f_t(\kappa)', y_{t-1}^{c})'$, we look at the limit results of the scalled estimators,

$$\Lambda^{-1}\widehat{\boldsymbol{\delta}} = \left(\sum_{t=1}^{T}\Lambda V_t' V_t \Lambda\right)^{-1}\sum_{t=1}^{T}\Lambda V_t' \Delta y_t^{c}$$

where $\Lambda = diag\left\{\frac{1}{\sqrt{T}}, \frac{1}{\sqrt{T}}, \frac{1}{T}\right\}$.

Thus, consider first the denominator,

$$\sum_{t=1}^{T}\Lambda V_t'V_t\Lambda = \begin{bmatrix} \frac{1}{T}\sum_{t=1}^{T} f_t(\kappa)'f_t(\kappa) & \frac{1}{T^{3/2}}\sum_{t=1}^{T} f_t(\kappa)y_{t-1}^c \\ \frac{1}{T^{3/2}}\sum_{t=1}^{T} y_{t-1}^c f_t'(\kappa) & \frac{1}{T^2}\sum_{t=1}^{T}\left(y_{t-1}^c\right)^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{T}\sum_{t=1}^{T}\sin^2\left(\frac{2\pi\kappa t}{T}\right) & \frac{1}{T}\sum_{t=1}^{T}\sin\left(\frac{2\pi\kappa t}{T}\right)\cos\left(\frac{2\pi\kappa t}{T}\right) & \frac{1}{T^{3/2}}\sum_{t=1}^{T}\sin\left(\frac{2\pi\kappa t}{T}\right)y_{t-1}^c \\ \frac{1}{T}\sum_{t=1}^{T}\sin\left(\frac{2\pi\kappa t}{T}\right)\cos\left(\frac{2\pi\kappa t}{T}\right) & \frac{1}{T}\sum_{t=1}^{T}\cos^2\left(\frac{2\pi\kappa t}{T}\right) & \frac{1}{T^{3/2}}\sum_{t=1}^{T}\cos\left(\frac{2\pi\kappa t}{T}\right)y_{t-1}^c \\ \frac{1}{T^{3/2}}\sum_{t=1}^{T}\sin\left(\frac{2\pi\kappa t}{T}\right)y_{t-1}^c & \frac{1}{T^{3/2}}\sum_{t=1}^{T}\cos\left(\frac{2\pi\kappa t}{T}\right)y_{t-1}^c & \frac{1}{T^2}\sum_{t=1}^{T}\left(y_{t-1}^c\right)^2 \end{bmatrix}$$

the following Lemma with the necessary limit results can be stated.

Lemma A.3 *Considering the DGP (1)-(2) under the null hypothesis, $H_0 : \rho = 1$, and assumption A (k fixed), it follows as $T \to \infty$ that,*

$$i) \quad \frac{1}{T}\sum_{t=1}^{T}\sin^2\left(\frac{2\pi\kappa t}{T}\right) \to \int_0^1 \sin^2\left(2\pi\kappa r\right)dr$$

$$ii) \quad \frac{1}{T}\sum_{t=1}^{T}\sin\left(\frac{2\pi\kappa t}{T}\right)\cos\left(\frac{2\pi\kappa t}{T}\right) \to 0$$

$$iii) \quad \frac{1}{T}\sum_{t=1}^{T}\cos^2\left(\frac{2\pi\kappa t}{T}\right) \to \int_0^1 \cos^2\left(2\pi\kappa r\right)dr$$

$$iv) \quad \frac{1}{T^{3/2}}\sum_{t=1}^{T}\sin\left(\frac{2\pi\kappa t}{T}\right)y_{t-1}^c \Rightarrow \sigma\int_0^1 \sin\left(2\pi\kappa r\right)W(r)dr - \sigma\mathbf{\Xi}_2\int_0^1 r\sin\left(2\pi\kappa r\right)dr$$

$$v) \quad \frac{1}{T^{3/2}}\sum_{t=1}^{T}\cos\left(\frac{2\pi\kappa t}{T}\right)y_{t-1}^c \Rightarrow \sigma\int_0^1 \cos\left(2\pi\kappa r\right)W(r)dr - \sigma\mathbf{\Xi}_2\int_0^1 r\cos\left(2\pi\kappa r\right)dr$$

$$vi) \quad \frac{1}{T^2}\sum_{t=1}^{T}\left(y_{t-1}^c\right)^2 \Rightarrow \sigma^2\int_0^1 W(r)^2dr - 2\sigma^2\mathbf{\Xi}_2\int_0^1 rW(r)dr + \frac{\sigma^2}{3}\mathbf{\Xi}_2^2.$$

where $\mathbf{\Xi}_2 = \frac{\mathfrak{W}}{1+c+\frac{c^2}{3}}$ *and* $\mathfrak{W} = (1-\bar{c})W(1) - 2\bar{c}\int_0^1 W(r)dr - \bar{c}^2\int_0^1 rW(r)dr.$

Proof of Lemma A.3

Noting that $\Delta y_t^{\bar{c}} = \Delta y_t - \Delta\mathbf{Z}_t\widehat{\boldsymbol{\alpha}}_{\bar{c}}$, where $\Delta y_t = \alpha_1 + \alpha_2\Delta\sin\left(\frac{2\pi\kappa t}{T}\right) + \alpha_3\Delta\cos\left(\frac{2\pi\kappa t}{T}\right) + \Delta x_t$ and $\Delta\mathbf{Z}_t = (0,1)'$. Consequently, $\Delta y_t^{\bar{c}} = \Delta y_t - \Delta\mathbf{Z}_t\widehat{\boldsymbol{\alpha}}_{\bar{c}} = \alpha_2\Delta\sin\left(\frac{2\pi\kappa t}{T}\right) + \alpha_3\Delta\cos\left(\frac{2\pi\kappa t}{T}\right) + \Delta x_t - \left(\widehat{\alpha}_{1,\bar{c}} - \alpha_1\right)$. Moreover,

$$
\begin{aligned}
y_t^{\bar{c}} &= y_t - \mathbf{Z}_t \widehat{\boldsymbol{\alpha}}_{\bar{c}} \\
&= \alpha_2 \sin\left(\frac{2\pi\kappa t}{T}\right) + \alpha_3 \cos\left(\frac{2\pi\kappa t}{T}\right) + x_t - \mathbf{Z}_t \left(\widehat{\boldsymbol{\alpha}}_{\bar{c}} - \boldsymbol{\alpha}\right).
\end{aligned}
$$

Thus, regarding the result in iv) it follows that,

$$
\begin{aligned}
& \frac{1}{T^{3/2}} \sum \sin\left(\frac{2\pi\kappa t}{T}\right) y_{t-1}^{\bar{c}} \\
= \ & \frac{1}{T^{3/2}} \sum \sin\left(\frac{2\pi\kappa t}{T}\right) \left[\alpha_2 \sin\left(\frac{2\pi\kappa (t-1)}{T}\right) + \alpha_3 \cos\left(\frac{2\pi\kappa (t-1)}{T}\right) + x_{t-1} - \mathbf{Z}_{t-1} \left(\widehat{\boldsymbol{\alpha}}_{\bar{c}} - \boldsymbol{\alpha}\right)\right] \\
= \ & \frac{1}{T^{3/2}} \sum \sin\left(\frac{2\pi\kappa t}{T}\right) x_{t-1} - \frac{1}{T^{3/2}} \sum \sin\left(\frac{2\pi\kappa t}{T}\right) \mathbf{Z}_{t-1} \left(\widehat{\boldsymbol{\alpha}}_{\bar{c}} - \boldsymbol{\alpha}\right) + o(1) \\
= \ & \frac{1}{T^{3/2}} \sum \sin\left(\frac{2\pi\kappa t}{T}\right) x_{t-1} - \frac{1}{T^{2}} \sum \sin\left(\frac{2\pi\kappa t}{T}\right) \mathbf{Z}_{t-1} \sqrt{T} \left(\widehat{\boldsymbol{\alpha}}_{\bar{c}} - \boldsymbol{\alpha}\right) + o(1) \\
\Rightarrow \ & \sigma \int_0^1 \sin\left(2\pi\kappa r\right) W(r) dr - \sigma \boldsymbol{\Xi}_2 \int_0^1 r \sin\left(2\pi\kappa r\right) dr \qquad \text{(A.24)}
\end{aligned}
$$

Note that the result $\frac{1}{T^{3/2}} \sum \sin\left(\frac{2\pi\kappa t}{T}\right) x_{t-1} \Rightarrow \sigma \int_0^1 \sin\left(2\pi\kappa r\right) W(r) dr$ follows from Bierens (1997, Lemma A.5). Similarly, for v), we observe that,

$$
\begin{aligned}
& \frac{1}{T^{3/2}} \sum \cos\left(\frac{2\pi\kappa t}{T}\right) y_{t-1}^{\bar{c}} \\
= \ & \frac{1}{T^{3/2}} \sum \cos\left(\frac{2\pi\kappa t}{T}\right) \left[\alpha_2 \sin\left(\frac{2\pi\kappa (t-1)}{T}\right) + \alpha_3 \cos\left(\frac{2\pi\kappa (t-1)}{T}\right) + x_{t-1} - \mathbf{Z}_{t-1} \left(\widehat{\boldsymbol{\alpha}}_{\bar{c}} - \boldsymbol{\alpha}\right)\right] \\
\Rightarrow \ & \sigma \int_0^1 \cos\left(2\pi\kappa r\right) W(r) dr - \sigma \boldsymbol{\Xi}_2 \int_0^1 r \cos\left(2\pi\kappa r\right) dr. \qquad \text{(A.25)}
\end{aligned}
$$

Note that in order to prove (A.24) and (A.25) the following results proved quite useful.

$$
\begin{aligned}
& \frac{1}{T} \sum_{t=1}^{T} \sin\left(\frac{2\pi\kappa t}{T}\right) \sin\left(\frac{2\pi\kappa(t-1)}{T}\right) \\
= \ & \frac{1}{T} \sum_{t=1}^{T} \frac{1}{2} \left(\cos\left(\frac{2\pi\kappa t}{T} - \frac{2\pi\kappa(t-1)}{T}\right) - \cos\left(\frac{2\pi\kappa t}{T} + \frac{2\pi\kappa(t-1)}{T}\right)\right) \\
= \ & \frac{1}{T} \sum_{t=1}^{T} \frac{1}{2} \left(\cos\left(\frac{2\pi\kappa}{T}\right) - \cos\left(\frac{4\pi\kappa t}{T} - \frac{2\pi\kappa}{T}\right)\right) \\
= \ & \frac{1}{T} \sum_{t=1}^{T} \frac{1}{2} \left(\cos\left(\frac{2\pi\kappa}{T}\right) - \cos\left(\frac{4\pi\kappa t}{T}\right) \cos\left(\frac{2\pi\kappa}{T}\right) - \sin\left(\frac{4\pi\kappa t}{T}\right) \sin\left(\frac{2\pi\kappa}{T}\right)\right) \\
= \ & \frac{\cos\left(\frac{2\pi\kappa}{T}\right)}{T} \sum_{t=1}^{T} \frac{1}{2} \left(1 - \cos\left(\frac{4\pi\kappa t}{T}\right)\right) - \frac{\sin\left(\frac{2\pi\kappa}{T}\right)}{T} \sum_{t=1}^{T} \frac{1}{2} \sin\left(\frac{4\pi\kappa t}{T}\right)
\end{aligned}
$$

Since,

$$\frac{\cos\left(\frac{2\pi\kappa}{T}\right)}{T}\sum_{t=1}^{T}\frac{1}{2}\left(1-\cos\left(\frac{4\pi\kappa t}{T}\right)\right)\rightarrow\frac{1}{2}$$

and

$$\frac{\sin\left(\frac{2\pi\kappa}{T}\right)}{T}\sum_{t=1}^{T}\frac{1}{2}\sin\left(\frac{4\pi\kappa t}{T}\right)\rightarrow 0,$$

thus,

$$\frac{1}{T}\sum_{t=1}^{T}\sin\left(\frac{2\pi\kappa t}{T}\right)\sin\left(\frac{2\pi\kappa(t-1)}{T}\right)\rightarrow\frac{1}{2}.$$

Furthermore,

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T}\sin\left(\frac{2\pi\kappa t}{T}\right)\cos\left(\frac{2\pi\kappa(t-1)}{T}\right)\\
=\ &\frac{1}{T}\sum_{t=1}^{T}\frac{1}{2}\left(\sin\left(\frac{4\pi\kappa t}{T}-\frac{2\pi\kappa}{T}\right)+\sin\left(\frac{2\pi\kappa}{T}\right)\right)\\
=\ &\frac{1}{T}\sum_{t=1}^{T}\frac{1}{2}\left(\sin\left(4\frac{\pi}{T}\kappa t\right)\cos\left(2\frac{\pi}{T}\kappa\right)+(1-\cos\left(4\frac{\pi}{T}\kappa t\right))\sin\left(2\frac{\pi}{T}\kappa\right)\right).
\end{aligned}$$

Since,

$$\frac{1}{T}\sum_{t=1}^{T}\frac{1}{2}\left(\sin\left(4\frac{\pi}{T}\kappa t\right)\cos\left(2\frac{\pi}{T}\kappa\right)\right)\rightarrow 0$$

and

$$\frac{1}{T}\sum_{t=1}^{T}\frac{1}{2}(1-\cos\left(4\frac{\pi}{T}\kappa t\right))\sin\left(2\frac{\pi}{T}\kappa\right)=\frac{1}{2}\frac{2\pi\kappa}{T}=\frac{\pi\kappa}{T}\rightarrow 0$$

it follows that

$$\frac{1}{T}\sum_{t=1}^{T}\sin\left(\frac{2\pi\kappa t}{T}\right)\cos\left(\frac{2\pi\kappa(t-1)}{T}\right)\rightarrow 0.$$

Finally, with respect to vi),

$$\begin{aligned}
\frac{1}{T^2}\sum\left(y_{t-1}^{\bar{c}}\right)^2 &= \frac{1}{T^2}\sum\left(\alpha_2\sin\left(\frac{2\pi\kappa\,(t-1)}{T}\right)+\alpha_3\cos\left(\frac{2\pi\kappa\,(t-1)}{T}\right)+x_{t-1}-\mathbf{Z}_{t-1}\left(\widehat{\boldsymbol{\alpha}}_{\bar{c}}-\boldsymbol{\alpha}\right)\right)^2\\
&= \frac{1}{T^2}\sum\left(x_{t-1}-\mathbf{Z}_{t-1}\left(\widehat{\boldsymbol{\alpha}}_{\bar{c}}-\boldsymbol{\alpha}\right)\right)^2+o(1)\\
&= \frac{1}{T^2}\sum x_{t-1}^2-\frac{2\left(\widehat{\boldsymbol{\alpha}}_{\bar{c}}-\boldsymbol{\alpha}\right)}{T^2}\sum x_{t-1}\mathbf{Z}_{t-1}+\frac{\left(\widehat{\boldsymbol{\alpha}}_{\bar{c}}-\boldsymbol{\alpha}\right)^2}{T^2}\sum\mathbf{Z}_{t-1}'\mathbf{Z}_{t-1}'+o(1)\\
&\Rightarrow \sigma^2\int_0^1 W(r)^2dr-2\sigma^2\mathbf{\Xi}_2\int_0^1 rW(r)dr+\frac{\sigma^2}{3}\mathbf{\Xi}_2^2.
\end{aligned}$$

This last result follows since the squares and cross products of the sine and cosine terms are all of at most order $O(T)$.■

Regarding the limits of $\Lambda V_t' \Delta y_t^c$ note that,

$$\begin{aligned}
\Lambda V_t' \Delta y_t^c &= \begin{bmatrix} \frac{1}{\sqrt{T}} \sum_{t=1}^T \sin\left(\frac{2\pi\kappa t}{T}\right) \Delta y_t^c \\ \frac{1}{\sqrt{T}} \sum_{t=1}^T \cos\left(\frac{2\pi\kappa t}{T}\right) \Delta y_t^c \\ \frac{1}{T} \sum_{t=1}^T y_{t-1}^c \Delta y_t^c \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{T}} \sum_{t=1}^T \sin\left(\frac{2\pi\kappa t}{T}\right) \left(\alpha_2 \Delta \sin\left(\frac{2\pi\kappa t}{T}\right) + \alpha_3 \Delta \cos\left(\frac{2\pi\kappa t}{T}\right) + \Delta x_t - (\widehat{\alpha}_{1,\bar{c}} - \alpha_1)\right) \\ \frac{1}{\sqrt{T}} \sum_{t=1}^T \cos\left(\frac{2\pi\kappa t}{T}\right) \left(\alpha_2 \Delta \sin\left(\frac{2\pi\kappa t}{T}\right) + \alpha_3 \Delta \cos\left(\frac{2\pi\kappa t}{T}\right) + \Delta x_t - (\widehat{\alpha}_{1,\bar{c}} - \alpha_1)\right) \\ \frac{1}{T} \sum_{t=1}^T y_{t-1}^c \left(\alpha_2 \Delta \sin\left(\frac{2\pi\kappa t}{T}\right) + \alpha_3 \Delta \cos\left(\frac{2\pi\kappa t}{T}\right) + \Delta x_t - (\widehat{\alpha}_{1,\bar{c}} - \alpha_1)\right) \end{bmatrix}.
\end{aligned}$$

Lemma A.4 *Considering the DGP (1)-(2) under the null hypothesis, $H_0 : \rho = 1$, and k fixed, it follows as $T \to \infty$ that,*

$$\begin{aligned}
\frac{1}{\sqrt{T}} \sum_{t=1}^T \sin\left(\frac{2\pi\kappa t}{T}\right) \Delta y_t^c &\Rightarrow \sigma \left(\sin(2\pi\kappa) W(1) - 2\pi\kappa \int_0^1 \cos(2\pi\kappa r) W(r) dr \right) \\
\frac{1}{\sqrt{T}} \sum_{t=1}^T \cos\left(\frac{2\pi\kappa t}{T}\right) \Delta y_t^c &\Rightarrow \sigma \left(W(1) + 2\pi\kappa \int_0^1 \sin(2\pi\kappa r) W(r) dr \right) \\
\frac{1}{T} \sum_{t=1}^T y_{t-1}^c \Delta y_t^c &\Rightarrow \sigma^2 \left(\int_0^1 W(r) dW(r) - \mathbf{\Xi}_2 \int_0^1 r dW(r) \right)
\end{aligned}$$

Proof of Lemma A.4

Following Enders and Lee (2004), we can establish that

$$\Delta \sin\left(\frac{2\pi\kappa t}{T}\right) = \frac{2\pi\kappa}{T} \cos\left(\frac{2\pi\kappa t}{T}\right)$$

and

$$\Delta \cos\left(\frac{2\pi\kappa t}{T}\right) = -\frac{2\pi\kappa}{T} \sin\left(\frac{2\pi\kappa t}{T}\right)$$

from which it follows that,

$$\Lambda V_t' \Delta y_t^c = \begin{bmatrix} \frac{1}{\sqrt{T}} \sum_{t=1}^T \sin\left(\frac{2\pi\kappa t}{T}\right) \left(\Delta x_t - (\widehat{\alpha}_{1,\bar{c}} - \alpha_1)\right) + o(1) \\ \frac{1}{\sqrt{T}} \sum_{t=1}^T \cos\left(\frac{2\pi\kappa t}{T}\right) \left(\Delta x_t - (\widehat{\alpha}_{1,\bar{c}} - \alpha_1)\right) + o(1) \\ \frac{1}{T} \sum_{t=1}^T y_{t-1}^c \left(\alpha_2 \Delta \sin\left(\frac{2\pi\kappa t}{T}\right) + \alpha_3 \Delta \cos\left(\frac{2\pi\kappa t}{T}\right) + \Delta x_t - (\widehat{\alpha}_{1,\bar{c}} - \alpha_1)\right) \end{bmatrix}.$$

Furthermore, since $\sum_{t=1}^T \sin\left(\frac{2\pi\kappa t}{T}\right) = \sum_{t=1}^T \cos\left(\frac{2\pi\kappa t}{T}\right) = 0$ when κ is an intenger, this still simplifies to

$$\Lambda V_t' \Delta y_t^c = \begin{bmatrix} \frac{1}{\sqrt{T}} \sum_{t=1}^T \sin\left(\frac{2\pi\kappa t}{T}\right) \Delta x_t + o(1) \\ \frac{1}{\sqrt{T}} \sum_{t=1}^T \cos\left(\frac{2\pi\kappa t}{T}\right) \Delta x_t + o(1) \\ \frac{1}{T} \sum_{t=1}^T y_{t-1}^c \Delta x_t - \sqrt{T} \left(\widehat{\alpha}_{1,\bar{c}} - \alpha_1\right) \frac{1}{T^{3/2}} \sum_{t=1}^T y_{t-1}^c + o_p(1) \end{bmatrix}. \tag{A.26}$$

Making use of the result in Bierens (1994, Lemma 9.6.3) - refered to earlier in the proof of theorem 3.1 - we obtain,

$$\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\cos\left(\frac{2\pi\kappa t}{T}\right)u_t \Rightarrow \sigma\left(W(1)+2\pi\kappa\int_0^1 \sin\left(2\pi\kappa r\right)W(r)dr\right)$$

and

$$\frac{1}{\sqrt{T}}\sum_{t=2}^{T}\sin\left(\frac{2\pi\kappa t}{T}\right)u_t \Rightarrow \sigma\left(\sin\left(2\pi\kappa\right)W(1)-2\pi\kappa\int_0^1 \cos\left(2\pi\kappa r\right)W(r)dr\right).$$

Regarding the last element of the vector (A.26), recall that

$$y_{t-1}^{c}=\left(\alpha_2\sin\left(\frac{2\pi\kappa\left(t-1\right)}{T}\right)+\alpha_3\cos\left(\frac{2\pi\kappa\left(t-1\right)}{T}\right)+x_{t-1}-\mathbf{Z}_{t-1}\left(\widehat{\boldsymbol{\alpha}}_{\overline{c}}-\boldsymbol{\alpha}\right)\right),$$

from which we observe that

$$\begin{aligned}\frac{1}{T^{3/2}}\sum_{t=1}^{T}y_{t-1}^{c} &= \frac{1}{T^{3/2}}\sum_{t=1}^{T}\left(x_{t-1}-\mathbf{Z}_{t-1}\left(\widehat{\boldsymbol{\alpha}}_{\overline{c}}-\boldsymbol{\alpha}\right)\right)\\ &\Rightarrow \sigma\left(\int_0^1 W(r)dr-\boldsymbol{\Xi}_2\frac{1}{2}\right)\end{aligned}$$

Thus,

$$\sqrt{T}\left(\widehat{\alpha}_{1,\overline{c}}-\alpha_1\right)\frac{1}{T^{3/2}}\sum_{t=1}^{T}y_{t-1}^{c}\Rightarrow\sigma^2\boldsymbol{\Xi}_2\left(\int_0^1 W(r)dr-\boldsymbol{\Xi}_2\frac{1}{2}\right).$$

Similarly,

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}y_{t-1}^{c}\Delta x_t &= \frac{1}{T}\sum_{t=1}^{T}\left(x_{t-1}-\mathbf{Z}_{t-1}\left(\widehat{\boldsymbol{\alpha}}_{\overline{c}}-\boldsymbol{\alpha}\right)\right)\Delta x_t\\ &\Rightarrow \sigma^2\left(\int_0^1 W(r)dW(r)-\boldsymbol{\Xi}_2\int_0^1 rdW(r)\right).\end{aligned}$$

■

Table 1: Critical Values for $t_\phi^{ERS_f}$, $t_\phi^{DF_f}$ *and* $t_\phi^{LM_f}$ Unit Root Tests

			$t_\phi^{ERS_f^\mu}$			$t_\phi^{ERS_f^\tau}$			$t_\phi^{DF_f^\mu}$			$t_\phi^{DF_f^\tau}$			$t_\phi^{LM_f}$	
T	κ	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10	0.01	0.05	0.10
	1	-3.778	-3.128	-2.755	-4.681	-4.090	-3.792	-4.470	-3.862	-3.531	-4.988	-4.377	-4.073	-4.689	-4.137	-3.830
	2	-3.249	-2.535	-2.137	-4.231	-3.599	-3.273	-3.914	-3.267	-2.907	-4.662	-3.994	-3.688	-4.184	-3.545	-3.218
100	3	-3.087	-2.319	-1.963	-4.014	-3.347	-3.014	-3.725	-3.068	-2.718	-4.413	-3.775	-3.449	-4.024	-3.304	-2.976
	4	-2.905	-2.236	-1.891	-3.898	-3.210	-2.889	-3.632	-2.965	-2.637	-4.281	-3.638	-3.283	-3.846	-3.185	-2.880
	5	-2.870	-2.185	-1.868	-3.789	-3.136	-2.819	-3.546	-2.926	-2.611	-4.205	-3.535	-3.229	-3.762	-3.122	-2.815
					-3.58	**-3.03**	**-2.74**	**-3.51**	**-2.89**	**-2.58**	**-4.05**	**-3.45**	**-3.15**	**-3.63**	**-3.06**	**-2.77**
	1	-3.710	-3.081	-2.712	-4.526	-3.982	-3.690	-4.381	-3.811	-3.495	-4.863	-4.318	-4.016	-4.618	-4.077	-3.800
200	2	-3.248	-2.432	-2.055	-4.164	-3.540	-3.195	-3.956	-3.265	-2.902	-4.639	-4.055	-3.694	-4.228	-3.577	-3.234
	3	-2.961	-2.197	-1.868	-3.980	-3.284	-2.935	-3.745	-3.054	-2.712	-4.437	-3.781	-3.450	-4.012	-3.335	-2.969
	4	-2.853	-2.158	-1.813	-3.835	-3.162	-2.828	-3.621	-2.974	-2.650	-4.278	-3.677	-3.333	-3.901	-3.221	-2.884
	5	-2.831	-2.113	-1.778	-3.731	-3.068	-2.756	-3.590	-2.932	-2.613	-4.229	-3.586	-3.258	-3.782	-3.146	-2.827
					-3.46	**-2.93**	**-2.64**	**-3.46**	**-2.88**	**-2.57**	**-3.98**	**-3.42**	**-3.13**	**-3.61**	**-3.04**	**-2.76**
	1	-3.585	-2.970	-2.610	-4.432	-3.901	-3.633	-4.367	-3.764	-3.467	-4.844	-4.268	-4.008	-4.559	-4.032	-3.764
	2	-3.044	-2.353	-1.957	-4.048	-3.422	-3.093	-3.836	-3.260	-2.912	-4.562	-3.979	-3.666	-4.142	-3.540	-3.200
1000	3	-2.900	-2.159	-1.792	-3.814	-3.212	-2.862	-3.667	-3.064	-2.712	-4.340	-3.757	-3.419	-3.925	-3.298	-2.960
	4	-2.769	-2.067	-1.723	-3.691	-3.088	-2.754	-3.608	-2.976	-2.664	-4.219	-3.616	-3.293	-3.817	-3.195	-2.872
	5	-2.739	-2.018	-1.691	-3.600	-3.011	-2.685	-3.554	-2.926	-2.611	-4.159	-3.558	-3.244	-3.752	-3.125	-2.822
					-3.48	**-2.89**	**-2.57**	**-3.43**	**-2.86**	**-2.57**	**-3.96**	**-3.41**	**-3.13**	**-3.58**	**-3.02**	**-2.75**

Note: The results in bold correspond to the critical values of the unit root test procedures as originally proposed, by Elliott, et al. (1996, Table I, p.825), Fuller (1996, Table 10.A.2, p.642) and Schmidt and Phillips (1992, Table IA, p.264), i.e. with no Fourier terms. The critical values for the test statistics $t_\phi^{ERS_f^\mu}$ and $t_\phi^{DF_f^\mu}$ were computed from test regressions with Fourier terms and a constant only, whereas the critical values for the test statistics $t_\phi^{ERS_f^\tau}$, $t_\phi^{DF_f^\tau}$ and $t_\phi^{LM_f}$ were computed from test regressions with Fourier terms, a constant and a time trend. 0.01, 0.05 and 0.10 correspond the 1%, 5% and 10% percentiles, respectively

Table 2: Empirical Size of Conventional ERS ($t_\phi^{ERS^\tau}$), DF ($t_\phi^{DF^\tau}$) and LM Type ($t_\phi^{LM^\tau}$) Unit Root Tests

			$T = 100$			$T = 200$		
κ	α_1	α_2	$t_\phi^{ERS^\tau}$	$t_\phi^{DF^\tau}$	$t_\phi^{LM^\tau}$	$t_\phi^{ERS^\tau}$	$t_\phi^{DF^\tau}$	$t_\phi^{LM^\tau}$
1	0	0	.051	.049	.050	.050	.051	.050
	0	5	.005	.007	.005	.011	.016	.016
	3	0	.024	.037	.021	.030	.042	.035
	3	5	.003	.003	.003	.009	.012	.013
2	0	0	.051	.049	.050	.050	.051	.050
	0	5	.000	.001	.000	.001	.004	.002
	3	0	.004	.010	.003	.016	.023	.013
	3	5	.000	.000	.000	.000	.000	.000
3	0	0	.051	.049	.050	.050	.051	.050
	0	5	.000	.001	.000	.000	.002	.000
	3	0	.001	.002	.001	.007	.010	.007
	3	5	.000	.000	.000	.000	.000	.000
4	0	0	.051	.049	.050	.050	.051	.050
	0	5	.000	.000	.000	.000	.002	.000
	3	0	.000	.001	.000	.005	.004	.002
	3	5	.000	.000	.000	.000	.000	.000
5	0	0	.051	.049	.050	.050	.051	.050
	0	5	.000	.000	.000	.000	.002	.000
	3	0	.000	.000	.000	.002	.003	.002
	3	5	.000	.000	.000	.000	.000	.000

Note: All results are based on a 5% nominal significance level. Critical values used for the $t_\phi^{ERS^\tau}$, $t_\phi^{DF^\tau}$ and $t_\phi^{LM^\tau}$ unit root tests were obtained from Elliott, *et al.* (1996, Table I, p.825), Fuller (1996, Table 10.A.2, p.642) and Schmidt and Phillips (1992, p.264).

Table 3: Empirical Power of Conventional ERS ($t_\phi^{ERS^\tau}$), DF ($t_\phi^{DF^\tau}$) and LM Type ($t_\phi^{LM^\tau}$) Unit Root Tests

			$T = 100$			$T = 200$		
κ	α_1	α_2	$t_\phi^{ERS^\tau}$	$t_\phi^{DF^\tau}$	$t_\phi^{LM^\tau}$	$t_\phi^{ERS^\tau}$	$t_\phi^{DF^\tau}$	$t_\phi^{LM^\tau}$
1	0	0	.557	.384	.497	.559	.393	.511
	0	5	.000	.000	.000	.000	.002	.002
	3	0	.085	.091	.044	.196	.185	.146
	3	5	.000	.000	.000	.001	.001	.001
2	0	0	.557	.384	.497	.559	.393	.511
	0	5	.000	.000	.000	.000	.001	.001
	3	0	.021	.008	.012	.111	.063	.088
	3	5	.000	.000	.000	.000	.000	.000
3	0	0	.557	.384	.497	.559	.393	.511
	0	5	.000	.000	.000	.000	.000	.000
	3	0	.007	.003	.006	.079	.027	.059
	3	5	.000	.000	.000	.000	.000	.000
4	0	0	.557	.384	.497	.559	.393	.511
	0	5	.000	.000	.000	.000	.000	.000
	3	0	.004	.001	.003	.059	.015	.043
	3	5	.000	.000	.000	.000	.000	.000
5	0	0	.557	.384	.497	.559	.393	.511
	0	5	.000	.000	.000	.000	.000	.000
	3	0	.004	.001	.002	.049	.010	.033
	3	5	.000	.000	.000	.000	.000	.000

See note under Table 2.

Table 4: Empirical power of Fourier ERS, DF and LM Type Tests (Known κ)

		$t_{\phi}^{ERS_f^{\tau}}$				$t_{\phi}^{LM_f^{\tau}}$				$t_{\phi}^{DF_f^{\tau}}$			
		c				c				c			
T	κ	-5	-10	-15	-20	-5	-10	-15	-20	-5	-10	-15	-20
100	1	.069	.122	.231	.401	.065	.112	.210	.370	.063	.099	.182	.327
	2	.102	.230	.429	.649	.101	.218	.400	.604	.088	.172	.320	.520
	3	.108	.262	.507	.751	.104	.241	.459	.695	.088	.185	.361	.596
	4	.109	.275	.533	.784	.104	.252	.481	.721	.083	.187	.379	.627
	5	.111	.283	.546	.797	.104	.254	.488	.732	.089	.195	.402	.657
200	1	.069	.126	.232	.402	.067	.116	.215	.372	.062	.103	.181	.313
	2	.097	.214	.406	.632	.091	.199	.372	.580	.078	.144	.264	.440
	3	.103	.256	.496	.740	.095	.227	.442	.674	.085	.174	.335	.560
	4	.104	.270	.523	.772	.097	.239	.464	.707	.080	.169	.346	.574
	5	.109	.286	.550	.801	.099	.245	.480	.729	.080	.174	.360	.596

Table 5: Empirical Size and Power of Fourier ERS, DF and LM Type Tests (Estimated κ)

				$t_{\phi}^{ERS_f^{\tau}}$					$t_{\phi}^{LM_f^{\tau}}$					$t_{\phi}^{DF_f^{\tau}}$				
				c					c					c				
T	α_1	α_2	κ	0	−5	−10	−15	−20	0	−5	−10	−15	−20	0	−5	−10	−15	−20
100	0	0	1	.063	.264	.438	.651	.836	.060	.284	.439	.632	.799	.070	.213	.346	.542	.747
	0	5		.051	.070	.122	.231	.401	.051	.065	.112	.210	.370	.052	.064	.099	.182	.327
	3	0		.057	.080	.131	.231	.400	.054	.072	.115	.210	.365	.062	.081	.123	.207	.346
	3	5		.051	.070	.122	.231	.401	.051	.065	.112	.210	.370	.052	.063	.099	.182	.327
	0	0	2	.063	.264	.438	.651	.836	.060	.284	.439	.632	.799	.070	.213	.346	.541	.747
	0	5		.050	.102	.230	.429	.649	.050	.101	.218	.400	.604	.050	.088	.172	.320	.520
	3	0		.060	.113	.236	.429	.650	.061	.113	.223	.401	.604	.067	.105	.183	.325	.523
	3	5		.050	.102	.230	.429	.649	.050	.101	.218	.400	.604	.050	.088	.172	.320	.520
	0	0	3	.063	.264	.438	.651	.836	.060	.284	.439	.632	.799	.070	.213	.346	.542	.747
	0	5		.050	.108	.262	.507	.751	.050	.104	.241	.459	.695	.050	.088	.185	.361	.596
	3	0		.031	.112	.264	.507	.752	.032	.106	.244	.460	.695	.032	.091	.187	.363	.597
	3	5		.050	.109	.262	.507	.751	.050	.104	.241	.459	.695	.050	.088	.185	.361	.596
	0	0	4	.063	.264	.438	.651	.836	.060	.284	.439	.632	.799	.070	.213	.346	.542	.747
	0	5		.050	.109	.275	.533	.784	.050	.104	.252	.481	.721	.050	.084	.187	.379	.627
	3	0		.038	.084	.277	.533	.784	.038	.082	.253	.481	.721	.038	.065	.187	.379	.627
	3	5		.050	.109	.275	.533	.784	.050	.104	.252	.481	.721	.050	.083	.187	.379	.627
	0	0	5	.063	.264	.438	.651	.836	.060	.284	.439	.632	.799	.070	.213	.346	.542	.747
	0	5		.050	.111	.283	.546	.797	.050	.104	.254	.488	.732	.050	.089	.195	.402	.657
	3	0		.043	.096	.284	.546	.797	.043	.092	.255	.488	.732	.040	.074	.196	.402	.657
	3	5		.050	.111	.283	.546	.797	.050	.104	.254	.488	.732	.050	.089	.195	.402	.657

Table 6: Empirical Size and Power of Fourier ERS, DF and LM Type Tests (Estimated κ)

				$t_{\phi}^{ERS_f}$					$t_{\phi}^{LM_f}$					$t_{\phi}^{DF_f}$				
				c					c					c				
T	α_1	α_2	κ	0	-5	-10	-15	-20	0	-5	-10	-15	-20	0	-5	-10	-15	-20
200	0	0	1	.068	.255	.413	.633	.816	.065	.262	.410	.609	.787	.073	.184	.302	.473	.678
	0	5		.055	.072	.126	.232	.402	.054	.069	.116	.215	.372	.056	.067	.104	.181	.313
	3	0		.058	.225	.339	.494	.674	.058	.232	.338	.482	.649	.068	.155	.244	.276	.543
	3	5		.054	.070	.127	.232	.402	.053	.068	.116	.215	.372	.054	.064	.103	.181	.313
	0	0	2	.068	.255	.413	.633	.816	.065	.262	.410	.609	.787	.073	.184	.302	.473	.678
	0	5		.052	.098	.215	.406	.632	.053	.092	.198	.372	.580	.052	.078	.145	.264	.440
	3	0		.081	.131	.241	.424	.638	.082	.126	.225	.387	.585	.078	.109	.176	.291	.456
	3	5		.050	.097	.214	.406	.632	.050	.091	.199	.372	.580	.050	.078	.144	.264	.440
	0	0	3	.068	.255	.413	.633	.816	.065	.262	.410	.609	.787	.073	.184	.302	.473	.678
	0	5		.050	.103	.256	.496	.740	.050	.095	.227	.442	.674	.050	.085	.174	.335	.560
	3	0		.043	.075	.285	.516	.747	.043	.075	.252	.458	.682	.041	.060	.199	.355	.571
	3	5		.050	.103	.256	.496	.740	.050	.095	.227	.442	.674	.050	.085	.174	.335	.560
	0	0	4	.068	.255	.413	.633	.816	.065	.262	.410	.609	.787	.073	.184	.302	.473	.678
	0	5		.039	.104	.270	.523	.772	.033	.097	.239	.464	.707	.040	.080	.169	.346	.574
	3	0		.026	.096	.230	.536	.778	.027	.090	.206	.476	.710	.023	.076	.152	.356	.580
	3	5		.050	.104	.270	.523	.772	.050	.097	.239	.464	.707	.050	.080	.169	.346	.574
	0	0	5	.068	.255	.413	.633	.816	.065	.262	.410	.609	.787	.073	.184	.302	.473	.678
	0	5		.040	.109	.286	.550	.801	.043	.099	.245	.480	.729	.041	.080	.174	.360	.596
	3	0		.038	.077	.254	.562	.806	.040	.073	.226	.490	.733	.035	.058	.155	.367	.600
	3	5		.050	.109	.286	.550	.801	.050	.099	.245	.480	.729	.050	.080	.174	.360	.596

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