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Local Identification in DSGE Models

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Abstract

The issue of parameter identification arises whenever structural models are estimated. This paper develops a simple condition for local identification in linearized DSGE models. The condition is necessary and sufficient for identification with likelihood-based methods under normality, or with limited information methods that utilize only second moments of the data. Using the methodology developed in the paper researchers can answer, prior to estimation, the following questions: which parameters are locally identified and which are not; is the identification failure due to data limitations, such as a lack of observations for some variables, or is it intrinsic to the structure of the model.

Keywords: DSGE models, Identification

JEL classification: C32, C51, C52, E32

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1 Introduction

Structural macroeconomic models are one of the important tools available to economic policy-makers. However, insofar as the quantitative implications of the models are of interest, it is crucial that the inputs to the models - their parameter values, have empirical credibility. This has prompted a very active research effort aimed at the estimation and empirical evaluation of dynamic stochastic general equilibrium (DSGE) models.

The empirical implications of DSGE models come from the restrictions they impose on the joint probability distribution of observed macroeconomic variables. Both full information methods, which use all of the restrictions, and limited information methods, which use only some of them, are applied in the literature to estimate the parameters of DSGE models. Whether estimation is possible in the first place is typically assumed and not verified formally. Examples of unidentified DSGE models can be found in Kim (2003), Beyer and Farmer (2004) and Cochrane (2007). While it has been recognized that the lack of identification is potentially a serious problem, the issue is rarely addressed in the empirical literature. This is partly due to the widespread use of Bayesian methods, which, as Canova and Sala (2009) point out, may serve to conceal identification problems when they exist. Another reason is that few models allow for direct verification of identifiability, as in the studies cited above. Larger models can only be solved numerically, thus making it impossible to derive explicitly the relationship between the deep parameters and the statistical model used to estimate them. As a result it appears that parameter identification can only be assessed indirectly and with the heavy use of numerical methods.

This paper presents a rank condition for local identification of the parameters in DSGE models. It is based on the observation that for local identification we only need the Jacobian matrix of the mapping from deep parameters to the parameters of the statistical model. Taking the latter to be the second moments of the observed data, I show that the Jacobian matrix can be derived analytically. The matrix having full rank is a necessary and sufficient condition for local identification when estimation is based on the second moments. Thus, the condition applies to likelihood-based estimation under normality, where information from all available second moments is used, as well as to limited information methods that utilize only some of the second moments. Examples of the latter approach include minimum distance estimation matching vector autoregression (VAR) parameters or impulse response coefficients.

Local identification by itself does not guarantee that a model is globally identified. Thus, even if the rank condition holds and the model is locally identified, there may exist multiple observationally equivalent model structures that cannot be distinguished with any amount of data. Unfortunately, it is generally impossible to establish global identification in non-linear models. Nevertheless, it is important to know if a model is locally identified for the following two reasons. First, local identification is sufficient for the asymptotic properties of classical estimators to hold (see Florens, Marimoutou, and

Péguin-Feissolle (2008)). Second, parameters that are globally unidentifiable everywhere in the parameter space, either because they do not appear in the likelihood function at all, or are indistinguishable from other parameters, are also locally unidentifiable. Problems of this nature are the most common cause for identification failures in DSGE models, including all of the examples mentioned above, and can be detected using the conditions developed in this paper.

An alternative approach for checking local identification in DSGE models, using the Fisher information matrix, was suggested in Iskrev (2008a). One advantage of the method proposed in this paper is that it is much easier to implement in practice. Furthermore, it applies to identification with limited as well as full information methods, unlike the information matrix approach which provides only a necessary condition for identification when limited information methods are used.

The remainder of the paper is organized as follows. In Section 2, I introduce the class of linearized DSGE models and the notation used throughout the paper. I also explain the relationship between the deep parameters and the statistical model used to identify them. The main result of the paper is in Section 3, where I present the rank condition for local identification and show that it can be evaluated analytically. I also discuss limitations of the data vs. purely model-related reasons for identification failures, and show how to distinguish between the two. In Section 4, I extend the rank condition to a limited information setting, and to estimation methods based on transformations of the second moments. The methodology is illustrated, in Section 5, with the help of a medium-scale DSGE model estimated by Smets and Wouters (2007). Concluding comments are given in Section 6.

2 DSGE Models

2.1 Structural model and reduced form

A DSGE model is summarized by a system of non-linear equations. Currently, most studies involving either simulation or estimation of DSGE models use linear approximations of the original models. That is, the model is first expressed in terms of stationary variables, and then linearized around the steady-state values of these variables. Once linearized, most DSGE models can be written in the following form:

$$\boldsymbol{\Gamma}_0(\boldsymbol{\theta})\boldsymbol{z}_t = \boldsymbol{\Gamma}_1(\boldsymbol{\theta})\,\mathrm{E}_t\,\boldsymbol{z}_{t+1} + \boldsymbol{\Gamma}_2(\boldsymbol{\theta})\boldsymbol{z}_{t-1} + \boldsymbol{\Gamma}_3(\boldsymbol{\theta})\boldsymbol{u}_t \tag{2.1}$$

where $\boldsymbol{z}_t$ is a $m-$dimensional vector of endogenous variables, and the structural shocks $\boldsymbol{u}_t$ are independent and identically distributed n-dimensional random vectors with $\mathrm{E}\,\boldsymbol{u}_t = \mathbf{0}$, $\mathrm{E}\,\boldsymbol{u}_t\boldsymbol{u}_t' = I_n$. The elements of the matrices $\boldsymbol{\Gamma}_0$, $\boldsymbol{\Gamma}_1$, $\boldsymbol{\Gamma}_2$ and $\boldsymbol{\Gamma}_3$ are functions of a $k-$dimensional vector of deep parameters $\boldsymbol{\theta}$, where $\boldsymbol{\theta}$ is a point in $\boldsymbol{\Theta} \subset \mathbb{R}^k$. The parameter space $\boldsymbol{\Theta}$ is defined as the set of all theoretically admissible values of $\boldsymbol{\theta}$.

There are several algorithms for solving linear rational expectations models (see

for instance Blanchard and Kahn (1980), Anderson and Moore (1985), Klein (2000), Christiano (2002), Sims (2002)). Depending on the value of $\boldsymbol{\theta}$, there may exist zero, one, or many stable solutions. Assuming that a unique solution exists, it can be cast in the following form

$$\boldsymbol{z}_t = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{z}_{t-1} + \boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{u}_t \tag{2.2}$$

where the $m \times m$ matrix $\boldsymbol{A}$ and the $m \times n$ matrix $\boldsymbol{B}$ are unique for each value of $\boldsymbol{\theta}$.

The model in (2.2) cannot be taken to the data directly since some of the variables in $\boldsymbol{z}_t$ are not observed. Instead, the solution of the model is expressed in a state space form, with transition equation given by (2.2), and a measurement equation

$$\boldsymbol{x}_t = \boldsymbol{C}(\boldsymbol{\theta})\boldsymbol{z}_t \tag{2.3}$$

where $\boldsymbol{x}_t$ is a l-dimensional vector of observed variables, and $\boldsymbol{C}$ is a $l \times m$ matrix, that may depend on $\boldsymbol{\theta}$.

Let $\boldsymbol{\Omega}(\boldsymbol{\theta}) = \boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{B}(\boldsymbol{\theta})'$, and $\boldsymbol{\tau} = [\mathrm{vec}(\boldsymbol{A})', \mathrm{vec}(\boldsymbol{C})', \mathrm{vech}(\boldsymbol{\Omega})']'$. To solve the linearized DSGE model in (2.1) means to find $\boldsymbol{\tau}$ given a value of $\boldsymbol{\theta}$.

2.2 Covariance structure

From (2.2)-(2.3) it follows that the unconditional first and second moments of $\boldsymbol{x}_t$ are given by

$$\mathrm{E}\,\boldsymbol{x}_t = \boldsymbol{0} \tag{2.4}$$

$$\mathrm{E}\,\boldsymbol{x}_{t+i}\boldsymbol{x}_t' = \Sigma_{\boldsymbol{x}}(i) \tag{2.5}$$

where[1]

$$\Sigma_{\boldsymbol{x}}(i) = \begin{cases} \boldsymbol{C}\Sigma_{\boldsymbol{z}}(0)\boldsymbol{C}' & \text{if } i = 0 \\ \boldsymbol{C}\boldsymbol{A}^i\Sigma_{\boldsymbol{z}}(0)\boldsymbol{C}' & \text{if } i > 0 \end{cases} \tag{2.6}$$

and $\Sigma_{\boldsymbol{z}}(0) = \mathrm{E}\,\boldsymbol{z}_t\boldsymbol{z}_t'$ solves the matrix equation

$$\Sigma_{\boldsymbol{z}}(0) = \boldsymbol{A}\Sigma_{\boldsymbol{z}}(0)\boldsymbol{A}' + \boldsymbol{\Omega} \tag{2.7}$$

[1]Notice that $\Sigma_{\boldsymbol{x}}(-i) = \Sigma_{\boldsymbol{x}}(i)'$.

Denote the observed data with $\boldsymbol{X}_T = [\boldsymbol{x}_1', \ldots, \boldsymbol{x}_T']'$, and let $\boldsymbol{\Sigma}_T$ be its covariance matrix, i.e.

$$\begin{aligned} \boldsymbol{\Sigma}_T &= \mathrm{E}\, \boldsymbol{X}_T \boldsymbol{X}_T' \\ &= \begin{pmatrix} \Sigma_{\boldsymbol{x}}(0), & \Sigma_{\boldsymbol{x}}(1)', & \ldots, & \Sigma_{\boldsymbol{x}}(T-1)' \\ \Sigma_{\boldsymbol{x}}(1), & \Sigma_{\boldsymbol{x}}(0), & \ldots, & \Sigma_{\boldsymbol{x}}(T-2)' \\ \ldots & \ldots & \ldots & \ldots \\ \Sigma_{\boldsymbol{x}}(T-1), & \Sigma_{\boldsymbol{x}}(T-2), & \ldots, & \Sigma_{\boldsymbol{x}}(0) \end{pmatrix} \end{aligned} \tag{2.8}$$

Assuming that the linearized DSGE model is determined everywhere in $\boldsymbol{\Theta}$, i.e. $\boldsymbol{\tau}$ is unique for each admissible value of $\boldsymbol{\theta}$, it follows that there exists a one-to-one relationship between the structural parameters $\boldsymbol{\theta}$ and the second moments of the data $\boldsymbol{\Sigma}_T$. In particular, if $\boldsymbol{\sigma}_T = [\mathrm{vech}(\Sigma_{\boldsymbol{x}}(0))', \mathrm{vec}(\Sigma_{\boldsymbol{x}}(1))', ..., \mathrm{vec}(\Sigma_{\boldsymbol{x}}(T-1))']'$ is a $(T-1)l^2 + l(l-1)/2$-dimensional vector collecting the unique elements of $\boldsymbol{\Sigma}_T$, then $\boldsymbol{\sigma}_T$ is a function of $\boldsymbol{\theta}$. If either $\boldsymbol{u}_t$ are Gaussian, or there are no distributional assumptions about the structural shocks, the model-implied restrictions on $\boldsymbol{\sigma}_T$ contain all information that can be used for the estimation of $\boldsymbol{\theta}$. The identifiability of $\boldsymbol{\theta}$ depends on whether the information is sufficient or not. This is the subject of the next section.

3 Identification

3.1 The rank condition

In a fully parametric setting a model is identified if different parameter values are associated with different probability distributions of the observed data. If the distribution is unknown and estimation is based on some statistics of the data, the model is identified by the estimation method if different parameter values imply different values of the population counterparts of the utilized statistics. Here I assume that the estimation of $\boldsymbol{\theta}$ is based on the second moments of the data collected in the vector $\boldsymbol{\sigma}_T$. Extending the analysis to functions of $\boldsymbol{\sigma}_T$ is straightforward, and will be discussed later.

Definition. *Suppose that the data $\boldsymbol{X}_T$ is generated by the model* (2.2)-(2.3) *with parameter vector $\boldsymbol{\theta}_0$. Then $\boldsymbol{\theta}_0$ is globally identified by the second moments of $\boldsymbol{X}_T$ if and only if*

$$\boldsymbol{\sigma}_T(\boldsymbol{\theta}) = \boldsymbol{\sigma}_T(\boldsymbol{\theta}_0) \Leftrightarrow \boldsymbol{\theta} = \boldsymbol{\theta}_0 \tag{3.1}$$

for any $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. If (3.1) *is true only for values $\boldsymbol{\theta}$ in an open neighborhood of $\boldsymbol{\theta}_0$, the identification of $\boldsymbol{\theta}_0$ is only local.*

Identifiability of the parameters $\boldsymbol{\theta}$ requires that the mapping from the population moments of the data - $\boldsymbol{\sigma}_T(\boldsymbol{\theta})$, to $\boldsymbol{\theta}$ is unique. If this is not the case, there exist different values of $\boldsymbol{\theta}$ that result in the same value of the population moments, and the true value of $\boldsymbol{\theta}$ cannot be determined even with an infinite number of observations. In general, there are no known global conditions for unique solutions of systems of non-linear equations,

and it is therefore difficult to show global identification of $\boldsymbol{\theta}$. Local identification, on the other hand, can be established with the help of the following condition

Theorem. *Suppose that $\boldsymbol{\sigma}_T$ is a continuously differentiable function of $\boldsymbol{\theta}$, and let $\boldsymbol{\theta}_0$ be a regular point of the Jacobian matrix $J(T) \equiv \frac{\partial \boldsymbol{\sigma}_T}{\partial \boldsymbol{\theta}'}$. Then $\boldsymbol{\theta}_0$ is locally identifiable if and only if $J(T)$ has a full column rank at $\boldsymbol{\theta}_0$.*

This result follows from Theorem 6 in Rothenberg (1971). A regular point of a matrix is a point around which there exists an open neighborhood where the rank of the matrix remains constant. For Jacobian matrices of analytic functions the set of irregular points has a measure of zero, and thus almost all points are regular (see Bekker and Pollock (1986)). Without this assumption the condition is only sufficient for local identification.

A necessary condition for identification is that the number of deep parameters does not exceed the number of unique parameters in the utilized second moments, i.e. $k \leq (T-1)l^2 + l(l-1)/2$. This is the usual order condition. A stronger necessary condition for identification will be presented in the next section.

The local identifiability of a point $\boldsymbol{\theta}_0$ can be established by verifying that the Jacobian matrix $J(T)$ has full column rank when evaluated at $\boldsymbol{\theta}_0$. Local identification at one point in $\boldsymbol{\Theta}$, however, does not guarantee that the model is locally identified everywhere in the parameter space. There may be some points where the model is locally identified, and others where it is not. Moreover, local identifiability everywhere in $\boldsymbol{\Theta}$ is necessary but not sufficient to ensure global identification. Nevertheless, as was pointed out already in Section 1, it is important to know whether a model is locally identified or not. Local identification makes possible the consistent estimation of $\boldsymbol{\theta}$, and is sufficient for the estimator to have the usual asymptotic properties. Perhaps more important in the context of DSGE models is that with the help of the Jacobian matrix we can detect problems that are a common cause for identification failures in these models. If, for instance, a deep parameter θ_j does not affect the solution of the model, it will be unidentifiable since its value is irrelevant for the statistical properties of the data generated by the model, and the second moments in particular. Consequently, $\frac{\partial \boldsymbol{\sigma}_T}{\partial \theta_j}$ - the column of $J(T)$ corresponding to θ_j, will be a vector of zeros for any T. Not only will the condition for local identification fail, but also it will be immediately clear which parameter is unidentified. Another type of identification failure occurs when two or more parameters enter in the solution in a manner which makes them indistinguishable, e.g. as a product or a ratio. As a result it is impossible to identify the parameters separately, and some of the columns of the Jacobian matrix are linearly dependent. An example of the first problem is the unidentifiability of the Taylor rule coefficients in a simple New Keynesian model pointed out in Cochrane (2007). An example of the second is the equivalence between the intertemporal and multisectoral investment adjustment cost parameters in Kim (2003). In these papers the problems are discovered by solving the models explicitly in terms of the deep parameters. That approach, however, is not feasible for larger models, which can only be solved numerically. As will be shown next, the Jacobian matrix in Theorem 3.1 is very easy to compute irrespectively of the size

of the model.

3.2 Computing the Jacobian matrix

Even when linearized DSGE models are typically impossible to solve analytically. This means that the mapping from $\boldsymbol{\theta}$ to $\boldsymbol{\tau}$ is not available in closed form. Since $\boldsymbol{\sigma}_T$ depends on $\boldsymbol{\theta}$ through $\boldsymbol{\tau}$, computing the Jacobian matrix by direct differentiation of $\boldsymbol{\sigma}_T$ can only be done by using numerical derivatives. Because of the strong nonlinearities involved, this may result in a very poor approximation of the true Jacobian matrix.

The use of numerical approximation can be avoided if $J(T)$ is computed using the chain rule, i.e.

$$J(T) = \underbrace{\frac{\partial \boldsymbol{\sigma}_T}{\partial \boldsymbol{\tau}'}}_{J_1(T)} \underbrace{\frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{\theta}'}}_{J_2} \tag{3.2}$$

Below I show that the two terms on the right-hand side in (3.2) can be computed analytically. In the derivations I use matrix derivative methods (see Magnus and Neudecker (1999) for more details), and the special matrices $\boldsymbol{K}_{mn}$, $\boldsymbol{D}_n$, and $\boldsymbol{D}_n^+$, with the following properties

$$\begin{aligned}
\boldsymbol{K}_{mn}\text{vec}(\boldsymbol{A}) &= \text{vec}(\boldsymbol{A}'), \text{ where } \boldsymbol{A} \text{ is } m \times n \text{ matrix} && (3.3)\\
\boldsymbol{D}_n\text{vech}(\boldsymbol{A}) &= \text{vec}(\boldsymbol{A}), \text{ where } \boldsymbol{A} \text{ is a symmetric } n \times n \text{ matrix} && (3.4)\\
\boldsymbol{D}_n^+\text{vec}(\boldsymbol{A}) &= \text{vech}(\boldsymbol{A}), \text{ where } \boldsymbol{A} \text{ is a symmetric } n \times n \text{ matrix} && (3.5)
\end{aligned}$$

$\boldsymbol{K}_{mn}$ is called the commutation matrix, and $\boldsymbol{D}_n$ - the duplication matrix. $\boldsymbol{D}_n^+$ is the Moore-Penroze inverse of $\boldsymbol{D}_n$. More about the properties of these matrices can be found in Magnus and Neudecker (1999).

3.2.1 Derivation of $J_1(T)$

The derivation of $J_1(T)$ is straightforward since from (2.6) and (2.7) we have each element of $\boldsymbol{\sigma}_T = [\text{vech}(\Sigma_x(0))', \text{vec}(\Sigma(1))', ..., \text{vec}(\Sigma(T-1))']'$ in terms of $\boldsymbol{A}(\boldsymbol{\tau})$, $\boldsymbol{C}(\boldsymbol{\tau})$, and $\boldsymbol{\Omega}(\boldsymbol{\tau})$. To simplify the presentation, the blocks of J_1 are derived separately as follows:

- for $\Sigma_{\boldsymbol{x}}(0)$ we have

$$\begin{aligned}
&\frac{\partial \text{vech}(\Sigma_{\boldsymbol{x}}(0))}{\partial \boldsymbol{\tau}'} = \boldsymbol{D}_l^+ \left(\Big(\boldsymbol{C}\Sigma_{\boldsymbol{z}}(0) \otimes \boldsymbol{I}_l \Big) + \Big(\boldsymbol{I}_l \otimes \boldsymbol{C}\Sigma_{\boldsymbol{z}}(0) \Big) \boldsymbol{K}_{lm} \right) \frac{\partial \text{vec}(\boldsymbol{C})}{\partial \boldsymbol{\tau}'} \qquad (3.6)\\
&+ \boldsymbol{D}_l^+ (\boldsymbol{C} \otimes \boldsymbol{C}) \boldsymbol{D}_m \frac{\partial \text{vech}(\Sigma_{\boldsymbol{z}}(0))}{\partial \boldsymbol{\tau}'}
\end{aligned}$$

- for $i = 1, \ldots, T-1$ the derivative of $\Sigma_{\boldsymbol{x}}(i)$ is

$$\frac{\partial \text{vec}(\Sigma_{\boldsymbol{x}}(i))}{\partial \boldsymbol{\tau}'} = \left(\Big(\boldsymbol{C}\Sigma_{\boldsymbol{z}}(0)\boldsymbol{A}^{i\prime} \otimes \boldsymbol{I}_l \Big) + \Big(\boldsymbol{I}_l \otimes \boldsymbol{C}\boldsymbol{A}^i\Sigma_{\boldsymbol{z}}(0) \Big) \boldsymbol{K}_{lm} \right) \frac{\partial \text{vec}(\boldsymbol{C})}{\partial \boldsymbol{\tau}'} \tag{3.7}$$
$$+ \Big(\boldsymbol{C}\Sigma_{\boldsymbol{z}}(0) \otimes \boldsymbol{C} \Big) \left(\sum_{s=1}^{i} (\boldsymbol{A}')^{i-s} \otimes \boldsymbol{A}^{s-1} \right) \frac{\partial \text{vec}(\boldsymbol{A})}{\partial \boldsymbol{\tau}'} + (\boldsymbol{C} \otimes \boldsymbol{C}\boldsymbol{A}^i)\boldsymbol{D}_m \frac{\partial \text{vech}(\Sigma_{\boldsymbol{z}}(0))}{\partial \boldsymbol{\tau}'}$$

- finally, the derivative of $\text{vech}(\Sigma_{\boldsymbol{z}}(0))$ used in (3.6) and (3.7) is

$$\frac{\partial \text{vech}(\Sigma_{\boldsymbol{z}}(0))}{\partial \boldsymbol{\tau}'} = \left(\boldsymbol{I}_{\frac{m(m+1)}{2}} - \boldsymbol{D}_m^+(\boldsymbol{A} \otimes \boldsymbol{A})\boldsymbol{D}_m \right)^{-1} \left(\Big(\boldsymbol{D}_m^+(\boldsymbol{A}\Sigma_{\boldsymbol{z}}(0) \otimes \boldsymbol{I}_m) \right. \tag{3.8}$$
$$\left. + \boldsymbol{D}_m^+(\boldsymbol{I}_m \otimes \boldsymbol{A}\Sigma_{\boldsymbol{z}}(0))\boldsymbol{K}_{mm} \Big) \frac{\partial \text{vec}(\boldsymbol{A})}{\partial \boldsymbol{\tau}'} + \frac{\partial \text{vech}(\boldsymbol{\Omega})}{\partial \boldsymbol{\tau}'} \right)$$

Details on the derivation of (3.7)-(3.8) are provided in the Appendix. Obviously, when $\boldsymbol{C}$ contains only constants $\frac{\partial \boldsymbol{C}}{\partial \boldsymbol{\tau}}$ is zero, and the first term on the right-hand side of (3.6) and (3.7) disappears.

3.2.2 Derivation of J_2

The derivative J_2 can be computed using the implicit function theorem as in Iskrev (2008a). An implicit function, with $\boldsymbol{\theta}$ and $\boldsymbol{\tau}$ as arguments, is provided by the restrictions the structural model (2.1) imposes on the reduced form (2.2). In particular, from (2.2) we have $\text{E}_t\, \boldsymbol{z}_{t+1} = \boldsymbol{A}\boldsymbol{z}_t$, and substituting in (2.1) yields:

$$(\boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_1\boldsymbol{A})\boldsymbol{z}_t = \boldsymbol{\Gamma}_2\boldsymbol{z}_{t-1} + \boldsymbol{\Gamma}_3\boldsymbol{u}_t \tag{3.9}$$

Combining the last equation with equation (2.2) gives to the following two matrix equations:

$$\boldsymbol{F}_1(\boldsymbol{\theta}, \boldsymbol{\tau}) = \Big(\boldsymbol{\Gamma}_0(\boldsymbol{\theta}) - \boldsymbol{\Gamma}_1(\boldsymbol{\theta})\boldsymbol{A}(\boldsymbol{\tau}) \Big) \boldsymbol{A}(\boldsymbol{\tau}) - \boldsymbol{\Gamma}_2(\boldsymbol{\theta}) = \mathbf{O} \tag{3.10}$$

$$\boldsymbol{F}_2(\boldsymbol{\theta}, \boldsymbol{\tau}) = \Big(\boldsymbol{\Gamma}_0(\boldsymbol{\theta}) - \boldsymbol{\Gamma}_1(\boldsymbol{\theta})\boldsymbol{A}(\boldsymbol{\tau}) \Big) \boldsymbol{\Omega}(\boldsymbol{\tau}) \Big(\boldsymbol{\Gamma}_0(\boldsymbol{\theta}) - \boldsymbol{\Gamma}_1\boldsymbol{A}(\boldsymbol{\tau}) \Big)' - \boldsymbol{\Gamma}_3(\boldsymbol{\theta})\boldsymbol{\Gamma}_3(\boldsymbol{\theta})' = \mathbf{O} \tag{3.11}$$

The system (3.10) - (3.11) is an implicit function with $\boldsymbol{\theta}$ and $\boldsymbol{\tau}$ as arguments. A third equation must be added if $\boldsymbol{C}$ in the measurement equation (2.3) is also a function of $\boldsymbol{\theta}$. When this is the case, the functional form of the mapping from $\boldsymbol{\theta}$ to the elements of $\boldsymbol{C}$ is known, and the third equation is of the form $\boldsymbol{F}_3(\boldsymbol{\theta}, \boldsymbol{\tau}) := \boldsymbol{C}(\boldsymbol{\tau}) - \boldsymbol{C}(\boldsymbol{\theta}) = \mathbf{O}$.[2]

Let $\boldsymbol{f}(\boldsymbol{\theta}, \boldsymbol{\tau}) := [\text{vec}(\boldsymbol{F}_1)', \text{vech}(\boldsymbol{F}_2)', \text{vec}(\boldsymbol{F}_3)']'$. Applying the implicit function theo-

[2] It is usually possible to express the reduced form model so that $\boldsymbol{C}$ is a constant matrix of ones and zeros only.

rem, we have:

$$\frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{\theta}'} = -\Big(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\tau}'}(\boldsymbol{\theta}, \boldsymbol{\tau}(\boldsymbol{\theta}))\Big)^{-1} \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\theta}'}(\boldsymbol{\theta}, \boldsymbol{\tau}(\boldsymbol{\theta})) \tag{3.12}$$

Hence, to compute $\frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{\theta}'}$ we need the derivatives of the elements of $\boldsymbol{f}$: $\mathrm{vec}(\boldsymbol{F}_1), \mathrm{vech}(\boldsymbol{F}_2)$ and $\mathrm{vec}(\boldsymbol{F}_3)$, with respect to $\boldsymbol{\tau}$ and $\boldsymbol{\theta}$. From (3.10), the derivatives of $\mathrm{vec}(\boldsymbol{F}_1)$ are

$$\frac{\partial \mathrm{vec}(\boldsymbol{F}_1)}{\partial \boldsymbol{\tau}'} = \Big(\boldsymbol{I}_m \otimes \boldsymbol{\Gamma}_0 - \boldsymbol{A}' \otimes \boldsymbol{\Gamma}_1 - \boldsymbol{I}_m \otimes \boldsymbol{\Gamma}_1 \boldsymbol{A}\Big) \frac{\partial \mathrm{vec}(\boldsymbol{A})}{\partial \boldsymbol{\tau}'} \tag{3.13}$$

and

$$\frac{\partial \mathrm{vec}(\boldsymbol{F}_1)}{\partial \boldsymbol{\theta}'} = \big(\boldsymbol{A}' \otimes \boldsymbol{I}_m\big) \frac{\partial \mathrm{vec}(\boldsymbol{\Gamma}_0)}{\partial \boldsymbol{\theta}'} - \big(\boldsymbol{A}'^2 \otimes \boldsymbol{I}_m\big) \frac{\partial \mathrm{vec}(\boldsymbol{\Gamma}_1)}{\partial \boldsymbol{\theta}'} - \frac{\partial \mathrm{vec}(\boldsymbol{\Gamma}_2)}{\partial \boldsymbol{\theta}'} \tag{3.14}$$

Similarly, from (3.11), the derivatives of $\mathrm{vec}(\boldsymbol{F}_2)$ are

$$\begin{aligned} \frac{\partial \mathrm{vech}(\boldsymbol{F}_2)}{\partial \boldsymbol{\tau}'} &= \boldsymbol{D}_m^+ \Big(\boldsymbol{\Gamma}_0 \otimes \big(\boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_1 \boldsymbol{A}\big) - \boldsymbol{\Gamma}_1 \boldsymbol{A} \otimes \big(\boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_1 \boldsymbol{A}\big)\Big) \boldsymbol{D}_n \frac{\partial \mathrm{vech}(\boldsymbol{\Omega})}{\partial \boldsymbol{\tau}'} \\ &- \boldsymbol{D}_m^+ \Big(\big(\boldsymbol{\Gamma}_0 \boldsymbol{\Omega} - \boldsymbol{\Gamma}_1 \boldsymbol{A} \boldsymbol{\Omega}\big) \otimes \boldsymbol{\Gamma}_1 + \big(\boldsymbol{\Gamma}_1 \otimes \boldsymbol{\Gamma}_0 \boldsymbol{\Omega} - \boldsymbol{\Gamma}_1 \otimes \boldsymbol{\Gamma}_1 \boldsymbol{A} \boldsymbol{\Omega}\big) \boldsymbol{K}_{mm}\Big) \frac{\partial \mathrm{vec}(\boldsymbol{A})}{\partial \boldsymbol{\tau}'} \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \frac{\partial \mathrm{vech}(\boldsymbol{F}_2)}{\partial \boldsymbol{\theta}'} &= \boldsymbol{D}_m^+ \Big(\big(\boldsymbol{\Gamma}_0 \boldsymbol{\Omega} - \boldsymbol{\Gamma}_1 \boldsymbol{A} \boldsymbol{\Omega}\big) \otimes \boldsymbol{I}_m + \big(\boldsymbol{I}_m \otimes \boldsymbol{\Gamma}_0 \boldsymbol{\Omega} - \boldsymbol{I}_m \otimes \boldsymbol{\Gamma}_1 \boldsymbol{A} \boldsymbol{\Omega}\big) \boldsymbol{K}_{mm}\Big) \frac{\partial \mathrm{vec}(\boldsymbol{\Gamma}_0)}{\partial \boldsymbol{\theta}'} \\ &- \boldsymbol{D}_m^+ \Big(\big(\boldsymbol{\Gamma}_0 \boldsymbol{\Omega} \boldsymbol{A}' - \boldsymbol{\Gamma}_1 \boldsymbol{A} \boldsymbol{\Omega} \boldsymbol{A}'\big) \otimes \boldsymbol{I}_m + \big(\boldsymbol{I}_m \otimes \boldsymbol{\Gamma}_0 \boldsymbol{\Omega} \boldsymbol{A}' - \boldsymbol{I}_m \otimes \boldsymbol{\Gamma}_1 \boldsymbol{A} \boldsymbol{\Omega} \boldsymbol{A}'\big) \boldsymbol{K}_{mm}\Big) \frac{\partial \mathrm{vec}(\boldsymbol{\Gamma}_1)}{\partial \boldsymbol{\theta}'} \\ &- \boldsymbol{D}_m^+ \Big(\boldsymbol{\Gamma}_3 \otimes \boldsymbol{I}_m + \big(\boldsymbol{I}_m \otimes \boldsymbol{\Gamma}_3\big) \boldsymbol{K}_{mn}\Big) \frac{\partial \mathrm{vec}(\boldsymbol{\Gamma}_3)}{\partial \boldsymbol{\theta}'} \end{aligned} \tag{3.16}$$

Finally, if $\boldsymbol{C}$ is not constant, we also need $\frac{\partial \mathrm{vec}(\boldsymbol{F}_3)}{\partial \boldsymbol{\tau}'} = \frac{\partial \mathrm{vec}(\boldsymbol{C})}{\partial \boldsymbol{\tau}'}$, and $\frac{\partial \mathrm{vec}(\boldsymbol{F}_3)}{\partial \boldsymbol{\theta}'} = \frac{\partial \mathrm{vec}(\boldsymbol{C})}{\partial \boldsymbol{\theta}'}$, which are straightforward to compute. More details on the derivation of (3.13)-(3.16) are provided in the Appendix.

The formulas in (3.8)-(3.7) and (3.13)-(3.16) reveal that in order to derive the Jacobian matrix $J(T)$ one needs seven matrix derivatives: of $\boldsymbol{A}, \boldsymbol{C}$ and $\boldsymbol{\Omega}$ with respect to $\boldsymbol{\tau}$, and of $\{\boldsymbol{\Gamma}_i\}_{i=0}^3$ with respect to $\boldsymbol{\theta}$. These derivatives are very easy to compute since the dependence of the reduced-form matrices on $\boldsymbol{\tau}$, and of the structural coefficient matrices on $\boldsymbol{\theta}$, is known from the canonical representation of the linearized model.[3] Thus, although the expressions involved in the computation of $J(T)$ are fairly complicated and require the use of software that can handle Kronecker products, the actual differentiation is simple and can be carried out by hand, as I show in the following example.

[3] In fact, it is easy to see that the derivatives of the reduced-form matrices are model-independent

3.2.3 Example

I use a simple version of the New-Keynesian model to illustrate the computation of the derivatives of the structural coefficient matrices with respect to $\boldsymbol{\theta}$. The log-linearized equilibrium conditions are summarized by the following equations:

$$\pi_t = \frac{\beta}{1+\beta\omega}E_t\pi_{t+1} + \frac{\omega}{1+\beta\omega}\pi_{t-1} + \frac{(1-\psi)(1-\psi\beta)}{\psi(1+\beta\omega)}x_t + \sigma_\varepsilon\varepsilon_t \tag{3.17}$$

$$x_t = \frac{1}{1+\lambda}E_t x_{t+1} + \frac{\lambda}{1+\lambda}x_{t-1} - \frac{1-\lambda}{(1+\lambda)\nu}\left(r_t - E_t\pi_{t+1}\right) + \sigma_u u_t \tag{3.18}$$

$$r_t = (1-\rho)\alpha_\pi\pi_t + (1-\rho)\alpha_x x_t + \rho r_{t-1} + \sigma_\zeta\zeta_t \tag{3.19}$$

where π_t is the inflation rate, x_t the output gap, and r_t the nominal interest rate. Equation (3.17) is a hybrid New Keynesian Phillips curve, (3.18) is an IS curve, and (3.19) is the policy rule of the central bank. The vector of deep parameters is $\boldsymbol{\theta} = [\psi, \beta, \omega, \lambda, \nu, \alpha_\pi, \alpha_x, \rho, \sigma_\varepsilon, \sigma_u, \sigma_\zeta]'$. Assuming that the structural shocks ε_t, u_t, and ζ_t are independent white noise processes, the canonical-form matrices are given by

$$\boldsymbol{\Gamma}_0 = \begin{bmatrix} 1 & -\frac{(1-\psi)(1-\psi\beta)}{\psi(1+\beta\omega)} & 0 \\ 0 & 1 & \frac{1-\lambda}{(1+\lambda)\nu} \\ -(1-\rho)\alpha_\pi & -(1-\rho)\alpha_x & 1 \end{bmatrix}; \boldsymbol{\Gamma}_1 = \begin{bmatrix} \frac{\beta}{1+\beta\omega} & 0 & 0 \\ \frac{1-\lambda}{(1+\lambda)\nu} & \frac{1}{1+\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

$$\boldsymbol{\Gamma}_2 = \begin{bmatrix} \frac{\omega}{1+\beta\omega} & 0 & 0 \\ 0 & \frac{\lambda}{1+\lambda} & 0 \\ 0 & 0 & \rho \end{bmatrix}; \boldsymbol{\Gamma}_3 = \begin{bmatrix} \sigma_\varepsilon & 0 & 0 \\ 0 & \sigma_u & 0 \\ 0 & 0 & \sigma_\zeta \end{bmatrix};$$

Differentiating with respect to $\boldsymbol{\theta}$ yields four 9×11-dimensional sparse matrices with the following non-zero components (the numbers in parenthesis denote the row and column of the corresponding matrix):

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and given by:

$$\frac{\partial \text{vec}(\boldsymbol{A})}{\partial \boldsymbol{\tau}'} = [\boldsymbol{I}_{m^2}, \mathbf{O}_{lm\times lm}, \mathbf{O}_{\frac{n^2(n+1)^2}{4}\times\frac{n^2(n+1)^2}{4}}]$$

$$\frac{\partial \text{vec}(\boldsymbol{C})}{\partial \boldsymbol{\tau}'} = [\mathbf{O}_{m^2\times m^2}, \boldsymbol{I}_{lm}, \mathbf{O}_{\frac{n^2(n+1)^2}{4}\times\frac{n^2(n+1)^2}{4}}]$$

$$\frac{\partial \text{vec}(\boldsymbol{\Omega})}{\partial \boldsymbol{\tau}'} = [\mathbf{O}_{m^2\times m^2}, \mathbf{O}_{lm\times lm}, \boldsymbol{I}_{\frac{n^2(n+1)^2}{4}}]$$

- for $\frac{\partial \text{vec}(\boldsymbol{\Gamma}_0)}{\partial \boldsymbol{\theta}'}$

$$\begin{array}{llllll} (3,6) & \rho-1 & (3,8) & \alpha_\pi & & \\ (4,1) & \frac{1-\beta\psi^2}{\psi^2(1+\beta\omega)} & (4,2) & \frac{(1-\psi)(\psi+\omega)}{\psi(1+\beta\omega)^2} & (4,3) & \frac{(\psi-1)(\psi\beta-1)}{\psi(1+\beta\omega)^2} \\ (6,7) & \rho-1 & (6,8) & \alpha_x & & \\ (8,4) & \frac{-2}{\nu(1+\lambda)^2} & (8,5) & \frac{\lambda-1}{\nu^2(1+\lambda)} & & \end{array}$$

- for $\frac{\partial \text{vec}(\boldsymbol{\Gamma}_1)}{\partial \boldsymbol{\theta}'}$

$$\begin{array}{llll} (1,2) & \frac{1}{(1+\beta\omega)^2} & (1,3) & \frac{-\beta^2}{(1+\beta\omega)^2} \\ (2,4) & \frac{-2}{\nu(1+\lambda)^2} & (2,5) & \frac{\lambda-1}{(1+\lambda)\nu^2} \\ (5,4) & \frac{-1}{(1+\lambda)^2} & & \end{array}$$

- for $\frac{\partial \text{vec}(\boldsymbol{\Gamma}_2)}{\partial \boldsymbol{\theta}'}$

$$\begin{array}{llll} (1,2) & \frac{-\omega^2}{(1+\beta\omega)^2} & (1,3) & \frac{1}{(1+\beta\omega)^2} \\ (5,4) & \frac{1}{(1+\lambda)^2} & & \\ (9,8) & 1 & & \end{array}$$

- for $\frac{\partial \text{vec}(\boldsymbol{\Gamma}_3)}{\partial \boldsymbol{\theta}'}$

$$\begin{array}{ll} (1,9) & 1 \\ (5,10) & 1 \\ (9,11) & 1 \end{array}$$

This example shows that the derivatives one needs in order to check the rank condition for identification are very easy to obtain from the canonical matrices of the linearized model. These matrices may be much larger in large-scale DSGE models, but they are rarely much more complicated functions of $\boldsymbol{\theta}$ than what we have in the example above.

3.3 Discussion

It should be emphasized that the conditions for (local) identification, given in Definition 3.1 and Theorem 3.1, involve the true values of the moments and reflect the understanding of identification as a population, not a finite sample issue.[4] Thus, the question whether a model is identified or not can in principle be addressed prior to confronting the model with a particular set of data. In practice, since the number of points

[4] In the words of Koopmans and Reiersøl (1950, p.170) "the study of identifiability proceeds from a hypothetical exact knowledge of the probability distribution of observed variables rather than from a finite sample of observations."

in $\boldsymbol{\Theta}$ is infinite, one can only check the rank condition for some parts of the parameter space. A procedure for doing this, using many random draws from $\boldsymbol{\Theta}$, is described in Section 5.2.

Although the identifiability of $\boldsymbol{\theta}$ does not depend on the characteristics of a particular set of observations, it does depend on how many and which of the variables in the model are observed, as well as on the number of available moments. There must be at least as many moments as there are deep parameters, in order to meet the order condition for identification. In general, using more moments may identify a model which is otherwise unidentifiable. Since the number of available moments is limited by the sample size, the latter is one aspect of the data that is relevant for identification. Furthermore, how many and which of the variables in the model are observed, are features of the data that are also relevant for identification. Again, more observed variables is generally better than fewer, although having more variables is not necessarily better than having a smaller number of different variables. For instance, having capital among the observed variables may be more useful for identification in a real business cycle model than observing several endogenous jump variables instead of capital.

One can easily check whether having more data would help with identification. A larger sample size would increase the value of T, and thus the dimension of $\boldsymbol{\sigma}_T$. The effect of having additional observed variables can be investigated by changing the dimension of $\boldsymbol{C}$, the matrix which selects the observed among all model variables (see equation 2.3). For instance, to find out the effect of observing capital on the identification of $\boldsymbol{\theta}$ one has to increase the number of rows in $\boldsymbol{C}$ with a vector which has 1 in the position of capital in $\boldsymbol{z}_t$, and zeros everywhere else.

In some models identification may fail for purely model-related reasons, not because of data limitations. This happens, for instance, if there are parameters that play no role in the equilibrium of the model, or if the effect of a parameter cannot be distinguished from that of other parameters. As was indicated in Section 3, such problems are common in the DSGE literature, and are not always easy to detect by inspecting the equations of a model. The factorization of the Jacobian matrix in equation (3.2) provides a simple method for doing that. Note that the second term, $\frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{\theta}'}$, captures the effect of perturbations in $\boldsymbol{\theta}$ on the parameters characterizing the equilibrium of the model. When this matrix is of less than full column rank some of the parameters are unidentified in the model, and therefore cannot be identified when the model is taken to the data, even if all state variables were observed. This leads to the following corollary to Theorem 1,

Corollary. *Let $\boldsymbol{\theta}_0$ be a regular point of the matrix J_2. Then $\boldsymbol{\theta}_0$ is locally identifiable only if the rank of J_2 at $\boldsymbol{\theta}$ is equal to k.*

This rank condition is not sufficient for local identification of $\boldsymbol{\theta}$ since, unless all state variables are observed, $\boldsymbol{\tau}$ is itself not identifiable.[5] Nevertheless, it is useful to check if the condition holds as a first step in the analysis of identification. As was discussed in

[5]Notice that $\boldsymbol{\tau}^* = [\text{vec}(\boldsymbol{D}\boldsymbol{A}\boldsymbol{D}^{-1})', \text{vec}(\boldsymbol{C}\boldsymbol{D}^{-1})', \text{vech}(\boldsymbol{D}\boldsymbol{\Omega}\boldsymbol{D}')']'$ for arbitrary non-singular matrix $\boldsymbol{D}$ is observationally equivalent to $\boldsymbol{\tau}$.

Section 3, by inspecting the columns of the matrix and the possible linear dependencies among them, one can gain a better understanding of the roles different deep parameters play in the model, and the causes for identification failures. For example, one will find that the columns of J_2 that correspond to the coefficients of the Taylor rule in Cochrane (2007) are zero vectors. In the model of Kim (2003), the columns of J_2 corresponding to the two investment adjustment cost parameters are exactly collinear. In both cases the above rank condition fails for any $\boldsymbol{\theta}$ in the parameter space, indicating a problem in the structure of the respective model.

The discussion so far has focused on the use of the Jacobian matrix to study identification. However, there are other purposes for which the formulas presented above may be useful. For instance, we can use an analytical Jacobian of transformation to construct priors about deep parameters on the basis of beliefs about second moments of the data. This method for eliciting priors for the parameters in DSGE models was developed in Del Negro and Schorfheide (2008) using dummy observations instead of the analytical change of variables. Having analytical derivatives is also very convenient when DSGE models are estimated using gradient-based optimization methods, or when inference is based on the usual first-order approximations. From an economic modeling perspective, one may wish to know the sign and size of the effect of small changes in some deep parameters on properties of the equilibrium outcomes, such as volatility or persistence of some endogenous variables. Such questions can be answered directly by using the expressions for analytical derivatives derived here.

4 Extensions

The main result in the last section, Theorem 3.1, can be extended in two obvious ways. First, instead of the complete covariance matrix of $\boldsymbol{X}_T$, estimation may be based on a smaller number of second moments. Without loss of generality, assume that the used moments are $\Sigma_{\boldsymbol{x}}(j),\ j = 1, \ldots, q < T$, and define $\boldsymbol{\sigma}_q$ as before. (Local) Identification of $\boldsymbol{\theta}$ in this limited information setting requires that the mapping from $\boldsymbol{\sigma}_q$ to $\boldsymbol{\theta}$ is (locally) unique. Theorem 3.1 with $J(q)$ instead of $J(T)$ provides a necessary and sufficient condition for local identification in this setting. Moreover, $J(q)$ with $q \leq T$ having full rank is a sufficient condition for identification with full information methods. Thus, finding that $J(q)$ has full rank for some small value of q makes it unnecessary to evaluate $J(T)$, which may be much more computationally expensive.

Second, the statistical model used to identify $\boldsymbol{\theta}$ could be extended to transformations of the second moments of $\boldsymbol{X}_T$, such as parameters of a finite-order VAR or impulse response coefficients. Such methods are common in the empirical DSGE literature, and it is therefore useful to know how to check for identification in that setting. Suppose that $\boldsymbol{\xi}$ is a r-dimensional vector, such that $\boldsymbol{\xi} = \boldsymbol{g}(\boldsymbol{\sigma}_T)$, and the functions $g_1(\boldsymbol{\sigma}_T), \ldots, g_r(\boldsymbol{\sigma}_T)$ are continuously differentiable. Then, in parallel to Theorem 3.1, a necessary and sufficient condition for local identification of $\boldsymbol{\theta}_0 \in \boldsymbol{\Theta}$ is that $\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{\theta}'}$ has full column rank at $\boldsymbol{\xi}_0 = \boldsymbol{g}(\boldsymbol{\sigma}_T(\boldsymbol{\theta}_0))$. Note that using the chain rule, we have

$$\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{\theta}'} = \frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{\sigma}_T{}'} \frac{\partial \boldsymbol{\sigma}_T}{\partial \boldsymbol{\theta}'} \tag{4.1}$$

The second term in the product on the right-hand side was derived in Section 3.2. To illustrate the derivation of $\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{\sigma}_T{}'}$, suppose that the statistical model used to estimate $\boldsymbol{\theta}$ is a VAR with p lags, i.e.

$$\boldsymbol{x}_t = \sum_{i=1}^{p} \boldsymbol{\Phi}_i^{(p)} \boldsymbol{x}_{t-i} + \boldsymbol{\varepsilon}_t^{(p)} \tag{4.2}$$

where $\varepsilon_t^{(p)}$ is uncorrelated with $\boldsymbol{x}_{t-i},\ i \geq 1$.
Let $\boldsymbol{\Phi}^{(p)} := [\boldsymbol{\Phi}_1^{(p)\prime}, \ldots, \boldsymbol{\Phi}_p^{(p)\prime}]'$ and $\boldsymbol{\Omega}^{(p)} := \mathrm{E}\, \boldsymbol{\varepsilon}_t^{(p)} \boldsymbol{\varepsilon}_t^{(p)\prime}$. Then we have

$$\boldsymbol{\xi} = [\mathrm{vec}(\boldsymbol{\Phi}^{(p)})', \mathrm{vech}(\boldsymbol{\Omega}^{(p)})']'$$

where $\boldsymbol{\Phi}^{(p)}$ and $\boldsymbol{\Omega}^{(p)}$ are given by

$$\boldsymbol{\Phi}^{(p)} = \begin{bmatrix} \Sigma_{\boldsymbol{x}}(0) & \Sigma_{\boldsymbol{x}}(1) & \ldots & \Sigma_{\boldsymbol{x}}(p-1) \\ \Sigma_{\boldsymbol{x}}(1)' & \Sigma_{\boldsymbol{x}}(0) & \ldots & \Sigma_{\boldsymbol{x}}(p-2) \\ \vdots & \vdots & & \vdots \\ \Sigma_{\boldsymbol{x}}(p-1)' & \Sigma_{\boldsymbol{x}}(p-2)' & \ldots & \Sigma_{\boldsymbol{x}}(0) \end{bmatrix}^{-1} \begin{bmatrix} \Sigma_{\boldsymbol{x}}(1) \\ \Sigma_{\boldsymbol{x}}(2) \\ \vdots \\ \Sigma_{\boldsymbol{x}}(p) \end{bmatrix}$$

and

$$\boldsymbol{\Omega}^{(p)} = \Sigma_{\boldsymbol{x}}(0) - \begin{bmatrix} \Sigma_{\boldsymbol{x}}(1)' \\ \Sigma_{\boldsymbol{x}}(2)' \\ \vdots \\ \Sigma_{\boldsymbol{x}}(p)' \end{bmatrix}' \begin{bmatrix} \Sigma_{\boldsymbol{x}}(0) & \Sigma_{\boldsymbol{x}}(1) & \ldots & \Sigma_{\boldsymbol{x}}(p-1) \\ \Sigma_{\boldsymbol{x}}(1)' & \Sigma_{\boldsymbol{x}}(0) & \ldots & \Sigma_{\boldsymbol{x}}(p-2) \\ \vdots & \vdots & & \vdots \\ \Sigma_{\boldsymbol{x}}(p-1)' & \Sigma_{\boldsymbol{x}}(p-2)' & \ldots & \Sigma_{\boldsymbol{x}}(0) \end{bmatrix}^{-1} \begin{bmatrix} \Sigma_{\boldsymbol{x}}(1) \\ \Sigma_{\boldsymbol{x}}(2) \\ \vdots \\ \Sigma_{\boldsymbol{x}}(p) \end{bmatrix}$$

Differentiating $\mathrm{vec}(\boldsymbol{\Phi}^{(p)})$ and $\mathrm{vech}(\boldsymbol{\Omega}^{(p)})$ with respect to $\boldsymbol{\sigma}_T$ gives $\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{\sigma}_T{}'}$.[6] We can similarly compute the Jacobian matrix for other functions of the second moments, such as impulse response coefficients.

5 Application: Identification in the Smets and Wouters (2007) model

In this section I apply the rank conditions developed above to a medium-scale DSGE model estimated in Smets and Wouters (2007) (SW07 henceforth). The model, based on the work of Smets and Wouters (2003) and Christiano, Eichenbaum, and Evans (2005), is an extension of the standard RBC model featuring a number of nominal frictions and

[6] Note that here $\boldsymbol{\xi}$ is a function of only the first p second moments. Thus $\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{\theta}'} = \frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{\sigma}_p{}'} \frac{\partial \boldsymbol{\sigma}_p}{\partial \boldsymbol{\theta}'}$.

real rigidities. I start with an outline of the main components of the model, and then turn to the identification of the parameters.

5.1 The model

The economy is populated by a continuum of households indexed by j, each having the following utility function

$$E_t\left[\sum_{s=0}^{\infty}\beta^s\frac{1}{1-\sigma_C}\left((C_{t+s}(j)-\lambda C_{t+s-1}(j))^{1-\sigma_C}\right)\exp\left(\frac{\sigma_c-1}{1+\sigma_l}L_{t+s}(j)^{1+\sigma_l}\right)\right], \tag{5.1}$$

where $C_{t+s}(j)$ is consumption, $L_{t+s}(j)$ is hours worked.

Households supply homogeneous labor services to labor unions indexed by l. Labor services are differentiated by a union, and sold to labor packers. Wage setting is subject to nominal rigidities with a Calvo mechanism whereby each period a union can set the nominal wage to the optimal level with constant probability equal to $1-\xi_w$. Unions that cannot adjust their nominal wage optimally, change it according to the following indexation rule

$$W_{t+s}(l)=\gamma W_{t-1}(l)\pi_{t-1}^{\iota_w}\pi_*^{(1-\iota_w)}, \tag{5.2}$$

where γ is the deterministic growth rate, ι_w measures the degree of wage indexation to past inflation, and π_* is the steady state rate of inflation.

Labor packers buy differentiated labor services $L_t(l)$ from unions, package and sell composite labor L_t, defined implicitly by

$$\int_0^1\mathcal{H}\left(\frac{L_t(l)}{L_t};\lambda_{w,t}\right)dl=1, \tag{5.3}$$

to the intermediate good sector firms. The function $\mathcal{H}$ is increasing, concave, and satisfies $\mathcal{H}(1)=1$; $\lambda_{w,t}$ is a stochastic exogenous process changing the elasticity of demand, and the wage markup over the marginal disutility from work.

In addition to supplying labor, households rent capital to the intermediate goods producers at rate $R_t^K(j)$. Households accumulate physical capital according to the following law of motion:

$$\bar{K}_t(j)=(1-\delta)\bar{K}_{t-1}(j)+\varepsilon_t^i\left[1-\mathcal{S}\left(\frac{I_t(j)}{I_{t-1}(j)}\right)\right]I_t(j), \tag{5.4}$$

where δ is the rate of depreciation, I_t is gross investment, and the investment adjustment cost function $\mathcal{S}$ satisfies $\mathcal{S}'>0$, $\mathcal{S}''>0$, and in steady state $\mathcal{S}=0$, $\mathcal{S}'=0$. ε_t^i represents the current state of technology for producing capital, and is interpreted as investment-specific technological progress (Greenwood, Hercowitz, and Krusell (2000)).

Households control the utilization rate $Z_t(j)$ of the physical capital they own, and pay $P_t a(Z_t(j))\bar{K}_{t-1}(j)$ in terms of consumption good when the capital intensity is $Z_t(j)$.

The income from renting capital to firms is $R_t^k K_t(j)$, where $K_t(j) = Z_t(j)\bar{K}_{t-1}(j)$ is the flow of capital services provided by the existing stock of physical capital $\bar{K}_{t-1}(j)$. The utility function (5.1) is maximized with respect to consumption, hours, investment, and capital utilization, subject to the capital accumulation equation (5.4), and the following budget constraint:

$$C_{t+s}(j) + I_{t+s}(j) + \frac{B_{t+s}(j)}{\varepsilon_{t+s}^b R_{t+s} P_{t+s}} - T_{t+s} = \frac{W_{t+s}(j)}{P_{t+s}} L_{t+s}(j) + \left(\frac{R_{t+s}^k Z_{t+s}(j)}{P_{t+s}} - a(Z_{t+s}(j)) \right) \bar{K}_{t+s-1}(j) + \frac{B_{t+s-1}(j)}{P_{t+s}} + \frac{\Pi_{t+s}(j)}{P_{t+s}}, \tag{5.5}$$

where B_{t+s} is a one-period nominal bond expressed on a discount basis, ε_t^b is an exogenous premium on the bond return, T_{t+s} is lump-sum taxes or subsidies, and Π_{t+s} is profit distributed by the labor union.

There is a perfectly competitive sector producing a single final good used for consumption and investment. The final good is produced from intermediate inputs $Y_t(i)$ using technology defined implicitly by

$$\int_0^1 \mathcal{G}\left(\frac{Y_t(i)}{Y_t}; \lambda_{p,t} \right) di = 1, \tag{5.6}$$

where $\mathcal{G}$ is increasing, concave, and $\mathcal{G}(1) = 1$; $\lambda_{p,t}$ is an exogenous stochastic process affecting the elasticity of substitution between different intermediate goods, also corresponding to a markup over marginal cost for intermediate good firms. Firms maximize profits given by

$$P_t Y_t - \int_0^1 P_t(i) Y_t(i) di, \tag{5.7}$$

where $P_t(i)$ is the price of intermediate good $Y_t(i)$.

Intermediate goods are produced in a monopolistically competitive sector. Each variety i is produced by a single firm using the following production technology:

$$Y_t(i) = \varepsilon_t^a K_t(i)^\alpha (\gamma^t L_t(i))^{1-\alpha} - \Phi \gamma^t, \tag{5.8}$$

where Φ is a fixed cost of production, and ε_t^a is the total factor productivity. As with wages, every period only a fraction $1 - \xi_P$ of intermediate firms can set optimally the price of the good they produce. The remaining ξ_p firms index their prices to past inflation according to

$$P_t(t) = \gamma P_{t-1}(i) \pi_{t-1}^{\iota_p} \pi_*^{(1-\iota_p)}, \tag{5.9}$$

where ι_p measures the degree of price indexation to past inflation.

The central bank sets the nominal interest rate according to the following rule

$$\frac{R_t}{R^*} = \varepsilon_t^r \left(\frac{R_{t-1}}{R^*}\right)^{\rho} \left[\left(\frac{\pi_t}{\pi^*}\right)^{r_\pi} \left(\frac{Y_t}{Y_t^*}\right)^{r_y}\right]^{1-\rho} \left(\frac{Y_t/Y_{t-1}}{Y_t^*/Y_{t-1}^*}\right)^{r_{\triangle y}} \tag{5.10}$$

where R^* is the steady state level of the gross nominal interest rate, r_t is a monetary policy shock, and Y^* is potential output, defined as the output in a flexible price and wage economy.

The government also collects lump-sum taxes in order to finance its consumption so as to respect the following budget constraint

$$P_t G_t + B_{t-1} = T_t + \frac{B_t}{R_t}, \tag{5.11}$$

where G_t is government consumption in terms of final good.

There are seven exogenous shocks in the model. Five of them - the risk premium, TFP, investment-specific technology, government purchases, and monetary policy - follow AR(1) processes; the remaining two shocks - to wage and price markup, follow ARMA$(1,1)$ processes.

The economy in the model is assumed to evolve along a deterministic growth path, with γ being the gross rate of growth. All growing variables - consumption, investment, capital, real wages, output and government spending, are detrended and then all equilibrium conditions are log-linearized around the deterministic steady state of the detrended variables. A detailed discussion of all log-linear equations can be found in SW07.

The linearized version of the model can be written as in (2.1) with $\boldsymbol{z}_t = \left[\boldsymbol{z}_t^{f\prime}, \boldsymbol{z}_t^{s\prime}\right]'$ being a $33-$dimensional vector, and the subvectors $\boldsymbol{z}_t^f$ and $\boldsymbol{z}_t^s$ are given by

$$\boldsymbol{z}_t^f = \left[c_t^f, l_t^f, w_t^f, q_t^f, i_t^f, r_t^{kf}, r_t^f, k_t^f, \bar{k}_{t-1}^f, y_t^f, z_t^f\right]'$$

and

$$\boldsymbol{z}_t^s = \left[c_t^s, l_t^s, \pi_t, w_t^s, q_t^s, i_t^s, r_t^{ks}, r_t^s, k_t^s, \bar{k}_{t-1}^s, y_t^s, z_t^s, \mathrm{mc}_t, \varepsilon_t^b, \varepsilon_t^i, \varepsilon_t^a, \varepsilon_t^g, \varepsilon_t^p, \varepsilon_t^w, \varepsilon_t^r, \eta_t^p, \eta_t^w\right]'$$

I use small letters to represent the percent deviation of the variables from their steady state levels.[7] $\boldsymbol{z}^f$ is a vector collecting the variables in the flexible price and wage version of the economy, and $\boldsymbol{z}^s$ collects the variables from the sticky price and wage economy. The vector of structural shocks is:

$$\boldsymbol{u}_t = [\eta_t^a, \eta_t^b, \eta_t^I, \eta_t^w, \eta_t^p, \eta_t^g, \eta_t^r]'$$

The coefficient matrices $\{\boldsymbol{\Gamma}_i\}_{i=0}^3$ in the canonical form (2.1) are functions of a

[7] q denotes the percent deviation of real value of capital from the steady state level of one.

39−dimensional vector of deep parameters $\boldsymbol{\theta}$, defined by[8]

$$\begin{aligned}\boldsymbol{\theta} = \; & [\delta, \lambda_w, g_y, \varepsilon_p, \varepsilon_w, \rho_{ga}, \beta, \mu_w, \mu_p, \alpha, \psi, \varphi, \sigma_c, \lambda, \Phi, \iota_w, \xi_w, \iota_p, \xi_p, \sigma_l, r_\pi, r_{\triangle y}, r_y, \rho, \rho_a, \rho_b, \\ & \rho_g, \rho_I, \rho_r, \rho_p, \rho_w, \gamma, \sigma_a, \sigma_b, \sigma_g, \sigma_I, \sigma_r, \sigma_p, \sigma_w]' \end{aligned} \tag{5.12}$$

It is assumed that the only observed variables are consumption, investment, output, wages, hours, inflation, and the nominal interest rate. Thus $\boldsymbol{x}_t$ is given by

$$\boldsymbol{x}_t = [c_t, l_t, \pi_t, w_t, i_t, r_t, y_t]' \tag{5.13}$$

and the remaining $39 - 7 = 32$ variables in $\boldsymbol{z}_t$ are treated as latent. Finally, matrix $\boldsymbol{C}$ in the measurement equation (2.3) is a 7×32 matrix constructed from the first seven rows of the 32×32 identity matrix.

5.2 Identification

In addition to a description of the model, to study parameter identification one needs to determine $\boldsymbol{\Theta}$ - the set of admissible values of the parameters. As was pointed out in Section 3, a model is identified if all points in $\boldsymbol{\Theta}$ are identifiable. One difficulty with determining $\boldsymbol{\Theta}$ for DSGE models is that it is usually impossible to know, before solving the model, for which values of $\boldsymbol{\theta}$ the model has either zero or many solutions. Such points are typically deemed as inadmissible, and therefore have to be excluded from $\boldsymbol{\Theta}$. A second problem arises from the fact that there are infinitely many points in $\boldsymbol{\Theta}$, and it is not feasible to check the identification condition for all of them. In view of these difficulties, one approach is to check the rank condition for identification at many randomly drawn points from $\boldsymbol{\Theta}'$, where $\boldsymbol{\Theta} \subset \boldsymbol{\Theta}'$, discarding values of $\boldsymbol{\theta}$ that do not imply a unique solution. The following procedure outlines the steps involved in this approach:

1. Draw randomly a point $\boldsymbol{\theta}^j$ from $\boldsymbol{\Theta}'$.
2. Check whether the model has a unique solution at $\boldsymbol{\theta}^j$. If not, discard $\boldsymbol{\theta}^j$ from $\boldsymbol{\Theta}$ and return to 1.
3. Evaluate the rank of J_2 at $\boldsymbol{\theta}^j$. If it is of less then full rank, go back to 1.
4. Evaluate the rank of $J(T)$ at $\boldsymbol{\theta}^j$.

[8] ϵ_w and ϵ_p are parameters measuring the curvature of the aggregation functions in the labor and final good sectors. They are defined as $\epsilon_p = \frac{\partial \ln(\kappa_p(1))}{\partial \ln(\tilde{P})}$, $\epsilon_w = \frac{\partial \ln(\kappa_w(1))}{\partial \ln(\tilde{W})}$, where $\kappa_p(x) = -\frac{\mathcal{G}'(x)}{x\mathcal{G}''(x)}$, $\kappa_w(y) = -\frac{\mathcal{H}'(y)}{y\mathcal{H}''(y)}$ are elasticities of demand for goods and labor services, and $\tilde{P}$ and $\tilde{W}$ are the relative price and wage. They measure the percent change in the elasticity of demand for goods and labor due to one percent change in the relative price/wage, evaluated in steady state. In the simple case, where the aggregator functions $\mathcal{H}$ and $\mathcal{G}$ have the Dixit-Stiglitz functional form, both parameters are equal to zero (see Eichenbaum and Fisher (2007)).

5. Repeat steps 1 through 4 N times, for some large N.

The set $\boldsymbol{\Theta}'$ contains all values of $\boldsymbol{\theta}$ that are theoretically plausible for the parameters in the model to take. In this application I define $\boldsymbol{\Theta}'$ using the prior distribution in SW07 (see Table B.1). Alternatively, one could treat all a priori admissible parameter values as equally likely, that is, assume uniform priors. The benefit of the former approach is that, by choosing the shape and parameters of the prior distribution, one can achieve a better coverage of the parts of the space that are believed to be more plausible. Steps 1 and 2 together provide a draw from $\boldsymbol{\Theta}$ by removing points of $\boldsymbol{\Theta}'$ where the model is indetermined or does not have a solution. Conditions for existence and uniqueness can be found in Sims (2002), and are automatically checked by most computer algorithms for solving linear rational expectations models. In step 3 one checks the necessary condition for identification. Finding that matrix J_2 is rank deficient at $\boldsymbol{\theta}^j$ implies that this point in $\boldsymbol{\Theta}$ is unidentifiable in the model. Finally, in step 4 one checks the necessary and sufficient condition for local identification of $\boldsymbol{\theta}^j$. Finding that J_2 has full rank, but $J(T)$ does not, means that $\boldsymbol{\theta}^j$ cannot be unidentified given the set of observed variables and the number of observations. As was discussed earlier, it is easy to check whether more data would help identification by including more variables in $\boldsymbol{x}_t$, increasing T, or both.

In some applications it may be preferable to start in step 4 by computing the Jacobian matrix for some small number of second moments, instead of all available moments. A good candidate would be the smallest number of moments for which the order condition for identification holds. In the SW07 model, which has 39 deep parameters and 7 observed variables, the order condition holds for any number of second moments greater than one.[9] As was pointed out in Section 4, identifiability with limited information is a sufficient condition for identification with full information, and the Jacobian matrix may be much easier to evaluate in the former case. A potential drawback is that, since the condition is only sufficient, one may have to repeat step 4 including additional second moments if the rank condition fails for a number smaller than T. In the analysis of the SW07 model I first check the rank of $J(2)$, and add additional moments when it is of less than full rank.

As a preliminary step in the identification analysis, I compute the Jacobian matrix J_2 at a few points in the parameter space, namely, the prior mean and the posterior mean and median reported in SW07. A rank deficient J_2 indicates that some deep parameters are unidentifiable for reasons that are inherent in the structure of the model. For the set of all deep parameters (see (5.12)), J_2 has 39 columns, while the rank is 36 at all points where it was evaluated. This rank deficiency is caused by linear dependence among the columns of J_2 corresponding to three sets of parameters:

(a) ϵ_p, ξ_p

(b) ϵ_w, ξ_w

[9]Note that the dimension of $\boldsymbol{\sigma}_2$ is 77, while that of $\boldsymbol{\sigma}_1$ is 28.

(c) δ, β, φ, λ, γ

In the case of (a) and (b), the lack of separate identification can be explained with the very similar roles the two curvature parameters - ϵ_p and ϵ_w, and the two Calvo parameters - ξ_p and ξ_w, play in the model. A high value of ϵ_p, for instance, implies that the elasticity of demand increases rapidly when a firm's relative price increases. This implies that it is optimal for the firm to increase its price by a smaller amount, compared to the case when ϵ_p is low. As a result, prices are adjusted less rapidly. The same outcome is observed when ξ_p - the probability that a firm is not able to adjust its price to the optimal level, is large. It should be noted, however, that, though similar, these parameters are not necessarily equivalent in the original model, as they become after linearization. The same applies to the wage parameters ϵ_w and ξ_w in (b). It is more difficult to relate the lack of identification of the parameters in (c) to their roles in the model, though one may expect some degree of similarity between, for instance, the role of patience (β) on one hand, and that of depreciation rate (δ), or investment adjustment cost (φ), on the other. In particular, the effect of higher β can be offset, at least partially, by increasing δ or φ. Unlike the parameters in (a) and (b), however, there is no exact equivalence between any two of the five parameters in (c), and any four of them would be identifiable if the fifth is known. Indeed, since the trend parameter γ can be identified using the growing observed variables, it should be treated as known when studying identification in the stationary model. Regarding the parameters in (a) and (b), I follow Smets and Wouters (2007) in assuming that the curvature parameters ϵ_w, and ϵ_p are known and are both equal to 10. Hence, in the following analysis I study the identification of a 36-dimensional vector $\boldsymbol{\theta}$, obtained by removing γ, ϵ_w, and ϵ_p from the list of parameters in (5.12).

The results can be summarized as follows. Approximately 96.8% of the 1 million draws from $\boldsymbol{\Theta}'$ are admissible, amounting to $968,318$ points from $\boldsymbol{\Theta}$. All but one of these points are identified in the model, i.e. result in J_2 with full rank. Two additional points do not pass the rank condition for identification with second moments for any value of T. In short, almost all of the points from parameter space are locally identifiable with any statistical model that utilizes at least the first two second moments of the variables listed in (5.13).

It is interesting to analyze which variables fail the identification conditions and why. In all three cases the rank conditions fail due to a linear dependence between the columns of the Jacobian matrix that correspond to the wage markup parameter λ_w, and the wage stickiness parameter ξ_w. In Smets and Wouters (2007) the authors maintain that λ_w cannot be identified, and therefore do not estimate it. The evidence presented here do not support that assertion. Even at the few points where the identification conditions fail, this is sensitive to the method used for determining the rank of a matrix. Without going into great details, in `Matlab`, which was used in the study, the rank of a matrix is determined as the number of singular values that exceed a certain tolerance

value. Using the default value of 8.9×10^{-4} results in rank deficiency.[10] Using the smaller threshold value of 1.7×10^{-4} results in Jacobian matrices with full rank. For comparison, the tolerance value must be set to less than 4.7×10^{-16} in order to overturn the conclusion regarding the lack of identification of either one of the three groups of parameters discussed above. In the light of these considerations, it appears more reasonable to conclude that λ_w is locally identified, but its identification is very weak in some parts of the parameter space.

To summarize, the objective in this section was to study parameter identification in the model estimated by Smets and Wouters (2007). Out of 39 parameters in total, 34 are locally identified from the restrictions implied by the linearized stationary model, and one is globally identified from the trend in the observed growing variables. The remaining four parameters are not separately identifiable, and any estimates of, say, price or wage stickiness parameters, is conditional on the assumed or calibrated value of the respective curvature parameter - ϵ_p or ϵ_w. Perhaps surprisingly, the results suggest that the identified parameters can, in principle, be estimated with limited information methods based on as few as two of the second moments of the observed variables. An example of such statistical model is a vector autoregression with only one lag. How accurate such estimates will be is, of course, a different matter.

6 Conclusion

This paper deals with parameter identification in DSGE models estimated with full or limited information methods based on the second moments of the data. A structural economic model is identified if there is a unique mapping from the parameters of the statistical model to the underlying economic parameters. Thus identification is not a feature of a particular sample of data, but a property of the economic model and the theory from which it is derived. As such, parameter identification can and should be verified prior to estimation. Using the conditions proposed in this paper, researchers can establish whether the parameters in their models are locally identified, and, if not, whether the identification failure is due to data limitations, such as a small number of observations or lack of observations for some variables, or to reasons that are intrinsic to the structural model.

It should be remembered that local identification does not guarantee that the parameters are globally identified. Unfortunately, global identification is difficult, if not impossible, to establish for the usually large and highly non-linear models estimated in the DSGE literature. Although only necessary for global identification, the conditions presented in this paper are useful for detecting problems which are common causes of global identification failures in DSGE models.

Another important aspect of identification in DSGE models that was not dealt with

[10]The default tolerance depends on the properties of the matrix - its dimension and largest singular value.

in this paper is the strength of identification. Finding that a model is identified only tells us that it is possible to recover the true value of the deep parameters from the true value of the population moments. In practice, the population moments have to be estimated, and it is important to know how estimation errors there propagate into the estimates of $\boldsymbol{\theta}$. Weak parameter identification leads to inaccurate estimates and unreliable inference even when the number of observations is large. Evidence suggesting that some popular DSGE models are weakly identified can be found in Canova and Sala (2009) and Iskrev (2008b). A more general treatment of the identification strength in DSGE models will appear in a separate paper.

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APPENDIX

A Derivation of J_1

Starting with $\Sigma_{\boldsymbol{x}}(0)$, from (2.6) we have

$$\Sigma_{\boldsymbol{x}}(0) = \boldsymbol{C}(\boldsymbol{\tau})\Sigma_{\boldsymbol{z}}(0)\boldsymbol{C}(\boldsymbol{\tau})'$$

and therefore the differential of $\Sigma_{\boldsymbol{x}}(0)$ is

$$\mathrm{d}\Sigma_{\boldsymbol{x}}(0) = \mathrm{d}\boldsymbol{C}(\boldsymbol{\tau})\Sigma_{\boldsymbol{z}}(0)\boldsymbol{C}(\boldsymbol{\tau})' + \boldsymbol{C}(\boldsymbol{\tau})\,\mathrm{d}\Sigma_{\boldsymbol{z}}(0)\boldsymbol{C}(\boldsymbol{\tau})' + \boldsymbol{C}(\boldsymbol{\tau})\Sigma_{\boldsymbol{z}}(0)\,\mathrm{d}\boldsymbol{C}(\boldsymbol{\tau})' \tag{A.1}$$

Vectorizing both sides of (A.1) we obtain

$$\begin{aligned}\mathrm{dvec}(\Sigma_{\boldsymbol{x}}(0)) &= (\boldsymbol{C}\Sigma_{\boldsymbol{z}}(0) \otimes \boldsymbol{I}_l)\,\mathrm{dvec}(\boldsymbol{C}) + (\boldsymbol{C} \otimes \boldsymbol{C})\,\mathrm{dvec}(\Sigma_{\boldsymbol{z}}(0)) \\ &+ (I \otimes \boldsymbol{C}\Sigma_{\boldsymbol{z}}(0))\,\mathrm{dvec}(\boldsymbol{C}')\end{aligned} \tag{A.2}$$

Using the duplication matrix and its inverse (see (3.4) and (3.5)) we get

$$\begin{aligned}\mathrm{dvech}(\Sigma_{\boldsymbol{x}}(0)) &= \boldsymbol{D}_l^+(\boldsymbol{C}\Sigma_{\boldsymbol{z}}(0) \otimes \boldsymbol{I}_l)\,\mathrm{dvec}(\boldsymbol{C}) + \boldsymbol{D}_l^+(\boldsymbol{C} \otimes \boldsymbol{C})\boldsymbol{D}_m\,\mathrm{dvech}(\Sigma_{\boldsymbol{z}}(0)) \\ &+ \boldsymbol{D}_l^+(\boldsymbol{I}_l \otimes \boldsymbol{C}\Sigma_{\boldsymbol{z}}(0))\,\mathrm{dvec}(\boldsymbol{C}')\end{aligned} \tag{A.3}$$

Collecting terms and using the relationship between differential and derivative leads to the expression in (3.6).

The derivation of (3.7) is similar; from (2.6) we have

$$\Sigma_{\boldsymbol{x}}(i) = \boldsymbol{C}(\boldsymbol{\tau})\boldsymbol{A}(\tau)^i\Sigma_{\boldsymbol{z}}(0)\boldsymbol{C}(\boldsymbol{\tau})'$$

and therefore

$$\mathrm{d}\Sigma_{\boldsymbol{x}}(i) = \mathrm{d}\boldsymbol{C}\boldsymbol{A}^i\Sigma_{\boldsymbol{z}}(0)\boldsymbol{C}' + \boldsymbol{C}\,\mathrm{d}(\boldsymbol{A}^i)\Sigma_{\boldsymbol{z}}(0)\boldsymbol{C}' + \boldsymbol{C}\boldsymbol{A}^i\,\mathrm{d}\Sigma_{\boldsymbol{z}}(0)\boldsymbol{C}' + \boldsymbol{C}\boldsymbol{A}^i\Sigma_{\boldsymbol{z}}(0)\,\mathrm{d}\boldsymbol{C}' \tag{A.4}$$

and

$$\begin{aligned}\mathrm{dvec}(\Sigma_{\boldsymbol{x}}(i)) &= (\boldsymbol{C}\Sigma_{\boldsymbol{z}}(0)\boldsymbol{A}^{i\prime} \otimes \boldsymbol{I}_l)\,\mathrm{dvec}(\boldsymbol{C}) + (\boldsymbol{C}\Sigma_{\boldsymbol{z}}(0) \otimes \boldsymbol{C})\,\mathrm{dvec}(\boldsymbol{A}^i) \\ &+ (\boldsymbol{C} \otimes \boldsymbol{C}\boldsymbol{A}^i)\boldsymbol{D}_m\,\mathrm{dvech}(\Sigma_{\boldsymbol{z}}(0)) + (\boldsymbol{I}_l \otimes \boldsymbol{C}\boldsymbol{A}^i\Sigma_{\boldsymbol{z}}(0))\,\mathrm{dvec}(\boldsymbol{C}')\end{aligned} \tag{A.5}$$

The expression in (3.7) is obtained by using (see exercise 13.18 (c) in Magnus and Abadir (2005))

$$\mathrm{dvec}(\boldsymbol{A}^i) = \left(\sum_{s=1}^{i}(\boldsymbol{A}')^{i-s} \otimes \boldsymbol{A}^{s-1}\right)\mathrm{dvec}(\boldsymbol{A}) \tag{A.6}$$

Finally, in order to evaluate (3.6) and (3.7), we need the derivative of $\mathrm{vech}(\Sigma_{\boldsymbol{z}}(0))$. From

(2.7) we have

$$\mathrm{d}\Sigma_{\boldsymbol{z}}(0) = \mathrm{d}\boldsymbol{A}\Sigma_{\boldsymbol{z}}(0)\boldsymbol{A}' + \boldsymbol{A}\,\mathrm{d}\Sigma_{\boldsymbol{z}}(0)\boldsymbol{A}' + \boldsymbol{A}\Sigma_{\boldsymbol{z}}(0)\,\mathrm{d}\boldsymbol{A}' + \mathrm{d}\boldsymbol{\Omega} \tag{A.7}$$

Vectorizing both sides, we get

$$\begin{aligned}\mathrm{dvec}(\Sigma_{\boldsymbol{z}}(0)) &= (\boldsymbol{A}\Sigma_{\boldsymbol{z}}(0)\otimes\boldsymbol{I}_m)\,\mathrm{dvec}(\boldsymbol{A}) + (\boldsymbol{A}\otimes\boldsymbol{A})\,\mathrm{dvec}(\Sigma_{\boldsymbol{z}}(0)) \\ &+ (\boldsymbol{I}_m\otimes\boldsymbol{A}\Sigma_{\boldsymbol{z}}(0))\,\mathrm{dvec}(\boldsymbol{A}') + \mathrm{dvec}(\boldsymbol{\Omega})\end{aligned} \tag{A.8}$$

Using (3.5) we have

$$\begin{aligned}\mathrm{dvech}(\Sigma_{\boldsymbol{z}}(0)) &= \boldsymbol{D}_m^+(\boldsymbol{A}\Sigma_{\boldsymbol{z}}(0)\otimes\boldsymbol{I}_m)\,\mathrm{dvec}(\boldsymbol{A}) + \boldsymbol{D}_m^+(\boldsymbol{A}\otimes\boldsymbol{A})\boldsymbol{D}_m\,\mathrm{dvech}(\Sigma_{\boldsymbol{z}}(0)) \\ &+ \boldsymbol{D}_m^+(\boldsymbol{I}_m\otimes\boldsymbol{A}\Sigma_{\boldsymbol{z}}(0))\,\mathrm{dvec}(\boldsymbol{A}') + \mathrm{dvech}(\boldsymbol{\Omega})\end{aligned} \tag{A.9}$$

Collecting terms and using (3.3) we obtain (3.8).

B Derivation of J_2

From the definition of $\boldsymbol{F}_1$ in (3.10) we have

$$\mathrm{d}\boldsymbol{F}_1 = \boldsymbol{\Gamma}_0\,\mathrm{d}\boldsymbol{A} - \boldsymbol{\Gamma}_1\,\mathrm{d}\boldsymbol{A}\boldsymbol{A} - \boldsymbol{\Gamma}_1\boldsymbol{A}\,\mathrm{d}\boldsymbol{A} \tag{B.1}$$

and

$$\mathrm{d}\boldsymbol{F}_1 = \mathrm{d}\boldsymbol{\Gamma}_0\boldsymbol{A} - \mathrm{d}\boldsymbol{\Gamma}_1\boldsymbol{A}^2 - \mathrm{d}\boldsymbol{\Gamma}_2 \tag{B.2}$$

Vectorizing (B.1) and (B.2) gives

$$\mathrm{dvec}(\boldsymbol{F}_1) = \boldsymbol{I}_m\otimes\boldsymbol{\Gamma}_0\,\mathrm{dvec}(\boldsymbol{A}) - \boldsymbol{A}'\otimes\boldsymbol{\Gamma}_1\,\mathrm{dvec}(\boldsymbol{A}) - \boldsymbol{I}_m\otimes\boldsymbol{\Gamma}_1\boldsymbol{A}\,\mathrm{dvec}(\boldsymbol{A})$$

and

$$\mathrm{dvec}(\boldsymbol{F}_1) = \boldsymbol{A}'\otimes\boldsymbol{I}_m\,\mathrm{dvec}(\boldsymbol{\Gamma}_0) - \boldsymbol{A}'^2\otimes\boldsymbol{I}_m\,\mathrm{dvec}(\boldsymbol{\Gamma}_1) - \mathrm{dvec}(\boldsymbol{\Gamma}_2)$$

which lead to (3.13) and (3.14).

Similarly, for $\boldsymbol{F}_2$ in (3.11) we have

$$\begin{aligned}\mathrm{d}\boldsymbol{F}_2 &= \boldsymbol{\Gamma}_0\,\mathrm{d}\boldsymbol{\Omega}\boldsymbol{\Gamma}_0' - \boldsymbol{\Gamma}_1\,\mathrm{d}\boldsymbol{A}\boldsymbol{\Omega}\boldsymbol{\Gamma}_0' - \boldsymbol{\Gamma}_1\boldsymbol{A}\,\mathrm{d}\boldsymbol{\Omega}\boldsymbol{\Gamma}_0' - \boldsymbol{\Gamma}_0\,\mathrm{d}\boldsymbol{\Omega}\boldsymbol{A}'\boldsymbol{\Gamma}_1' - \boldsymbol{\Gamma}_0\boldsymbol{\Omega}\,\mathrm{d}\boldsymbol{A}'\boldsymbol{\Gamma}_1' \\ &+ \boldsymbol{\Gamma}_1\,\mathrm{d}\boldsymbol{A}\boldsymbol{\Omega}\boldsymbol{A}'\boldsymbol{\Gamma}_1' + \boldsymbol{\Gamma}_1\boldsymbol{A}\,\mathrm{d}\boldsymbol{\Omega}\boldsymbol{A}'\boldsymbol{\Gamma}_1' + \boldsymbol{\Gamma}_1\boldsymbol{A}\boldsymbol{\Omega}\,\mathrm{d}\boldsymbol{A}'\boldsymbol{\Gamma}_1'\end{aligned} \tag{B.3}$$

and

$$\begin{aligned}\mathrm{d}\boldsymbol{F}_2 = {}& \mathrm{d}\boldsymbol{\Gamma}_0\boldsymbol{\Omega}\boldsymbol{\Gamma}_0' + \boldsymbol{\Gamma}_0\boldsymbol{\Omega}\,\mathrm{d}\boldsymbol{\Gamma}_0' - \mathrm{d}\boldsymbol{\Gamma}_1\boldsymbol{A}\boldsymbol{\Omega}\boldsymbol{\Gamma}_0' - \boldsymbol{\Gamma}_1\boldsymbol{A}\boldsymbol{\Omega}\,\mathrm{d}\boldsymbol{\Gamma}_0' - \mathrm{d}\boldsymbol{\Gamma}_0\boldsymbol{\Omega}\boldsymbol{A}'\boldsymbol{\Gamma}_1' - \\ & \boldsymbol{\Gamma}_0\boldsymbol{\Omega}\boldsymbol{A}'\,\mathrm{d}\boldsymbol{\Gamma}_1' + \mathrm{d}\boldsymbol{\Gamma}_1\boldsymbol{A}\boldsymbol{\Omega}\boldsymbol{A}'\boldsymbol{\Gamma}_1' + \boldsymbol{\Gamma}_1\boldsymbol{A}\boldsymbol{\Omega}\boldsymbol{A}'\,\mathrm{d}\boldsymbol{\Gamma}_1' - \mathrm{d}\boldsymbol{\Gamma}_3\boldsymbol{\Gamma}_3 - \boldsymbol{\Gamma}_3\,\mathrm{d}\boldsymbol{\Gamma}_3'\end{aligned} \tag{B.4}$$

Vectorizing (B.3) and (B.4) gives

$$\operatorname{dvec}(\boldsymbol{F}_2) = \Big(\boldsymbol{\Gamma}_0 \otimes \boldsymbol{\Gamma}_0 - \boldsymbol{\Gamma}_0 \otimes \boldsymbol{\Gamma}_1\boldsymbol{A} - \boldsymbol{\Gamma}_1\boldsymbol{A} \otimes \boldsymbol{\Gamma}_0 + \boldsymbol{\Gamma}_1\boldsymbol{A} \otimes \boldsymbol{\Gamma}_1\boldsymbol{A}\Big)\boldsymbol{D}_n \operatorname{dvech}(\boldsymbol{\Omega}) \tag{B.5}$$
$$- \Big(\boldsymbol{\Gamma}_0\boldsymbol{\Omega} \otimes \boldsymbol{\Gamma}_1 + \big(\boldsymbol{\Gamma}_1 \otimes \boldsymbol{\Gamma}_0\boldsymbol{\Omega}\big)\boldsymbol{K}_{mm} - \boldsymbol{\Gamma}_1\boldsymbol{A}\boldsymbol{\Omega} \otimes \boldsymbol{\Gamma}_1 - \big(\boldsymbol{\Gamma}_1 \otimes \boldsymbol{\Gamma}_1\boldsymbol{A}\boldsymbol{\Omega}\big)\boldsymbol{K}_{mm}\Big) \operatorname{dvec}(\boldsymbol{A})$$

and

$$\operatorname{dvec}(\boldsymbol{F}_2) = \Big(\boldsymbol{\Gamma}_0\boldsymbol{\Omega} \otimes \boldsymbol{I}_m + \big(\boldsymbol{I}_m \otimes \boldsymbol{\Gamma}_0\boldsymbol{\Omega}\big)\boldsymbol{K}_{mm} - \boldsymbol{I}_m \otimes \boldsymbol{\Gamma}_1\boldsymbol{A}\boldsymbol{\Omega}\boldsymbol{K} - \boldsymbol{\Gamma}_1\boldsymbol{A}\boldsymbol{\Omega} \otimes \boldsymbol{I}_m\Big) \operatorname{dvec}(\boldsymbol{\Gamma}_0)$$
$$- \Big(\boldsymbol{\Gamma}_0\boldsymbol{\Omega}\boldsymbol{A}' \otimes \boldsymbol{I}_m + \big(\boldsymbol{I}_m \otimes \boldsymbol{\Gamma}_0\boldsymbol{\Omega}\boldsymbol{A}'\big)\boldsymbol{K}_{mm} - \big(\boldsymbol{I}_m \otimes \boldsymbol{\Gamma}_1\boldsymbol{A}\boldsymbol{\Omega}\boldsymbol{A}'\big)\boldsymbol{K}_{mm} - \boldsymbol{\Gamma}_1\boldsymbol{A}\boldsymbol{\Omega}\boldsymbol{A}' \otimes \boldsymbol{I}_m\Big) \operatorname{dvec}(\boldsymbol{\Gamma}_1)$$
$$- \Big(\boldsymbol{\Gamma}_3 \otimes \boldsymbol{I}_m + \big(\boldsymbol{I}_m \otimes \boldsymbol{\Gamma}_3\big)\boldsymbol{K}_{mm}\Big) \operatorname{dvec}(\boldsymbol{\Gamma}_3) \tag{B.6}$$

Table B.1: Prior Distribution of θ

Parameter	Distr.	Mean	Stdd.
α	$\mathcal{N}$	0.300	0.050
ψ	$\mathcal{B}$	0.500	0.150
φ	$\mathcal{N}$	4.000	1.500
σ_c	$\mathcal{N}$	1.500	0.375
h	$\mathcal{B}$	0.700	0.100
$100(\beta^{-1}-1)$	$\mathcal{G}$	0.250	0.100
Φ	$\mathcal{N}$	1.250	0.125
ι_w	$\mathcal{B}$	0.500	0.150
ξ_w	$\mathcal{B}$	0.500	0.100
ι_p	$\mathcal{B}$	0.500	0.150
ξ_p	$\mathcal{B}$	0.500	0.100
σ_l	$\mathcal{N}$	2.000	0.750
r_π	$\mathcal{N}$	1.500	0.250
$r_{\triangle y}$	$\mathcal{N}$	0.125	0.050
r_y	$\mathcal{N}$	0.125	0.050
ρ	$\mathcal{B}$	0.750	0.100
γ	$\mathcal{N}$	0.400	0.100
δ	$\mathcal{B}$	0.025	0.005
λ_w	$\mathcal{N}$	1.500	0.250
g_y	$\mathcal{N}$	0.180	0.050
ρ_{ga}	$\mathcal{B}$	0.500	0.250
ρ_a	$\mathcal{B}$	0.500	0.200
ρ_b	$\mathcal{B}$	0.500	0.200
ρ_g	$\mathcal{B}$	0.500	0.200
ρ_I	$\mathcal{B}$	0.500	0.200
ρ_r	$\mathcal{B}$	0.500	0.200
ρ_p	$\mathcal{B}$	0.500	0.200
ρ_w	$\mathcal{B}$	0.500	0.200
μ_w	$\mathcal{B}$	0.500	0.200
μ_p	$\mathcal{B}$	0.500	0.200
σ_a	$\mathcal{IG}$	0.100	2.000
σ_b	$\mathcal{IG}$	0.100	2.000
σ_g	$\mathcal{IG}$	0.100	2.000
σ_I	$\mathcal{IG}$	0.100	2.000
σ_r	$\mathcal{IG}$	0.100	2.000
σ_p	$\mathcal{IG}$	0.100	2.000
σ_w	$\mathcal{IG}$	0.100	2.000

Note:$\mathcal{N}$ is Normal distribution, $\mathcal{B}$ is Beta-distribution, $\mathcal{G}$ is Gamma distribution, $\mathcal{IG}$ is Inverse Gamma distribution.

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