COMPUTING GENERAL EQUILIBRIUM MODELS WITH OCCUPATIONAL CHOICE AND FINANCIAL FRICTIONS

António Antunes
Tiago Cavalcanti
Anne Villamil

Please address correspondence to
António Antunes
Economic Research Department
Banco de Portugal, Av. Almirante Reis no. 71, 1150-012 Lisboa, Portugal;
Tel.: 351 21 312 8246, Email: aantunes@bportugal.pt

Computing General Equilibrium Models
with Occupational Choice and Financial Frictions[*]

António Antunes[†]   Tiago Cavalcanti[‡]   Anne Villamil[§]

September 29, 2006

Abstract

This paper establishes the existence of a stationary equilibrium and a procedure to compute solutions to a class of dynamic general equilibrium models with two important features. First, occupational choice is determined endogenously as a function of heterogeneous agent type, which is defined by an agent's managerial ability and capital bequest. Heterogeneous ability is exogenous and independent across generations. In contrast, bequests link generations and the distribution of bequests evolves endogenously. Second, there is a financial market for capital loans with a deadweight intermediation cost and a repayment incentive constraint. The incentive constraint induces a non-convexity. The paper proves that the competitive equilibrium can be characterized by the bequest distribution and factor prices, and uses the monotone mixing condition to ensure that the stationary bequest distribution that arises from the agent's optimal behavior across generations exists and is unique. The paper next constructs a direct, non-parametric approach to compute the stationary solution. The method reduces the domain of the policy function, thus reducing the computational complexity of the problem.

JEL Classification: C62; C63; E60; G38.

Keywords: Existence; Computation; Dynamic general equilibrium; Non-convexity.

1   Introduction

This paper establishes the existence of a stationary equilibrium and a procedure to compute solutions to dynamic general equilibrium models with occupational choice and financial frictions. Occupational choice models are common in macroeconomics and there is a voluminous literature on financial market frictions.[1] These models often have non-convexities which give rise to discontinuous stochastic behavior (e.g., Antunes, Cavalcanti and Villamil, 2006); standard fixed point existence arguments that require

continuity are not applicable.[2] Hopenhayn and Prescott (1992) remedy this problem by proving existence of stationary equilibria for stochastically monotone processes. They use the Knaster-Tarski fixed point theorem to prove existence of fixed point mappings on compact sets of measures that are increasing with respect to a stochastic ordering (monotone). Our contribution is two-fold. First, we show how the Hopenhayn and Prescott result can be applied to this class of dynamic general equilibrium models to prove existence of a stationary equilibrium. Second, we construct a direct, non-parametric approach to compute the stationary solution. Our method reduces the domain of the policy function, thus reducing the computational complexity of the problem.

The class of models that we consider have two important features. First, occupational choice is determined endogenously as a function of heterogeneous agent type. Agents are endowed with different innate abilities to manage a firm (cf., Lucas, 1978) and different bequests (cf., Antunes et al., 2006). Heterogeneous ability is exogenous, in the sense that managerial ability is drawn from a fixed distribution, and is independent within and across generations. In contrast, agents choose consumption and bequests to maximize preferences subject to lifetime wealth. Bequests thus connect generations across time periods and the distribution of bequests evolves endogenously. Second, there is a financial market for capital with two frictions: a deadweight cost to intermediate loans and an incentive constraint to ensure loan repayment. The incentive constraint induces a non-convexity. We characterize the competitive equilibrium, and then use a condition derived by Hopenhayn and Prescott, monotone mixing, to ensure that the optimal stationary bequest distribution that arises from the stochastic optimization problem exists and is unique.

The paper proceeds as follows. Section 2 contains the model. Section 3 describes optimal consumption and production behavior. On the production side, agents choose an occupation (to manage a firm or work) and firm finance (if a manager). Consumers choose consumption and bequests, where bequests link agents across periods. Section 4 specifies the competitive equilibrium and proves existence of a stationary equilibrium. We show that there is a unique stationary equilibrium that is fully characterized by a time invariant bequest distribution and associated equilibrium factor prices. We use the monotone mixing condition from Hopenhayn and Prescott (1992) Theorem 2. In our context, this condition characterizes two types of mobility in the bequest distribution: Given that ability is independent across generations, there is a positive probability that a future descendent of an agent changes occupation (i.e., from worker to entrepreneur or from entrepreneur to worker). Thus the economy experiences occupation mobility, but from any initial bequest distribution and any interest rate, convergence to a unique invariant bequest distribution occurs. Finally, section 5 contains the numerical solution method.

2 The Model

Consider an economy with a continuum of measure one agents who live for one period. Each agent reproduces another such that population is constant. There is one good each period that can be used for consumption or production, or left to the next generation as a bequest. Time is discrete and infinite, with $t = 0, 1, 2, ...$

---

[2]Standard Brouwer-Kakutani type fixed point theorems cannot be used because the fixed point mapping is not necessarily continuous or upper or lower semi-continuous (cf., Krasa and Yannelis (1994)). The Knaster-Tarski fixed point theorem is non-topological.

2.1 Preferences, Endowments, Technology and Frictions

2.1.1 Preferences

In period t, agent i's utility is defined over personal consumption and a bequest to offspring, denoted by c_t^i and b_{t+1}^i, respectively. Assume the utility function has the form

$$U^i = u(c_t^i, b_{t+1}^i). \tag{1}$$

$u(\cdot, \cdot)$ is twice continuously differentiable, strictly concave and increasing in both arguments. We also assume that the utility function satisfies the Inada conditions.[3] Preferences are for the bequest and not the offspring's utility (cf., Banerjee and Newman, 1993).

2.1.2 Heterogeneous Endowments

Each period, agents are distinguished by their publicly known endowments of initial wealth and ability as entrepreneurs, denoted by (b_t^i, x_t^i). The bequest is inherited from the previous generation and generates a given initial bequest distribution $\Upsilon(b)$. Each individual's talent for managing, x^i, is drawn from a continuous cumulative probability distribution function $\Gamma(x)$, with $x \in [\underline{x}, \overline{x}]$, cf., Lucas (1978). Assume that managerial talent is not hereditary. Each individual will choose to be either a worker or an entrepreneur. Entrepreneurs create jobs and manage firm labor n; workers are employed by entrepreneurs at wage w_t. For notational convenience, in the remainder of the paper we drop agent superscript i.

2.1.3 Production

Managers operate a technology that uses labor, n, and capital, k, to produce a single consumption good, y, where

$$y = xf(k, n). \tag{2}$$

$f(\cdot, \cdot)$ is twice continuously differentiable, strictly concave and increasing in both arguments. Function $f(\cdot, \cdot)$ is also homogenous of degree less than one and satisfies the Inada conditions. Capital fully depreciates between periods. Managers can operate only one project. The labor and capital markets are competitive, with prices w and r, respectively.

2.1.4 Capital Market Frictions, τ and ϕ

The capital market has two frictions:

τ: Agents deposit bequest b in a financial intermediary and earn competitive return r. The intermediary lends the resources to entrepreneurs. The part of the loan that is fully collateralized by b costs r; the remainder costs $r_l = r + \tau$, where τ is the deadweight cost of intermediation.

ϕ: Borrowers cannot commit *ex-ante* to repay, but there is an exogenous enforcement technology. An agent who defaults on a loan incurs penalty ϕ, which is the percentage of output forfeited net of wages. This is equivalent to an additive utility punishment, and reflects the strength of contract enforcement: As $\phi \to 1$ the penalty is strong; as $\phi \to 0$ it is weak.[4]

---

[3]Both Cobb-Douglas and CES utility functions satisfy these restrictions.

[4]As is common in the literature, we choose a proportional punishment for convenience. See Krasa and Villamil (2000) and Krasa, Sharma and Villamil (2005) for extended analysis of enforcement and debt contracts.

3 Optimal Behavior

3.1 Entrepreneurs

Agents who have sufficient resources and managerial ability to become entrepreneurs choose the level of capital and the number of employees to maximize profit subject to a technological constraint and (possibly) a credit market incentive constraint. Consider first the problem of an entrepreneur for a given level of capital k and wages w:

$$\pi(k, x; w) = \max_n x f(k, n) - wn. \tag{3}$$

This yields the labor demand of each entrepreneur: $n(k, x; w)$. This labor demand is differentiable, continuous in all arguments, increasing in k and x, and decreasing in w. Moreover, $\lim_{w \to 0} n(k, x; w) = \infty$ and $\lim_{w \to \infty} n(k, x; w) = 0$.

Substituting $n(k, x; w)$ into (3) yields the entrepreneur's profit function for a given level of capital, $\pi(k, x; w)$. The profit function is differentiable, continuous in all arguments, increasing in k and x, and decreasing in w. We now consider two problems: when initial wealth is sufficient to fully finance a firm and when it may not be. Let a be the amount of self-financed capital and l be the amount of funds borrowed from a financial intermediary.[5]

Unconstrained Problem. When initial wealth is sufficient for the agent to start her own business without resorting to credit finance (i.e., $b > a$ and $l = 0$), entrepreneurs solve:

Problem 1

$$\max_{k \geq 0} \pi(k, x; w) - (1 + r)k. \tag{4}$$

This gives the optimal physical capital level, $k^*(x; w, r)$, which is continuous in all arguments, and strictly decreasing in factor prices w and r. It can be also shown that $\lim_{r \to -1} k^*(x; w, r) = \infty$ and $\lim_{r \to \infty} k^*(x; w, r) = 0$. There is no credit market incentive constraint because the firm is entirely self-financed (i.e., no repayment problem exists).

Constrained Problem. When the entrepreneur's initial wealth may not be sufficient to finance the firm (i.e., $b \geq a$ and $l \geq 0$), the agent will consider loans from the credit market. Since no agent can commit to repay a loan, these debt contracts must be self-enforcing.

The entrepreneur now maximizes the net income from operating the project:

Problem 2

$$V(b, x; w, r) = \max_{b \geq a \geq 0, \ l \geq 0} \pi(a + l, x; w) - (1 + r)a - (1 + r + \tau)l \tag{5}$$

subject to:

$$\phi \pi(a + l, x; w) \geq (1 + r + \tau)l. \tag{6}$$

Feasibility constraint $b \geq a \geq 0$, implicit in the objective, states that the amount of (non-trivial) self finance, a, cannot exceed the entrepreneur's bequest, b. Problem 2 yields optimal policy functions $a(b, x; w, r)$ and $l(b, x; w, r)$ that define the size of each firm,

$$k(b, x; w, r) = a(b, x; w, r) + l(b, x; w, r).$$

Restriction (6) is an incentive feasibility constraint which guarantees that ex-ante repayment promises will be honored, cf., Kehoe and Levine (1993).[6]

---

[5]Equivalently, a can be interpreted as the part of a loan that is fully collateralized by personal assets and l as the uncollateralized part.

[6]The restriction requires the percentage of firm profit the financial intermediary seizes in default to be at least as high

3.1.1 Solutions to the Entrepreneur's Problem

The Lagrangian associated with problem 2 is

$$L = \pi(a + l, x; w) - (1 + r)a - (1 + r + \tau)l$$
$$+ \lambda(\phi\pi(a + l, x; w) - (1 + r + \tau)l) + \chi(b - a).$$

The Kuhn-Tucker conditions are:

$$\frac{\partial L}{\partial l} = \pi_1(a + l, x; w) - (1 + r + \tau) + \lambda(\phi\pi_1(a + l, x; w) - (1 + r + \tau)) \leq 0, \tag{7}$$

$$\frac{\partial L}{\partial a} = \pi_1(a + l, x; w) - (1 + r) + \lambda\phi\pi_1(a + l, x; w) - \chi \leq 0, \tag{8}$$

$$\lambda(\phi\pi(a + l, x; w) - (1 + r + \tau)l) = 0, \tag{9}$$

$$\chi(b - a) = 0, \tag{10}$$

$$l \geq 0, \quad \frac{\partial L}{\partial l}l = 0, \quad a \geq 0, \quad \frac{\partial L}{\partial a}a = 0, \quad \lambda \geq 0, \quad \chi \geq 0,$$

plus the incentive compatibility constraint (6) and constraint $b \geq a$. Constrained entrepreneurs are those for which $l > 0$ holds. It is optimal for entrepreneurs to put their entire wealth in their project. To see this, assume that constrained entrepreneurs do not put their entire wealth in the project; that is, $0 \leq a < b$. Then, from (10), $\chi = 0$, and from (7) at equality and (8) it follows that $(1+r)\lambda + (1+\lambda)\tau \leq 0$, which is a contradiction. Therefore, if entrepreneurs are credit constrained, $a = b$.

There are four cases of the Kuhn-Tucker conditions to consider:

1. $0 < a < b$, and $l = 0$. Then from (9) and (10), $\chi = \lambda = 0$ and

$$a = k^*(x; r, w). \tag{11}$$

2. $0 < a = b$, and $l = 0$, but $\phi\pi(a + l, x; w) - (1 + r + \tau)l > 0$. This case arises because intermediation implies a discrete jump in costs. We have $\lambda = 0$ and χ (which is non-negative) given by equation (8) at equality:

$$\chi = \pi_1(a + l, x; w) - (1 + r). \tag{12}$$

   The interpretation is straightforward: while the entrepreneur would invest more if she had a higher bequest, the incremental profit from borrowing is non-positive, as can be seen in equation (7). The entrepreneur's marginal profit exceeds $1 + r$ but is smaller than $1 + r + \tau$.

3. $0 < a = b$, and $l > 0$, but $\phi\pi(a + l, x; w) - (1 + r + \tau)l > 0$. Then from (9), $\lambda = 0$, and by (7) and (8) at equality it follows that $\chi = \tau$. Therefore,

$$l + b = k^*(x; r + \tau, w), \tag{13}$$

   where $k^*(x; r + \tau, w)$ is an unconstrained maximizer of π if the interest rate is $r_l = r + \tau$.

4. $0 < a = b$ and $l > 0$, but $\phi\pi(a + l, x; w) - (1 + r + \tau)l = 0$. This is the credit-constrained case. The total loan $l(b, x; r, w)$ is given by the solution to the previous equation with a substituted by b, and $\chi = \tau(1 + \lambda)$.

Entrepreneurs invest their entire wealth in their firm as long as $b \leq k^*(x; w, r)$. This follows immediately from the fact that the cost of self-finance is lower than using a financial intermediary. Firm size k

---

as the repayment obligation.

of an entrepreneur (b, x) is such that

$$k \leq b + \frac{\phi}{1+r+\tau}\pi(b+l, x; w). \tag{14}$$

The arguments of k and l are omitted for readability. Thus, firm size is limited by an agent's inheritance, b, and the capital market frictions, τ and ϕ.

The following lemma characterizes the value function and policy functions:

Lemma 1 *For any $x \in [\underline{x}, \overline{x}]$, $w > 0$ and $r > -1$, the value function $V(b, x; w, r)$ and the associated policy function $l(b, x; w, r)$ have the following properties:*

1. *$V(b, x; w, r)$ is continuous and differentiable in x, w and r. If $x > 0$, it is also strictly increasing in x and strictly decreasing in w and r.*

2. *For $b < k^*(x; w, r)$, $V(b, x; w, r)$ is continuous, differentiable and strictly increasing in b. For $b \geq k^*(x; w, r)$, $V(b, x; w, r)$ is constant in b.*

3. *For credit constrained agents, $l(b, x; w, r)$ is strictly increasing in b for $b < k^*(x; w, r)$; $l(b, x; w, r) = 0$ for $b \geq k^*(x; w, r)$.*

Proof. Continuity of $V(b, x; w, r)$ follows from the Maximum Theorem and differentiability, cf., Theorem 4.11 of Stokey and Lucas (1989). From the Lagrangian and the Envelope Theorem it is easily seen that, provided $x > 0$,

$$V_2(b, x; w, r) = \pi_2(b+l, x; w)(1+\lambda\phi) > 0,$$
$$V_3(b, x; w, r) = \pi_3(b+l, x; w)(1+\lambda\phi) < 0,$$
$$V_4(b, x; w, r) = -a - (1+\lambda)l < 0,$$

We omit the arguments of a, l and λ for readability. If $b < k^*(x; w, r)$, the net income from entrepreneurship would increase if b increased: cases 3 and 4 imply that $\chi > 0$; case 2 implies that $\chi = \pi_1(b, x; w) - (1+r)$, which is positive because π_1 is decreasing in k and $k^*(x; w, r)$ is found by equating this expression to 0; therefore, $\chi > 0$. By the Envelope Theorem,

$$V_1(b, x; w, r) = \chi > 0.$$

When $b \geq k^*(x; w, r)$, then by the definition of $k^*(x; w, r)$ the net income from entrepreneurship cannot increase and $V_1(b, x; w, r) = 0$ and $l = 0$. When agents are credit constrained, the incentive constraint holds with equality and

$$\phi\pi(b+l, x; w) = (1+r+\tau)l.$$

Thus,

$$\frac{\partial l}{\partial b} = \frac{\phi\pi_1(k, x; w)}{1+r+\tau-\phi\pi_1(k, x; w)}.$$

By condition (7), we have that $(1+r+\tau) - \phi\pi_1(k, x; w) = \frac{\pi_1(k,x;w)-(1+r+\tau)}{\lambda}$. Since this is for constrained agents, $\lambda > 0$ and $\pi_1(k, x; w)$ is greater than $1+r+\tau$. Therefore,

$$\frac{\partial l}{\partial b} = \lambda\frac{\phi\pi_1(k, x; w)}{\pi_1(k, x; w) - (1+r+\tau)} > 0.$$

∎

3.1.2 Occupational choice

The occupational choice of each agent determines lifetime income. Define $\Omega = [0, \infty) \times [\underline{x}, \overline{x}]$. For any $w, r > 0$, an agent (b, x) will become an entrepreneur if $(b, x) \in E(w, r)$, where

$$E(w, r) = \{(b, x) \in \Omega : V(b, x; w, r) \geq w\}. \tag{15}$$

Let $E^c(w, r)$ denote the complement set of $E(w, r)$ in Ω. Obviously, if $(b, x) \in E^c(w, r)$, then agents are workers. The following lemma characterizes the occupational choice for a given bequest and entrepreneurial ability.

Lemma 2 *Define $b_e(x; w, r)$ as the curve in set Ω such that $V(b, x; w, r) = w$. Then there exists an $x^*(w, r)$ such that $\frac{\partial b_e(x; w, r)}{\partial x} < 0$ for $x > x^*(w, r)$ and $\frac{\partial b_e(x; w, r)}{\partial x} = -\infty$ for $x = x^*(w, r)$.*

1. For all $x > x^*$, if $b < b_e(x; w, r)$, then $(b, x) \in E^c(w, r)$.

2. For all $x > x^*$, if $b \geq b_e(x; w, r)$, then $(b, x) \in E(w, r)$.

Proof. We must show that there exists an $x^*(w, r)$ such that $\frac{\partial b_e(x; w, r)}{\partial x} < 0$ for $x > x^*(w, r)$, and $\frac{\partial b_e(x; w, r)}{\partial x} = -\infty$ for $x = x^*(w, r)$. Observe that at all points where $b_e(x; w, r)$ is differentiable,

$$V(b, x; w, r) = w$$

defines $b_e(x; w, r)$ such that

$$\frac{\partial b_e}{\partial x}(x; w, r) = -\frac{V_2(b, x; w, r)}{V_1(b, x; w, r)},$$

where $V_1(b, x; w, r)$ and $V_2(b, x; w, r)$ are derived in the proof of Lemma 1.

First consider the case where $x < x^*(w, r)$ for $V(b, x; w, r) \geq w$ when agents are not borrowing constrained. We wish to characterize $(b, x) \in E^c(w, r)$.[7] The proof of Lemma 1 establishes that when agents have $b > k^*(x; w, r)$ and thus are not borrowing constrained, $V_1(b, x; w, r) = 0$. As a consequence, $\frac{\partial b_e(x; w, r)}{\partial x} = -\infty$ for $x = x^*(w, r)$. The critical $x^*(w, r)$ is clearly independent of b, which implies that agents prefer to be workers rather than managers for $x < x^*(w, r)$ even when $V(b, x; w, r) \geq w$.

Now we wish to characterize $(b, x) \in E(w, r)$. Consider the case where agents are borrowing constrained and $x \geq x^*(w, r)$. Now

$$\frac{\partial b_e}{\partial x}(x; w, r) = -\frac{V_2(b, x; w, r)}{V_1(b, x; w, r)} < 0,$$

because the proof of Lemma 1 establishes that $V_1(b, x; w, r) > 0$ and $V_2(b, x; w, r) > 0$. It follows immediately that for all $x > x^*$ we have $\frac{\partial b_e(x; w, r)}{\partial x} < 0$. If $b < b_e(x; w, r)$, then $(b, x) \in E^c(w, r)$ and if $b \geq b_e(x; w, r)$ we have that $(b, x) \in E(w, r)$. ∎

Lemma 2 indicates that agents are workers when the quality of their project is low, i.e., $x < x^*(w, r)$. For $x \geq x^*(w, r)$ agents may become entrepreneurs, depending on whether they are credit constrained or not. For very low bequests agents might be workers even though their entrepreneurial ability is higher than $x^*(w, r)$. The negative association between $b_e(x; w, r)$ and x indicates that managers with better managerial ability need a lower level of initial wealth to run a firm. This is intuitive since profits are increasing in managerial ability.

---

[7]Note that if $V(b, x; w, r) < w$ it is optimal for the agent to become a worker and $(b, x) \in E^c(w, r)$.

3.2 Consumers

In period t, the lifetime wealth of an agent characterized by (b_t, x_t) is given by

$$Y_t = Y(b_t, x_t; w_t, r_t) = \max\{w_t, V(b_t, x_t; w_t, r_t)\} + (1 + r_t)b_t. \tag{16}$$

Lifetime wealth is thus a function of agent-specific b_t and x_t, and economy-wide w_t and r_t. Given lifetime wealth, agents choose consumption and bequests to maximize preferences (1). This problem defines optimal policies for consumption, $c_t = c(Y_t)$, and bequest, $b_{t+1} = b(Y_t)$. Policy functions $c(\cdot)$ and $b(\cdot)$ are clearly continuous, differentiable, and increasing in Y_t. We assume that bequests cannot be negative because parents cannot borrow from their descendants future income (cf., Banerjee and Newman, 1993 or Lloyd-Ellis and Bernhardt, 2000).

4 Competitive Equilibrium

We now define and prove existence of a competitive equilibrium. We use Hopenhayn and Prescott (1992), Theorem 2 to establish the existence of a stationary equilibrium through a stochastic monotonicity condition. They use the Knaster-Tarski fixed point theorem to prove existence of fixed point mappings on compact sets of measures that are increasing with respect to a stochastic ordering (monotone). As Hopenhayn and Prescott show, stochastic monotonicity arises in economic models from monotone decision rules, which result from agents' optimizing behavior. They establish a monotone mixing condition under which the optimal stationary policies in a dynamic stochastic problem are unique. The intuition is as follows. Consider two bequest distributions. A monotone mapping and its iterates preserve the ordering of the two distributions, but after finitely many iterations some mass in the distributions reverse order. The monotone mixing condition implies that this can only occur in the limit if the two distributions coincide.

4.1 Notation and Definitions

We begin by introducing some useful notation:

$Z = [\underline{b}, \overline{b}]$ is the set of possible bequests.

$(Z, \mathcal{B})$ is a measurable space with Borel algebra $\mathcal{B}$ for the set.

$\Lambda(Z, \mathcal{B})$ is the set of all possible probability measures defined on measurable space $(Z, \mathcal{B})$.[8]

For any (b_t, A) in $(Z, \mathcal{B})$, measure P_t defines a non-stationary transition probability function

$$P_t(b_t, A) = \Pr\{b_{t+1} \in A | b_t\}.$$

$B(Z)$ is the set of real-valued bounded functions defined on Z.

$h : B(Z) \to B(Z)$ is a bounded and non-decreasing function.

Function P_t assigns a probability to event A for the descendant of an agent with bequest b_t who does not yet know x_t. We must determine the probability that next period's bequest lies in set A, given that the current bequest is b_t. Function P_t is important because it affects the law of motion of the bequest distribution,

$$\Upsilon_{t+1} = \int P_t(b_t, A)\Upsilon_t(db_t).$$

The following definitions will be useful:

---

[8]Υ_t, which specifies the probability of each event in $\mathcal{B}$ at time t, belongs to $\Lambda(Z, \mathcal{B})$.

Definition 1 Let $(Z, \mathcal{B})$ be a measurable space. A *transition function* is a function $P : Z \times \mathcal{B} \to [0, 1]$ such that

1. for each $z \in Z$, $P(z, \cdot)$ is a probability measure on $(Z, \mathcal{B})$; and

2. for each $A \in \mathcal{B}$, $P(\cdot, A)$ is a $\mathcal{B}$-measurable function.

Definition 2 For any $\mathcal{B}$-measurable function h, define Th by $(Th)(b) = \int h(b')P(b, db')$, $\forall b \in Z$, where operator $T : B(Z) \to B(Z)$ is associated with transition function P.

Definition 3 For any probability measure λ on $(Z, \mathcal{B})$, define $(T^*\lambda)(A) = \int P(b, A)\lambda(db)$, $\forall A \in \mathcal{B}$, where operator $T^* : \Lambda(Z, \mathcal{B}) \to \Lambda(Z, \mathcal{B})$ and $\lambda \to T^*(\lambda)$ is associated with transition function P.

Definition 4 A transition function P on $(Z, \mathcal{B})$ is *monotone* if the associated operator T has the property that for every nondecreasing function $h : Z \to R$, the function Th is also nondecreasing.

Definition 1 establishes a 1-step transition function; we will generalize this to an n-step transition function for a sequence of bequest shocks in n successive periods. Definition 2 gives the expected value of h next period given that the current state is b. Definition 3 gives the probability that the state next period is in A, given that the current state is drawn according to λ. Operators T and T^* are well-defined (see Stokey and Lucas (1989), Corollary and Theorem 8.2 respectively, p. 215). In Proposition 5 the law of motion will be defined recursively by applying operator T^* on Υ_0 to get the n-step transition function.

We now define an equilibrium.

Definition 5 Given distributions Γ and Υ_t, an equilibrium at date t is a w_t, r_t, $n(x; w_t, r_t)$, $l(b, x; w_t, r_t)$, $a(b, x; w_t, r_t)$, $k(b, x; w_t, r_t)$, $c_t = c(\cdot)$, and $b_{t+1} = b(\cdot)$ such that:

A. Given w_t, r_t, an agent of type (b, x) chooses an occupation to maximize lifetime wealth, (16).

B. Given w_t, r_t, the technology constraint and frictions, entrepreneurs choose n to maximize profits, (3).

C. $l(b, x; w_t, r_t)$ and $a(b, x; w_t, r_t)$ solve (5) and $k(b, x; w_t, r_t) = a(b, x; w_t, r_t) + l(b, x; w_t, r_t)$.

D. Given lifetime wealth, (16), each agent maximizes utility, (1).

E. The labor market clears:

$$\iint\limits_{z \in E^c(w_t, r_t)} \Upsilon_t(db_t)\Gamma(dx_t) = \iint\limits_{z \in E(w_t, r_t)} n(x; w_t, r_t)\Upsilon_t(db_t)\Gamma(dx_t).$$

F. The aggregate supply of funds for investment is given by initial wealth:

$$\iint b_t \Upsilon_t(db_t)\Gamma(dx_t) = \iint\limits_{z \in E(w_t, r_t)} k(b_t, x_t; w_t, r_t)\Upsilon_t(db_t)\Gamma(dx_t).$$

In the model the only connection between periods is bequests.[9] Thus, we wish to show that there is a unique stationary equilibrium summarized by $(\Upsilon, r^*, w(r^*))$. Specifically, we must establish the existence

---

[9]Bequest b_{t+1} is chosen optimally in the consumer's problem and connects generations. Bequest distribution Υ_t thus evolves endogenously across periods. In contrast to bequests, managerial talent x_t is not inherited; realization x_t is drawn independently across agents and generations from the fixed distribution Γ each period.

of a unique time invariant distribution Υ and associated equilibrium wage w and interest rate r, such that from any initial distribution Υ_0, the operator $T^*\Upsilon_t$ converges to a unique Υ. Thus, competitive equilibrium conditions A through F are summarized by Υ, w, and r.

4.2   Preliminary Results

We first show that for any finite interest rate r, there exists a positive and unique wage rate in each period t that clears the labor market.

Lemma 3 *Assume that ability cumulative distribution function Γ is continuous on $[\underline{x}, \overline{x}]$, free disposal of bequests, and an interest rate $r \in I$, where $I = [-1, \bar{r}]$ and $\bar{r} < \infty$. Then:*

1. *There exists $0 < w(r) < \infty$, continuous in r, that clears the labor market.*

2. *There exists $\underline{w}(r) > 0$ such that $\underline{w}(r) \leq w(r)$ for all bequest probability measures Υ.*

3. *There exists $\underline{w} > 0$ such that $\underline{w} \leq \underline{w}(r)$ for all bequest probability measures Υ and interest rates $r \in I$.*

Proof. The lower bound of interval I is -1. Any lower interest rate would imply a negative end-of-period value for initial inheritance; agents who chose to be workers would dispose of b.

Given bequest and ability distributions, Υ and Γ, define the labor excess demand function by

$$\text{LED}(r, w) = \iint\limits_{E(r,w)} (1 + n(b, x; r, w))\Upsilon(db)\Gamma(dx) - 1 \, .$$

Functions $n(b, x; r, w)$ and $V(b, x; r, w)$ are continuous in w and r (see Lemma 1). Probability distribution $\Upsilon\Gamma$ has no points with positive mass probability, which implies that the measure of set $E(r, w)$ varies smoothly.[10] Therefore, function $\text{LED}(r, w)$ is continuous. In addition, $n(b, x; r, w)$ and $V(b, x; r, w)$ are strictly decreasing in w. As $w \to 0$, $V(b, x; r, w)$ is unbounded and no agent wishes to become a worker. It follows that $\text{LED}(r, w) > 0$. When w increases sufficiently, $\text{LED}(r, w) < 0$, since all agents wish to become workers. Therefore, by continuity of $\text{LED}(r, w)$, there must be some $0 < w(r) < \infty$ such that $\text{LED}(r, w(r)) = 0$. Function $w(r)$ is continuous by continuity of $\text{LED}(r, w)$.

We now show $\underline{w} \leq \underline{w}(r) \leq w(r)$ for some $\underline{w}, \underline{w}(r) > 0$: Consider an initial bequest distribution that assigns a zero bequest to all agents. Assume that at rate r, the intermediary can borrow abroad any amount needed to fulfill the internal demand for capital or lend any excess supply of capital. We have shown that equilibrium wage rate $\underline{w}(r)$ is positive and finite. Since the wage rate is positive, next period's bequests will all be positive. Therefore, the set of possible occupational choices cannot shrink, and may expand. This implies that for wage $\underline{w}(r)$, excess demand is nonnegative, $\text{LED}(r, \underline{w}(r)) \geq 0$, which in turn means that for this new bequest distribution the wage rate that clears the labor market is higher than $\underline{w}(r)$. This function is again continuous. Consequently, $\underline{w}(r)$ is the lowest equilibrium wage rate when the interest rate is r. By continuity of $\underline{w}(r)$ and compactness of I we can define $\underline{w} > 0$ such that $\underline{w} \leq \underline{w}(r)$ for all r in I. ∎

Given that the wage rate is positive, we can show that for any interest rate $r \in I$ the distribution of bequests is bounded. Define the share of consumption as

$$\gamma(Y) = \frac{c(Y)}{Y} \, .$$

---

[10]Although Υ might have positive mass probability points, $\Upsilon\Gamma$ does not because Γ is continuous.

The fact that the marginal utility of bequests is positive and unbounded as bequests go to zero, and that the same is true for consumption, implies that $\{\gamma(Y), \forall Y \in \mathbb{R}^+\}$ has a supremum smaller than 1, which we designate by γ, and an infimum higher that 0. From here on, define the upper bound of I as $\overline{r} = (1 - \gamma)^{-1} - 1 - \varepsilon$, where ε is a small positive constant.

Lemma 4 *Assume that Γ is continuous on $[\underline{x}, \overline{x}]$, free disposal of bequests, and $r \in I$, where $I = [-1, \overline{r}]$. For any $r \in I$ and initial bequest probability measure Υ_0 such that each agent receives a positive bequest, the set of possible bequests Z is compact.*

Proof. We wish to show that Z is closed and bounded. The bequest function of an agent of type (b, x) is, in case she is an entrepreneur,

$$g(b, x; r, w) = b(V(b, x; r, w) + (1 + r)b), \tag{17}$$

where $b(\cdot)$ is the bequest function (see section 3.2). Function g is strictly decreasing in w for any entrepreneur.

Consider an upper bound. An upper bound for any "sustainable" bequest is $\overline{b}(r)$ such that

$$\overline{b}(r) = g(\overline{b}(r), \overline{x}; r, \underline{w}(r)) = (1 - \gamma(\overline{Y}))\overline{Y}. \tag{18}$$

This is the bequest that, at the lowest wage possible, the most productive entrepreneur will leave to the next generation given that she received the same bequest. Suppose that $r = -1$. Then by (11), the unconstrained level of capital is infinite. All entrepreneurs will invest all their bequest and some might resort to borrowing.

Function $g(b, \overline{x}; -1, \underline{w}(-1))$ is strictly increasing in b, since by the Envelope Theorem its derivative with respect to b is $g_1(b, \overline{x}; -1, \underline{w}(-1)) = b'(\overline{Y})(\chi + 1 + r)|_{r=-1} = b'(\overline{Y})\chi$, where we defined $\overline{Y} = V(b, \overline{x}; r, \underline{w}) + (1 + r)b$. Given that agents prefer a higher bequest, $\chi > 0$, and $b'(\cdot) > 0$ (see section 3.2), the claim follows.

Now we consider what happens to the function as $b \to 0$ and $b \to \infty$.

(i) When $b \to 0$, the entrepreneur will be in either case 3 or 4 of the Kuhn-Tucker conditions in the entrepreneur's problem in section 3.1.1.[11] It follows immediately from case 3 that when $\lambda = 0$ and (7) and (8) hold at equality, then χ equals the marginal cost of funds, τ; as for case 4, when (7), (8) and the incentive compatibility constraint hold at equality, then χ equals $\tau(1 + \lambda)$. So, for both cases, $\chi \geq \tau$. Moreover, $\lim_{b \to 0} g(b, \overline{x}; -1, \underline{w}(-1)) > 0$, as the entrepreneur's credit limit is always strictly positive. To see this, notice that (14) at equality with $r = -1$, $b = 0$ and $k = l$ yields $\tau l = \phi\pi(l, \overline{x}; \underline{w}(-1))$. Since $\pi(0, \overline{x}; \underline{w}(-1)) = 0$ and π is increasing in l, and the limit of π_1 is infinity as l goes to zero, the equation has a strictly positive solution, which is the credit limit.

(ii) When $b \to \infty$, case 2 applies because the entrepreneur does not borrow. Case 2 establishes that the marginal profit of funds is π_1, which goes to zero as b goes to infinity.[12]

Results (i) and (ii) imply that the slope and intercept of $g(b, \overline{x}; -1, \underline{w}(-1))$ at the origin are strictly positive and, as b increases, the slope decreases to zero. It follows that equation $b = g(b, \overline{x}; -1, \underline{w}(-1))$ holds for some sufficiently large b. Thus, there is a finite upper bound $\overline{b}(-1)$ for all bequests when $r = -1$. Finally, $\overline{b}(r)$ is continuous by continuity of $V(b, x; r, w)$, and compactness of I implies that there is a $\overline{b} = \max_{r \in I}\{\overline{b}(r)\}$.

By the same argument, there is a positive lower bound for all possible bequests, $\underline{b}$. ∎

---

[11] Cases 1 and 2 involve no borrowing, so as b goes to zero the scale of the firm goes to zero and the entrepreneur will be better off if she borrows. We are abstracting from the occupational consequences of lowering b – the agent at some point might prefer becoming a worker.

[12] As b increases, π_1 decreases until it reaches τ (case 3). As b increases beyond the level defined by (13), case 2 applies and π_1 falls below τ, thus making it unprofitable to borrow.

4.3 Existence of a Unique, Stationary Equilibrium

We now prove monotonicity of the bequest distribution. Since the state space is compact, this implies the existence of a stationary distribution (see Corollary 2 of Hopenhayn and Prescott (1992)). Specifically, Corollary 2 requires Z to have a minimum element, which follows from compactness, and T^* to be increasing, which follows from monotonicity (see our Definition 4). We also show that the transition probability function satisfies the monotonic mixing condition. We then apply Theorem 2 of Hopenhayn and Prescott (1992) to show uniqueness and convergence of the stationary distribution.

Proposition 5 *Let the conditions of Lemma 4 be satisfied. Then for any $r \in I$ there exists a unique invariant distribution Υ. In addition, for any initial bequest distribution Υ_0, the bequest distribution converges to Υ.*

Proof. Lemma 4 establishes that set Z is compact. From Definition 3, $\forall A \in \mathcal{B}$ operator $T^* : \Lambda(Z, \mathcal{B}) \to \Lambda(Z, \mathcal{B})$ is given by

$$(T^*\Upsilon_t)(A) = \int P_t(b_t, A)\Upsilon_t(db_t) \tag{19}$$

We wish to show that operator T^* has a unique fixed point $T^*\Upsilon = \Upsilon$ for any Borel subset $A \in \mathcal{B}$, given the initial bequest distribution Υ_0.[13] Of course, $T^*\Upsilon_t = \Upsilon_{t+1}$.

In order to find a fixed point, first note that w_t is well defined for every distribution Υ_t and any $r \in I$. Second, $b_{t+1} = \max\{w_t + (1+r)b_t, g(b_t, x_t; r, w_t)\}$, which is increasing in b_t (cf., (16)) and Z is compact (cf., Lemma 4), therefore it has a minimum element. Operator $(Th)(b_t) = \int h(b_{t+1})P_t(b_t, db_{t+1})$, defined for any h, is the conditional expectation of function h at $t+1$ given that the state at t is b_t. For any wage rate w_t, $g(b_t, x_t; r, w_t)$ is bounded and increasing in b_t and x_{t+1} is independent of b_t. Then the conditional expectation of $h(b_{t+1})$ on b_t is also increasing and bounded provided that h is increasing.[14] Function Th is increasing, thus T^* is increasing and P_t is a monotonic transition function (cf., Definition 4 and Stokey and Lucas, 1989, pages 220 and 379). Because Z is compact and T^* is increasing, the conditions of Corollary 2 of Hopenhayn and Prescott (1992) hold and there is a fixed point for map T^*.

We now show that P_t satisfies the monotone mixing condition. Define the n-step transition function beginning at t, $P_{t+n}(b_t, A) = \Pr\{b_{t+n} \in A|b_t\}$. We must show that transition function P_{t+n} satisfies, for all t,

$$P_{t+N}(\underline{b}, [b_a, \overline{b}]) > \epsilon \quad \text{and} \quad P_{t+N}(\overline{b}, [\underline{b}, b_a]) > \epsilon$$

for some $b_a \in Z$, $\epsilon > 0$, and $N \in \mathbb{N}$.[15] We omit subscript t and fix r for simplicity and without loss of generality. Let w be the wage associated with the fixed point of map T^*, Υ. Define:

$\underline{b} = b(w + (1+r)\underline{b})$: minimum stationary bequest;

$b_a = b(w + (1+r)\underline{b}) + \varrho$ for some small $\varrho > 0$.

There are two types of mobility in distribution Υ: We must show there is a positive probability that the Nth descendent of an agent with $b = \underline{b}$ receives a bequest above b_a and that an agent with $b = \overline{b}$ receives a bequest below b_a. An agent's descendants will have bequests in the neighborhood of $\underline{b}$ in finite time because they will earn at least the wage rate. Since the measures of sets $E(r, w)$ and $E^c(r, w)$ are non-zero and constant (because the labor market clears with wage w), and ability is independent across generations, we now show there is a positive probability of occupation change:

---

[13]$(T^*\Upsilon_t)(A)$ is the probability that next period's bequest lies in A, given the current bequest distribution. Map T^* is well defined, i.e., $(T^*\Upsilon_t)(A) \in \Lambda(Z, \mathcal{B})$. Stokey and Lucas (1989) Theorem 8.2 shows $T^* : \Lambda(Z, \mathcal{B}) \to \Lambda(Z, \mathcal{B})$.

[14]Given the equilibrium wage rate w_t, an agent's descendant is never worse off in terms of the expected value of any increasing function of b_{t+1} if, for any $db > 0$, the agent's state were $b_t + db$ instead of b_t.

[15]In our context the monotone mixing condition means that there is a positive probability that the Nth descendent changes from a worker to an entrepreneur or vice versa due to independent draws of random ability x. The N-step transition function, P_{t+N}, indicates the probability of going from point b_t to set A in exactly N periods. See Stokey (1988), p. 219.

(i) Worker to Entrepreneur: Suppose by way of contradiction that agents with ability in the neighborhood of $\overline{x}$ and bequests in the neighborhood of $\underline{b}$ cannot have descendants that become entrepreneurs. Since all agents' descendants face a positive probability of having a bequest in the neighborhood of $\underline{b}$ in finite time due to successive low x's, this implies that the measure of agents (workers) in the neighborhood of $\underline{b}$ is 1, a contradiction to the fact that $E(r,w)$ has a non-zero measure. Therefore, agents in the neighborhood of $(\overline{x},\underline{b})$ have descendants that become entrepreneurs. Moreover, this can occur in the following generation. This implies that they can also have bequests higher than $b_a > \underline{b}$ as long as they have a sufficiently high x, in which case they have a credit limit sufficiently high to become entrepreneurs.

(ii) Entrepreneur to Worker: Starting from $b = \overline{b}$, a succession of low x's leaves the agent's descendants with bequests lower than b_a. They will become workers and remain so until they get a sufficiently high x.

The conditions of Hopenhayn and Prescott (1992), Theorem 2 are satisfied, thus there exists a unique time invariant distribution Υ and associated equilibrium wage w, such that from any initial distribution Υ_0 and any interest rate $r \in I$, the operator $T^*\Upsilon_t$ converges to a unique Υ. ∎

Now, we show that each invariant distribution Υ is associated with a positive and unique pair $(w^*(r^*), r^*)$ that clears the labor and capital markets.

Proposition 6 *Let the conditions of Proposition 5 be satisfied. Then there exists a finite and unique interest rate $r^* \in I$ that clears the capital market in the stationary equilibrium. Triplet $(\Upsilon, r^*, w(r^*))$ completely characterizes the stationary equilibrium.*

Proof. Define the capital excess demand as

$$\text{KED}(r,w) = \iint\limits_{E(r,w)} k(b,x;r,w)\Upsilon(db)\Gamma(dx) - \int b\Upsilon(db). \tag{20}$$

We must prove that this is zero for some pair $(r^*, w(r^*))$, where $w(r^*)$ is the stationary equilibrium wage rate, as established in Proposition 5. Notice that independence across generations and continuity of Γ implies that function $\text{KED}(r,w)$ is continuous.

Suppose again that $r = -1$. Then either $\text{KED}(-1, w(-1))$ is non-positive or positive. In the first case, total bequests are enough to finance total demand for capital. The equilibrium interest rate is $r^* = -1$ and bequests are discarded because they have no value at the end of the period. For the second case, that is, $\text{KED}(-1, w(-1))$ is positive, we need to show that, as the interest rate goes to the upper bound of I, $\text{KED}(r, w(r))$ becomes negative. To see this, notice that

$$\text{KED}(r,w(r)) < \iint\limits_{E(r,w)} k(b,x;r,w(r))\Upsilon(db)\Gamma(dx) - \frac{(1-\gamma)\underline{w}}{1-(1-\gamma)(1+r)},$$

since all agents receive at least $w(r) \geq \underline{w}$ and $1-\gamma$ is the lowest possible share of income transmitted to the next generation. As $r \to (1-\gamma)^{-1} - 1 - \varepsilon$, the second term on the right-hand side of this expression can be made arbitrarily large in absolute terms.[16] Since bequests become uniformly arbitrarily large, the first term grows to the point where none of the entrepreneurs is ever constrained and capital is always at the unconstrained optimum for all entrepreneurs. Therefore, for sufficiently small ε, the right-hand side of the inequality is negative, implying that $\text{KED}(r, w(r))$ is negative. We have shown that, for the case

---

[16]In (20), given x, $\int b\Upsilon(db)$ is the supply of loans (see condition F in Definition 5). The corresponding term in the inequality is the smallest supply of loans: the bequest if all agents were workers who received the lowest possible wage, $\underline{w}$, at the lowest transmission rate, $1-\gamma$. To compute this bequest, observe that for such a worker $b_{t+1} = (1-\gamma)(\underline{w} + (1+r)b_t)$. In a stationary equilibrium, $b_{t+1} = b_t$ and the result follows.

when $KED(r, w(r)) > 0$ with $r = -1$, $KED(r, w(r)) < 0$ as $r \to (1 - \gamma)^{-1} - 1 - \varepsilon$. By continuity, there must be some r^* such that $KED(r^*, w^*(r^*))$ is zero.

It remains to show that for each stationary distribution Υ, the pair $(r^*, w^*(r^*))$ is unique. Define $\underline{b}$ and $\bar{b}$ as the lower and upper bounds of the support of the stationary bequest distribution, respectively. They are defined implicitly by the following equations:

$$\underline{b} = (1 - \gamma(Y_l))Y_l \tag{21}$$

$$\bar{b} = (1 - \gamma(Y_u))Y_u \,, \tag{22}$$

where $Y_l = ((1 + r)\underline{b} + w^*)$ and $Y_u = ((1 + r)\bar{b} + V(\bar{b}, \bar{x}, r^*, w^*))$. A worker whose ancestors have been workers will have bequests in the neighborhood of $\underline{b}$. An entrepreneur with ability in the neighborhood of $\bar{x}$ whose ancestors have had ability in the neighborhood of $\bar{x}$ will have bequests in the neighborhood of $\bar{b}$.

Since the stationary equilibrium distribution of b is unique, any change in pair (r^*, w^*) that is still a stationary equilibrium must not change the bequest distribution. Hence, $\underline{b}$ and $\bar{b}$ must not change. Therefore, equation (22) defines a price schedule $w(r)$ consistent with the same stationary bequest distribution whose derivative is

$$w'(r) = - \left. \frac{-\gamma'(Y_u)\left(\bar{b} + \frac{\partial V}{\partial r}\right)Y_u + (1 - \gamma(Y_u))\left(\bar{b} + \frac{\partial V}{\partial r}\right)}{-\gamma'(Y_u)\frac{\partial V}{\partial w}Y_u + (1 - \gamma(Y_u))\frac{\partial V}{\partial w}} \right|_{(\bar{b}, \bar{x}, r, w)} = - \left. \frac{\bar{b} + \frac{\partial V}{\partial r}}{\frac{\partial V}{\partial w}} \right|_{(\bar{b}, \bar{x}, r, w)} . \tag{23}$$

Another implication of a unique stationary equilibrium distribution is the fact that the entrepreneurs' profits along the distribution in (b, x) must not change. Therefore, in the neighborhood of $(\bar{b}, \bar{x})$, the value function of entrepreneurs must not change as we change the prices. This implies that $V(\bar{b}, \bar{x}, r, w)$ must be constant for any change in prices consistent with a unique stationary equilibrium. By the Implicit Function Theorem, we thus have another schedule $w(r)$ with derivative equal to

$$w'(r) = - \left. \frac{\frac{\partial V}{\partial r}}{\frac{\partial V}{\partial w}} \right|_{(\bar{b}, \bar{x}, r, w)} . \tag{24}$$

If the equilibrium prices are not unique, this derivative must equal (23). This is possible only if $\bar{b} = 0$, a contradiction to the fact that $w^*(r^*) > 0$. ∎

5 Numerical Solution Method

We use a direct, non-parametric approach to compute the stationary solution $(\Upsilon, r^*, w^*(r^*))$. A non-parametric approach, in contrast to a parametric approach, makes no assumptions about the the shape of the endogenously determined bequest distribution Υ. As we show, provided the (exogenous) ability cumulative distribution Γ is continuous, Υ is stationary and unique. However, it might be a complicated function of the model parameters and exogenous functions, particularly when it is the only link between generations and we have the non-convex enforcement restriction. This is a powerful justification for not using parametric specifications of Υ. The method may also be used to compute the equilibrium during the transition to the stationary equilibrium.

The strategy is as follows. Consider a large number of ex ante identical individuals. Suppose that each of these individuals has a fixed bequest and entrepreneurial ability, as well as policy functions. Individuals interact in a "synthetic" marketplace. Moreover, financial intermediaries and entrepreneurs interact so as to clear the capital market. Once the equilibrium bequest distribution and wage and interest rates are determined, we calculate next period's initial wealth and repeat the procedure. The stationary

distribution of bequests, wage and interest rate are found when they vary less than a small amount from one period to the next.

In order to solve the equilibrium numerically it is important to define parametric forms for the utility function, production function, and ability distribution. We use the following functions:

1. Utility function: $u(c,b) = c^\gamma b^{1-\gamma}$, with $\gamma \in (0,1)$.

2. Production function: $f(k,n) = k^\alpha n^\beta$, with $\alpha, \ \beta > 0$, and $\alpha + \beta < 1$.

3. Ability distribution: $\Gamma(x) = x^{\frac{1}{\epsilon}}$, with $\epsilon > 0$. This distribution implies that $x \in [0,1]$.

One must also define the value of six parameters: $(\gamma, \alpha, \beta, \epsilon, \phi, \tau)$. Antunes et al. (2006) choose the following parameter values $(\gamma, \alpha, \beta, \epsilon, \phi, \tau) = (0.94, 0.35, 0.55, 4.422, 0.26, 0.1907)$. See Antunes et al. (2006) for a full explanation of how these parameters were calibrated and how the model matches key statistics of the United States economy.

5.1 Determining prices

Policy functions are the solutions to the static problem (16), where $V(b,x;w,r)$ is given by (5). This, in principle, points to a four-dimensional domain for the policy functions. However, it can be shown that two of the dimensions can be collapsed into one, as the entrepreneur's problem can be written as a function of $h = \frac{x}{w^\beta}$, so $V(b,x;w,r) = \bar{V}(b,h;r)$. This reduces the computational complexity of the problem.

For a given initial bequest distribution, $\Upsilon_0(b)$, the equilibrium wage and interest rate are found by iteration. Start with an initial value for prices, w_0 and r_0. Given these prices, agents decide whether to become entrepreneurs or workers based on (16) and (5). For instance, if agent j is characterized by pair (b_j, x_j), she calculates her net output as an entrepreneur, $V(b_j, h_j; r_0)$, where $h_j = \frac{x_j}{w_0^\beta}$, and compares it with w_0. She becomes an entrepreneur if $V(b_j, h_j; r_0) > w_0$, and a worker if $V(b_j, h_j; r_0) < w_0$. The individual is indifferent between both occupational choices if equality prevails.

We choose a large number N of workers and determine their occupational choices through this process. Denote by $\mathcal{I}_j^w$ and $\mathcal{I}_j^e$ the indicator functions for workers and entrepreneurs, respectively. Average excess labor supply is

$$X_0 = \frac{\sum_{j=1}^N \left(\mathcal{I}_j^w - \mathcal{I}_j^e n(b_j, h_j; r_0) \right)}{N}, \tag{25}$$

where $n(b,h;r)$ is the policy function for labor demand. Likewise, capital excess supply as a fraction of total initial assets is given by

$$Y_0 = \frac{\sum_{j=1}^N \left(b_j - \mathcal{I}_j^e k(b_j, h_j; r_0) \right)}{\sum_{j=1}^N b_j}, \tag{26}$$

where $k(b,h;r)$ is the policy function for capital.

The price update is performed through

$$\begin{bmatrix} w_1 \\ r_1 \end{bmatrix} = \begin{bmatrix} w_0 \\ r_0 \end{bmatrix} + \sigma \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_2 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}, \tag{27}$$

where the a_{lm} are suitably chosen parameters[17] and $0 < \sigma \leq 1$ is a weight attached to the previous iteration. This parameter is used to avoid excessive fluctuation of successive iterations. The process is repeated until X_0 and Y_0 are sufficiently low.

---

[17]We assume $a_{12} = a_{21} = 0$ throughout.

5.2 Policy functions

This method is not feasible if the policy functions are recalculated every time that prices are updated, as this would involve solving an optimization problem in each iteration. Execution time would then become larger by one or two orders of magnitude.

This problem is solved using interpolation. For this purpose, first calculate the policy functions k and n (and the value function V) for a grid of the (b, h, r) space. These functions are used unchanged in the entire algorithm. This removes the computational burden from each iteration to the beginning of the algorithm. In each iteration, the program performs a table lookup instead of a complete optimization algorithm.

5.3 The stationary bequest distribution

The stationary bequest distribution is found by computing equilibria for successive generations. The algorithm is stopped when prices remain stable from one generation to the next. Given the convergence properties of the model, stable prices imply a stable bequest distribution.[18]

It should be emphasized that a direct Monte Carlo approach is not easily applicable to our model. Stachurski (2005) analyzes the general framework where the endogenous variable X_t, which takes values in $S \subset \mathbb{R}^k$, is given by the recursive relation $X_t = H_t(X_{t-1}, W_t)$, where the shocks W_t are iid and taken from a known distribution, and the initial distribution of the endogenous variable is also known.[19] It would in principle be possible to draw sequences of shocks $(W_1, \ldots, W_T)$ so that, using the recursive law above, one could obtain the distribution of X_T. (This approach could even be improved if the conditional of X_t given X_{t-1} were known.) This process is not feasible in our case because in each period H_t depends on the equilibrium prices of factors, and there is not a direct method to calculate them without complete knowledge of the distribution function of X_{t-1}. (For the same reason, it is not possible to easily calculate the distribution of X_t conditional on X_{t-1}.) In other words, in order to calculate H_t, we need to know the entire distribution of X_{t-1}. This suggests that an agent-based approach, such as ours, is more appropriate.

Figure 1 presents the sorted bequests in two consecutive periods plotted against each other. The curve is approximately a straight line, as expected. From this we see that the distribution is stable. The shape of the stationary bequest distribution is depicted in Figure 2. The straight section is a consequence of the uniform wage rate.

---

[18]This does not imply that an agent always gets the same bequest.
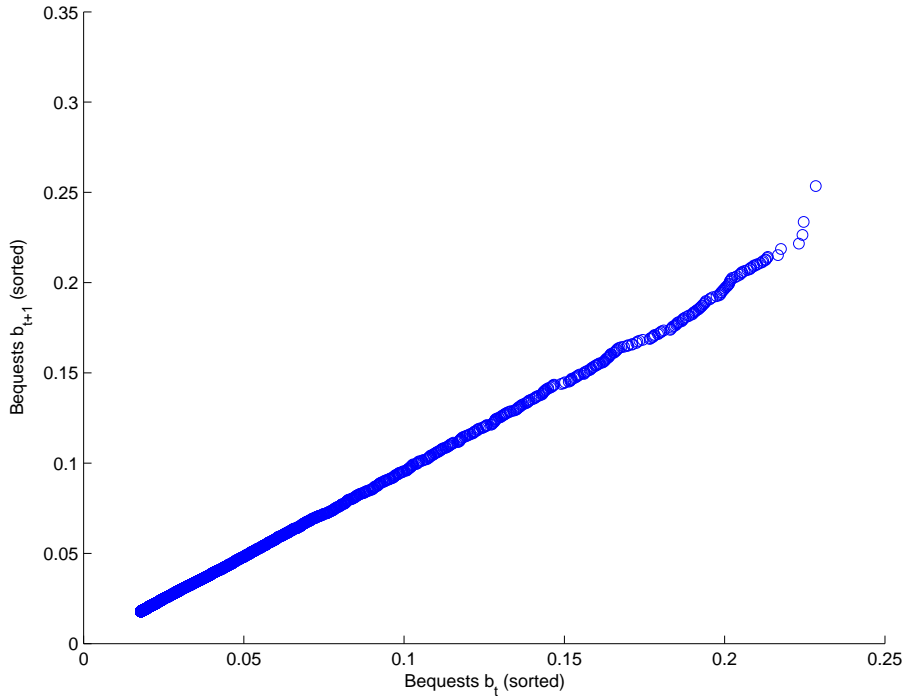[19]In our case, X_t is b_t and W_t is x_t.

Figure 1: Values of b_t (sorted) plotted against b_{t+1} (sorted). $N = 100000$, only 10000 depicted.

6 Concluding Remarks

We have proved the existence of a stationary equilibrium for a class of dynamic general equilibrium models with agent heterogeneity, which leads to endogenous occupational choice, and loan market frictions. The loan repayment incentive constraint induces a non-convexity which makes standard fixed point arguments that require continuity inapplicable. We obtain two main theoretical results. Under the conditions stated, the first proposition proves the existence of a unique steady-state equilibrium for any fixed interest rate which clears the labor market. Under the conditions stated, the second proposition proves existence of a unique steady-state equilibrium that clears the capital market and the labor market. We also describe how to compute the steady state solution.

The second proposition is of additional interest because it makes it possible to extend the literature on occupational choice models with financial market imperfections in an important way. The existing literature, for example Banerjee and Newman (1993), Lloyd-Ellis and Bernhardt (2000), Antunes and Cavalcanti (2005) and Amaral and Quintin (2006), assumes a small open economy. This case corresponds to our first proposition where the interest rate is given. Our second proposition applies to the alternative case of an endogenously determined interest rate, which to our knowledge has been neglected. In Antunes et al. (2006) we show that this general equilibrium effect is important, both qualitatively and quantitatively, and that it has an important implication for policy: It suggests that financial reform (e.g., reforms designed to strengthen contract enforcement such as bankruptcy law revisions) should be accompanied by policies which increase capital mobility, which affect the interest rate. Otherwise, the financial reforms will often have a minor quantitative effect on efficiency due to general equilibrium adjustments.
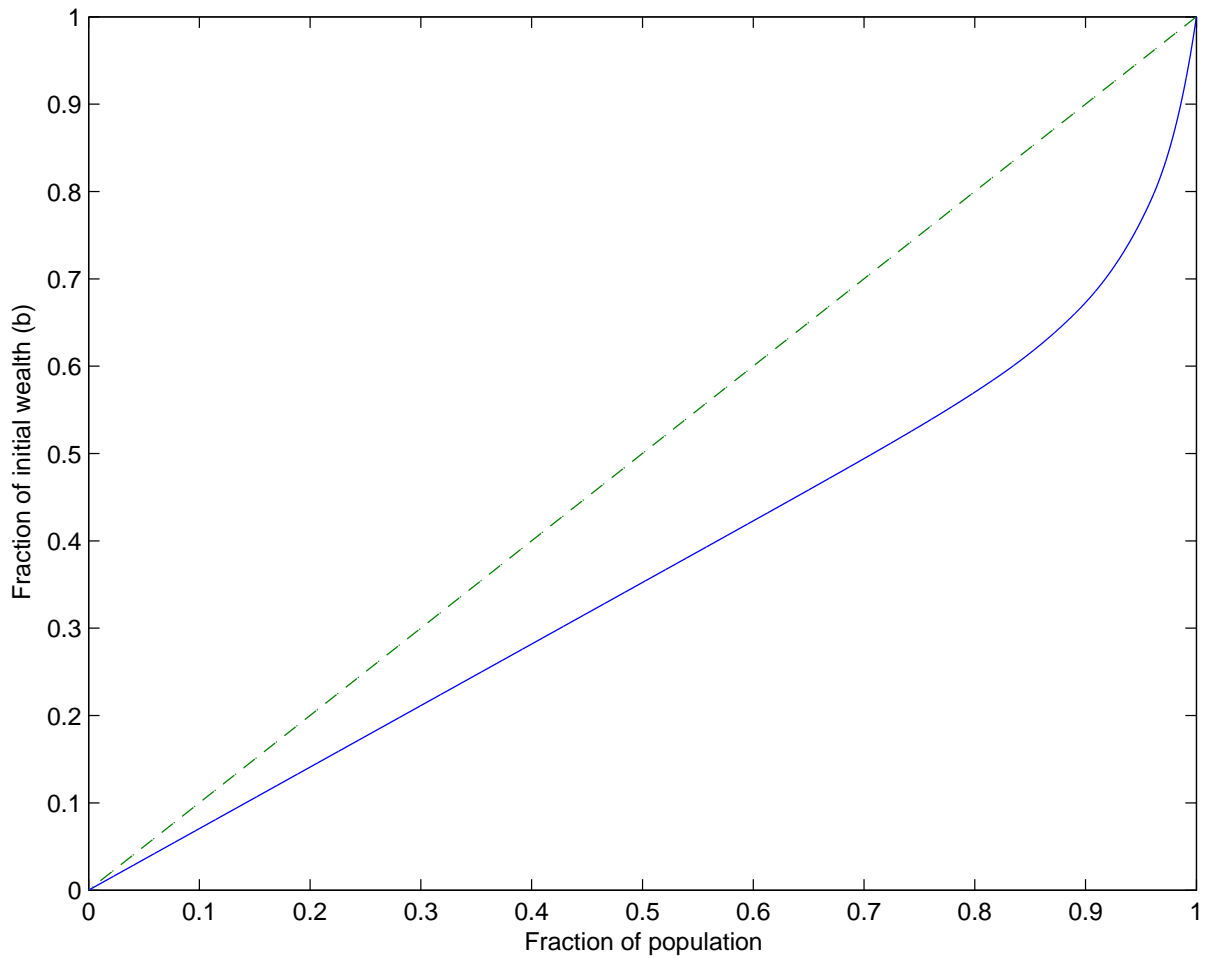
Figure 2: Lorenz curve of the stationary bequest distribution.

References

Amaral, P. and Quintin, E. (2006), 'A competitive model of the informal sector'. Forthcoming in *Journal of Monetary Economics*.

Antinolfi, G. and Huybens, E. (1998), 'Capital accumulation and real exchange rate behavior in a small open economy with credit market frictions', *Economic Theory* **12**, 461–488.

Antunes, A. and Cavalcanti, T. (2005), 'Start up costs, limited enforcement, and the hidden economy'. Forthcoming in *European Economic Review*.

Antunes, A., Cavalcanti, T. and Villamil, A. (2006), 'The effect of financial repression & enforcement on entrepreneurship and economic development', *Working Paper, University of Illinois at Urbana-Champaign* .

Banerjee, A. V. and Newman, A. F. (1993), 'Occupational choice and the process of development', *Journal of Political Economy* **101**(2), 274–298.

Betts, C. and Bhattacharya, J. (1998), 'Unemployment, credit rationing, and capital accumulation: a tale of two frictions', *Economic Theory* **12**, 489–518.

Boyd, J. and Smith, B. (1998), 'The evolution of debt and equity markets in economic development', *Economic Theory* **12**, 519–560.

Erosa, A. (2001), 'Financial intermediation and occupational choice in development', *Review of Economic Dynamics* **4**(2), 303–334.

Hopenhayn, H. A. and Prescott, E. C. (1992), 'Stochastic monotonicity and stationary distributions for dynamic economies', *Econometrica* **60**(6), 1387–1406.

Kehoe, T. and Levine, D. (1993), 'Debt-constrained asset markets', *Review of Economic Studies* **60**, 865–888.

Krasa, S., Sharma, T. and Villamil, A. P. (2005), 'Debt contracts and cooperative improvements', *Journal of Mathematical Economics* **41**, 857–874.

Krasa, S. and Villamil, A. P. (2000), 'Optimal contracts when enforcement is a decision variable', *Econometrica* **68**, 119–134.

Krasa, S. and Yannelis, N. (1994), 'An elementary proof of the Knaster-Kuratowski-Mazurkiewitz-Shapley theorem', *Economic Theory* **4**(1), 467–471.

Lloyd-Ellis, H. and Bernhardt, D. (2000), 'Inequality and economic development', *Review of Economic Studies* **67**(1), 147–168.

Lucas, Jr, R. E. (1978), 'Asset prices in an exchange economy', *Econometrica* **46**, 1429–1445.

Stachurski, J. (2005), Computing the distributions of economic models via simulation, Manuscript, University of Melbourne.

Stokey, N. L. (1988), 'Learning by doing and the introduction of new goods', *Journal of Political Economy* **96**, 701–717.

Stokey, N. L. and Lucas, Jr, R. E. (1989), *Recursive Methods in Economic Dynamics*, Harvard University Press, Cambridge, Massachusetts. With Edward C. Prescott.