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**UNIQUE EQUILIBRIUM WITH SINGLE MONETARY
INSTRUMENT RULES**

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Unique Equilibrium with Single Monetary Instrument Rules.*

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Abstract

We consider standard cash-in-advance monetary models and show that there are interest rate or money supply rules such that equilibria are unique. The existence of these single instrument rules depends on whether the economy has an infinite horizon or an arbitrarily large but finite horizon.

Key words: Monetary policy; interest rate rules; unique equilibrium.

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1. Introduction

In this paper we revisit the issue of multiplicity of equilibria when monetary policy is conducted with either the interest rate or the money supply as the instrument of

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policy. There has been an extensive literature on this topic starting with Sargent and Wallace (1975), including a recent literature on local and global determinacy in models with nominal rigidities. We show that it is possible to implement a unique equilibrium with an appropriately chosen interest rate feedback rule, and similarly with a money supply feedback rule of the same type. This is a surprising result because while it is well known that interest rate feedback rules can deliver a locally unique equilibrium, it is no less known that they generate multiple equilibria globally.

We show that the reason for the results is the model assumption of an infinite horizon. In finite horizon economies, the number of degrees of freedom in conducting policy does not depend on the way policy is conducted. The number is the same independently of whether interest rates are set as constant functions of the state, or as backward, current or forward functions of endogenous variables.

In analogous finite horizon economies, the number of degrees of freedom in conducting policy can be counted exactly. The equilibrium is described by a system of equations where the unknowns are the quantities, prices and policy variables. There are more unknowns than variables, and the difference is the number of degrees of freedom in conducting policy. It is a necessary condition for there to be a unique equilibrium that the same number of exogenous restrictions on the policy variables be added to the system of equations. Single instrument policies are not sufficient restrictions. They always generate multiple equilibria. This is no longer the case in the infinite horizon economy, as we show in this paper.

Whether the appropriate description of the world is an infinite horizon economy or the limit of finite horizon economies, thus, makes a big difference for this particular issue of policy interest, i. e. whether policy conducted with a single instrument, such as the nominal interest rate, is sufficient to determine a unique competitive equilibrium.

As already mentioned, after Sargent and Wallace (1975) and McCallum (1981), there is a large literature on multiplicity of equilibria when the government follows either an interest rate rule or a money supply rule. This includes the literature on local determinacy that identifies conditions on preferences, technology, timing of markets, and policy rules, under which there is a unique local equilibrium (see Bernanke and Woodford (1997), Clarida, Gali and Gertler (1999, 2000), Carlstrom and Fuerst (2001, 2002), Benhabib, Schmit-Grohe and Uribe (2001a), Dopor (2001) among others). This literature has in turn been criticized by recent work on global stability that makes the point that the conditions for local deter-

minacy are also conditions for global indeterminacy (see Benhabib, Schmit-Grohe and Uribe (2001b) and Christiano and Rostagno, 2002).

Our modelling approach is close to Adao, Correia and Teles (2003) for the case with sticky prices. In this paper we show that even at the optimal zero interest rate rule there is still room for policy to improve welfare since it is possible to use money supply to implement the optimal allocation in a large set of implementable allocations. This paper is also close to Adao, Correia and Teles (2004) where we show that it is possible to implement unique equilibria in environments with flexible prices and prices set in advance by pegging state contingent interest rates as well as the initial money supply. Bloise, Dreze and Polemarchakis (2004) and Nakajima and Polemarchakis (2005) are also related research.

We assume that fiscal policy is endogenous. Exogeneity of fiscal policy could be used, as in the fiscal theory of the price level to determine unique equilibria.

The paper proceeds as follows: In Section 2, we consider a simple cash in advance economy with flexible prices. In Section 3 we analyze a simple example to discuss the properties of the equilibria obtained when a single monetary policy instrument is used. In Section 4, we show that there are single instrument feedback rules that implement a unique equilibrium. In Section 5 we show that in analogous finite horizon environments the single instrument rules would generate multiple equilibria. In Section 6, we show that the results generalize to the case where prices are set in advance. Section 7 contains concluding remarks.

2. A model with flexible prices

We first consider a simple cash in advance economy with flexible prices. The economy consists of a representative household, a representative firm behaving competitively, and a government. The uncertainty in period $t \geq 0$ is described by the random variable $s_t \in S_t$ and the history of its realizations up to period t (state or node at t), $(s_0, s_1, \dots, s_t)$, is denoted by $s^t \in S^t$. The initial realization s_0 is given. We assume that the history of shocks has a discrete distribution. The number of states in period t is Φ_t .

Production uses labor according to a linear technology. We impose a cash-in-advance constraint on the households' transactions with the timing structure described in Lucas and Stokey (1983). That is, each period is divided into two subperiods, with the assets market operational in the first subperiod and the goods market in the second.

2.1. Competitive equilibria

Households The households have preferences over consumption C_t , and leisure L_t , described by the expected utility function:

$$U = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(C_t, L_t) \right\} \quad (2.1)$$

where β is a discount factor. The households start period t with nominal wealth $\mathbb{W}_t$. They decide to hold money, M_t , and to buy B_t nominal bonds that pay $R_t B_t$ one period later. R_t is the gross nominal interest rate at date t . They also buy $B_{t,t+1}$ units of state contingent nominal securities. Each security pays one unit of money at the beginning of period $t+1$ in a particular state. Let $Q_{t,t+1}$ be the beginning of period t price of these securities normalized by the probability of the occurrence of the state. Therefore, households spend $E_t Q_{t,t+1} B_{t,t+1}$ in state contingent nominal securities. Thus, in the assets market at the beginning of period t they face the constraint

$$M_t + B_t + E_t Q_{t,t+1} B_{t,t+1} \leq \mathbb{W}_t \quad (2.2)$$

Consumption must be purchased with money according to the cash in advance constraint

$$P_t C_t \leq M_t. \quad (2.3)$$

At the end of the period, the households receive the labor income $W_t N_t$, where $N_t = 1 - L_t$ is labor and W_t is the nominal wage rate and pay lump sum taxes, T_t . Thus, the nominal wealth households bring to period $t+1$ is

$$\mathbb{W}_{t+1} = M_t + R_t B_t + B_{t,t+1} - P_t C_t + W_t N_t - T_t \quad (2.4)$$

The households' problem is to maximize expected utility (2.1) subject to the restrictions (2.2), (2.4), (3.4), together with a no-Ponzi games condition on the holdings of assets.

The following are first order conditions of the households problem:

$$\frac{u_L(t)}{u_C(t)} = \frac{W_t}{P_t} \frac{1}{R_t} \quad (2.5)$$

$$\frac{u_C(t)}{P_t} = R_t E_t \left[\frac{\beta u_C(t+1)}{P_{t+1}} \right] \quad (2.6)$$

$$Q_{t,t+1} = \beta \frac{u_C(t+1)}{u_C(t)} \frac{P_t}{P_{t+1}}, t \geq 0 \quad (2.7)$$

From these conditions we get $E_t Q_{t,t+1} = \frac{1}{R_t}$. Condition (2.5) sets the intratemporal marginal rate of substitution between leisure and consumption equal to the real wage adjusted for the cost of using money, R_t . Condition (2.6) is an intertemporal marginal condition necessary for the optimal choice of risk-free nominal bonds. Condition (2.7) determines the price of one unit of money at time $t+1$, for each state of nature s^{t+1} , normalized by the conditional probability of occurrence of state s^{t+1} , in units of money at time t .

Firms The firms are competitive and prices are flexible. The production function of the representative firm is linear

$$Y_t \leq A_t N_t$$

The equilibrium real wage is

$$\frac{W_t}{P_t} = A_t. \quad (2.8)$$

Government The policy variables are taxes, T_t , interest rates, R_t , money supplies, M_t , state noncontingent public debt, B_t . We can define a policy as a mapping for the policy variables $\{T_t, R_t, M_t, B_t, t \geq 0, \text{ all } s^t\}$, that maps sequences of quantities, prices and policy variables into sets of sequences of the policy variables. Defining a policy as a correspondence allows for the case where the government is not explicit about some of the policy variables. Lucas and Stokey (1983) define policy as sequences of numbers for some of the variables. Adao, Correia and Teles (2003) define policy as sequences of numbers for all the policy variables. Here we allow for more generic functions (correspondences) for all the policy variables. We do not allow for targeting rules that can be defined as mappings from prices, quantities and policy variables to prices and quantities.

The period by period government budget constraints are

$$M_t + B_t = M_{t-1} + R_{t-1}B_{t-1} + P_{t-1}G_{t-1} - P_{t-1}T_{t-1}, t \geq 0$$

Let $Q_{t+1} \equiv Q_{0,t+1}$, with $Q_0 = 1$. If $\lim_{T \rightarrow \infty} E_t Q_{T+1} \mathbb{W}_{T+1} = 0$

$$\sum_{s=0}^{\infty} E_t Q_{t,t+s+1} M_{t+s} (R_{t+s} - 1) = \mathbb{W}_t + \sum_{s=0}^{\infty} E_t Q_{t,t+s+1} P_{t+s} [G_{t+s} - T_{t+s}] \quad (2.9)$$

Market clearing Market clearing in the goods and labor market requires

$$C_t + G_t = A_t N_t,$$

and

$$N_t = 1 - L_t.$$

We have already imposed market clearing in the money and debt markets.

Equilibrium A competitive equilibrium is a sequence of policy variables, quantities and prices such that the private agents maximize given the sequences of policy variables and prices, the budget constraint of the government is satisfied and the policy sequence is in the set defined by the policy.

The equilibrium conditions for the variables $\{C_t, L_t, R_t, M_t, B_t, T_t, Q_{t,t+1}\}$ are the resources constraint

$$C_t + G_t = A_t(1 - L_t), t \geq 0 \quad (2.10)$$

the intratemporal condition that is obtained from the households intratemporal condition (2.11) and the firms optimal condition (2.8)

$$\frac{u_C(t)}{u_L(t)} = \frac{R_t}{A_t}, t \geq 0 \quad (2.11)$$

as well as the cash in advance constraints (3.4), the intertemporal conditions (2.6) and (2.7), and the budget constraints (2.9).

3. Example

In this section we consider a particular utility function to discuss the properties of equilibria when the central bank chooses either the interest rate or the money supply as the sole instrument of monetary policy. We discuss the properties of the equilibria, paying particular attention to the so called local determinacy property of the equilibrium. Local determinacy means that in the neighborhood of an equilibrium there is no other equilibrium.

We also consider an interest rate feedback rule as the literature is currently dominated by a rule-based approach to monetary policy. We review what is meant by an interest rate feedback rule guaranteeing local determinacy and show that local determinacy is achieved if the interest rate feedback rule satisfies the Taylor

principle. The Taylor principle is verified if in response to an increase in inflation the increase in the nominal interest rate is higher.

To simplify the presentation we take $G_t = 0$ and the utility function $u(C_t, L_t) = C_t + v(L_t)$, with $v(L_t)$ increasing in L_t , $\lim_{L_t \rightarrow 0} v'(L_t) = \infty$ and $\lim_{L_t \rightarrow 1} v'(L_t) = 0$. We consider 3 monetary policies: a constant interest rate, a constant growth rate for the money supply and an interest rate feedback rule. For the sake of simplicity we consider the deterministic environment, i.e. $s^t = s^{t+1}$ for all t . The stochastic environment is considered in the appendix.

The equilibrium conditions for the variables $\{C_t, L_t, P_t, M_t, R_t\}$ are: the household's intratemporal and intertemporal conditions

$$\frac{1}{v'(L_t)} = \frac{R_t}{A} \quad (3.1)$$

and

$$\frac{1}{P_t} = R_t \beta \frac{1}{P_{t+1}}, \quad (3.2)$$

the feasibility condition

$$C_t = A(1 - L_t), \quad (3.3)$$

and the cash in advance condition

$$M_t \geq P_t C_t, \text{ with equality if } R_t > 1. \quad (3.4)$$

It will be useful for the discussion below to remember that from (3.1) and (3.3) there is a positive relation between L_t and R_t and a negative relation between C_t and L_t .

3.1. Constant interest rate

Here we assume that the central bank chooses to maintain a constant interest rate equal to $\mathbb{R} \geq 1$. In this case C_t and L_t are pin down by (3.1) and (3.3). The inflation, π_t , is pin down by (3.2), $\pi_t = \mathbb{R}\beta$. Any positive real number is an equilibrium P_0 . Thus, there is a multiplicity of equilibrium price sequences and as a consequence from (3.4) a multiplicity of equilibrium money sequences. The literature has a jargon for this result, it is said that the outcome of setting the interest rate is real determinacy and nominal indeterminacy. All the equilibria are locally undetermined as for any equilibrium price level there is another equilibrium price level in its neighborhood. In a stochastic environment with nominal frictions,

like sticky prices or sticky wages, the monetary policy of setting the interest rate is less interesting since it leads to multiplicity of the real allocations. We clarify this issue in the appendix.

3.2. Constant money growth

Here we study the equilibria when the central bank chooses M_0 and a constant rate of money growth of the form $M_t = \Theta^t M_0$, where $\Theta > \frac{1}{\beta}$. There are many equilibria. In order to show that, we find it useful to define real money as $m_t \equiv \frac{M_t}{P_t}$, and replace (3.1) in (3.2)

$$m_{t+1} = \Upsilon(L_t) m_t, \text{ where } \Upsilon(L_t) = \frac{\Theta v'(L_t)}{A\beta}. \quad (3.5)$$

There are two steady states: one with $\frac{m_{t+1}}{m_t} = 1$, $R_t = \frac{\Theta}{\beta} > 1$, C_t and $L_t (= \bar{L})$ that solve (3.1) and (3.3) for $R_t = \frac{\Theta}{\beta}$, and P_t satisfying (3.4) with equality. There is another steady state with $R_t = 1$, C_t and $L_t (= \tilde{L})$ that solve (3.1) and (3.3) for $R_t = 1$, $\frac{m_{t+1}}{m_t} = \Upsilon(\tilde{L}) > 1$ and $\frac{P_{t+1}}{P_t} = \frac{\Theta}{\Upsilon(\tilde{L})}$. In this steady state inflation is determined but the initial price level is not since (3.4) may not be binding when $R_t = 1$.

The remaining equilibria can be divided according to the value of leisure in period zero, L_0 . There are many equilibria with $L_0 > \bar{L}$. From (3.5) we get $\frac{m_1}{m_0} = \Upsilon(L_0) < 1$. Thus, from (3.3) and the fact that (3.4) holds with equality in period 1 we obtain $L_1 < L_0$ which implies $\Upsilon(L_1) < \Upsilon(L_0)$. Proceeding in this way we obtain m_t and C_t approaching zero and L_t approaching 1. From (3.1) and the fact that m_t approaches zero we obtain R_t and P_t approaching infinity. Inflation $\frac{P_{t+1}}{P_t} = \frac{\Theta}{\Upsilon(L_t)}$ approaches infinity as the denominator approaches zero.

There are also equilibria with $\tilde{L} < L_0 < \bar{L}$. By (3.1), $R_0 > 1$, which means that (3.4) holds with equality. From (3.5), (3.3) and the assumption that $m_t = c_t$ we get sequences $\{L_t, C_t, m_t\}$. Let t^* be the first period such that L_{t^*} obtained from the process just described satisfies (3.1) with $R_t \leq 1$. The elements of the sequence up to t^* are part of the equilibrium, but the ones after are not. In period t^* (3.4) does not hold with equality which implies that $R_{t^*} = 1$. This means that in periods $t^*, t^* + 1, t^* + 2, \dots$ the equilibrium L_t must satisfy (3.1) for $R_t = 1$, which we denoted by $\tilde{L}$ and the equilibrium C_t solves (3.3) for $L_t = \tilde{L}$. Also $\frac{m_{t+1}}{m_t} = \Upsilon(\tilde{L}) > 1$ for $t \geq t^*$, and inflation is constant, $\frac{P_{t+1}}{P_t} = \frac{\Theta}{\Upsilon(\tilde{L})}$ for $t \geq t^*$, m_t approaches infinity and P_t approaches zero.

The steady state associated with $R_t = \frac{\Theta}{\beta}$, for all t , is locally determined and the steady state associated with $R_t = 1$, for all t , is locally undetermined.

3.3. Interest rate feedback rule

Now we study the equilibria when the central bank follows an interest rate feedback rule. Let $\mathbb{R}$ be a steady state equilibrium interest rate and let Π be the corresponding steady state equilibrium inflation rate. Then, $\mathbb{R} = \frac{\Pi}{\beta}$, where $\frac{1}{\beta}$ is the real interest rate. Assume that the central bank conducts a pure current nonlinear Taylor rule:¹

$$R_t = \mathbb{R} \left(\frac{\pi_t}{\Pi} \right)^{\tau\beta},$$

where $\tau\beta \geq 1$ (the Taylor principle), and $\pi_t \equiv \frac{P_t}{P_{t-1}}$. After substituting the Taylor rule in the intertemporal condition of the household, (3.2), we get

$$z_{t+1} = (z_t)^{\tau\beta},$$

where $z_t = \frac{\pi_t}{\Pi}$. By recursive substitution we get

$$z_{t+k} = (z_t)^{k\tau\beta}, \text{ for all } k \text{ and } t. \quad (3.6)$$

There is no condition to pin down the initial value for inflation. Since the initial inflation level can be any value there is an infinity of equilibrium trajectories for the inflation rate. Nevertheless, they can be typified in 3 classes. Either inflation is constant, $\pi_t = \Pi$, or there is an hyperinflation, $\pi_t \rightarrow \infty$, or inflation is approaching zero, $\pi_t \rightarrow 0$. This is easy to verify. If $\pi_0 = \Pi$, then (3.6) implies that $\pi_t = \Pi$ for all t . If $\pi_0 > \Pi$, then (3.6) implies that $\pi_{t+1} > \pi_t$ and $\pi_t \rightarrow \infty$, since $\tau\beta > 1$. If $\pi_0 < \Pi$, then (3.6) implies that $\pi_{t+1} < \pi_t$ and $\pi_t \rightarrow 0$, since $\tau\beta > 1$.

Thus, when the central bank follows a Taylor rule that obeys the Taylor principle it is able to get local determinacy. In a neighborhood of the steady state inflation Π there is no other equilibrium inflation trajectory. But we have just seen that there is an infinity of other equilibria for inflation which converge to zero

¹Usually the Taylor rule is presented in its linearized form. As can be verified the linearized version is,

$$R_t - \mathbb{R} = \tau (\pi_t - \Pi).$$

or to infinity. These results beg two interrelated questions: Why is local determinacy such an interesting property? Or why has most of the literature assumed that undesirable equilibria do not happen? We do not know the answer to these questions.

It is easy to verify, using an argument similar to the one above, that if the Taylor rule did not obey the Taylor principle, i.e. $\tau\beta < 1$, there would be just two types of equilibrium. The steady state and an infinity of equilibria converging to the steady state. At first sight it would seem that it would be preferable that a central bank would follow a Taylor rule that did not satisfy the Taylor principle, as "undesirable" equilibria, hyperinflations or hyperdeflations would not be possible. This conclusion is not correct because whenever there is multiplicity of equilibria it may be possible that sunspots can cause large fluctuations in inflation. Inflation can fluctuate randomly just because agents come to believe this will happen.

Why do we get so many equilibria? Is it possible that we are forgetting equilibrium conditions? There are no more equilibrium conditions over these variables. The so called transversality conditions are satisfied since in our economy there are government bonds. Moreover, since our fiscal authority has a Ricardian policy the government's infinite-horizon budget constraint does not provide additional information. In particular it cannot be used to obtain the initial price level as it is done in the fiscal theory of price level literature.

There may be institutions that we have ignored in the model, which can be used to eliminate some of these "undesirable" equilibria. For instance, in some models an hyperinflation can be eliminated if the central bank has sufficient real resources and can commit to buy back its currency if the price level exceeds a certain level. This is known as fractional real backing of the currency (see Obstfeld and Rogoff (1983)). We are not going to pursue this issue here.

4. Single instrument feedback rules.

In this section we assume that policy is conducted with either interest rate or money supply feedback rules. We show that there are single instrument feedback rules that implement a unique equilibrium for the allocation and prices. The proposition for an interest rate feedback rule follows:

Proposition 4.1. *When the fiscal policy is endogenous and monetary policy is*

conducted with the interest rate feedback rule

$$R_t = \frac{\xi_t}{E_t \frac{\beta u_C(t+1)}{P_{t+1}}},$$

ξ_t is an exogenous variable, there is a unique equilibrium.

Proof: Suppose policy is conducted with the interest rate feedback rule $R_t = \frac{\xi_t}{E_t \frac{\beta u_C(t+1)}{P_{t+1}}}$. Then the intertemporal and intratemporal conditions, (2.6) and (2.11) can be written as

$$\frac{u_C(t)}{P_t} = \xi_t, t \geq 0 \quad (4.1)$$

$$\frac{u_C(t)}{u_L(t)} = \frac{\frac{\xi_t}{\beta E_t \xi_{t+1}}}{A_t}, t \geq 0 \quad (4.2)$$

These conditions together with the cash in advance conditions, (3.4), and the resource constraints, (2.10), determine uniquely the variables C_t , L_t , P_t and M_t .

The budget constraints (4.4) are satisfied for multiple paths of the taxes and state noncontingent debt levels ■

The forward looking interest rate feedback rules that guarantee uniqueness of the equilibrium resemble the rules that appear to be followed by central banks. The nominal interest rate reacts positively both to the forecast of future consumption and to the forecast of the future price level. In this there is a difference to the feedback rules that are usually considered in that it depends on the future price level rather than inflation.

Depending on the exogenous process for ξ_t , with this feedback rule it is possible to decentralize any feasible allocation distorted by the nominal interest rate. The first best allocation, at the Friedman rule of a zero nominal interest rate, can also be implemented. With $\xi_t = \frac{1}{\beta^t}$, $t \geq 0$, condition (4.2) becomes

$$\frac{u_C(t)}{u_L(t)} = \frac{1}{A_t}, t \geq 0$$

which, together with the resource constraint (2.10) gives the first best allocation $C_t = C(A_t, G_t)$, $L_t = L(A_t, G_t)$. The price level $P_t = P(A_t, G_t)$ can be obtained using (4.1), i.e.

$$\frac{u_C(C(\cdot), L(\cdot))}{P_t} = \frac{1}{\beta^t}, t \geq 0,$$

and the money supply is obtained using the cash-in-advance constraint, $M_t = P(A_t, G_t)C(A_t, G_t)$.

Allocations where inflation is zero can also be implemented even if in this flexible price environment they are not desirable. There are multiple such allocations with nominal interest rates satisfying

$$R_t = \frac{u_C(\bar{C}(A_t, G_t, R_t), \bar{L}(A_t, G_t, R_t))}{\beta E_t u_C(\bar{C}(A_{t+1}, G_{t+1}, R_{t+1}), \bar{L}(A_{t+1}, G_{t+1}, R_{t+1}))}, t \geq 0$$

where the functions $\bar{C}$ and $\bar{L}$ are the solution for C_t and L_t of the system of equations given by (2.11) and (2.10).

For each path of the nominal interest rate, $\{R_t\}$, associated with zero inflation, there is a unique path for $\{\xi_t\}$ up to a constant term,

$$\frac{u_C(\bar{C}(A_t, G_t, R_t), \bar{L}(A_t, G_t, R_t))}{\bar{P}} = \xi_t, t \geq 0.$$

In an economy where the use of money is becomes negligible which corresponds to a cash-in-advance condition

$$v_t P_t C_t \leq M_t. \quad (4.3)$$

where $v_t \rightarrow 0$, there is a single path for the nominal interest rate consistent with zero inflation,

$$R_t = \frac{u_C(\bar{C}(A_t, G_t), \bar{L}(A_t, G_t))}{\beta E_t u_C(\bar{C}(A_{t+1}, G_{t+1}), \bar{L}(A_{t+1}, G_{t+1}))}, t \geq 0$$

An analogous proposition to Proposition 3.1 is obtained when policy is conducted with a particular money supply feedback rule.

Proposition 4.2. *When the fiscal policy is endogenous and the policy is conducted with the money supply feedback rule,*

$$M_t = \frac{\beta R_{t-1} C_t u_C(t)}{\xi_{t-1}}$$

there is a unique equilibrium.

Proof: Suppose policy is conducted according to the money supply rule $M_t = \frac{\beta R_{t-1} C_t u_C(t)}{\xi_{t-1}}$. Then, the equilibrium conditions

$$P_t C_t = \frac{\beta R_{t-1} C_t u_C(t)}{\xi_{t-1}}$$

obtained using the cash in advance conditions (3.4),

$$\frac{u_C(t)}{P_t} = \xi_t$$

obtained from the intertemporal conditions (2.6), in addition to the resource constraints, (2.10) and the intratemporal conditions (2.11) determine uniquely the four variables, C_t , h_t , P_t , R_t in each period $t \geq 0$ and state s^t .

The taxes and debt levels satisfy the budget constraint (4.4) ■

The result that there are single instrument feedback rules that implement a unique equilibrium is a surprising one. In fact it is well known that interest rate rules may implement a determinate equilibrium, but not a unique global equilibrium. To illustrate this, consider the case where monetary policy is conducted with constant functions for the policy variables. We will show that in that case an interest rate policy generates multiple equilibria. That result is directly extended to the case where the interest rate is a function of contemporaneous or past variables.

4.1. Conducting policy with constant functions.

In this section, we show that in general when policy is conducted with constant functions for the policy instruments, it is necessary to determine exogenously both interest rates and money supplies.

The equilibrium conditions are the resources constraints, (2.10), the intratemporal conditions (2.11), the cash in advance constraints (3.4), the intertemporal conditions (2.6) and the budget constraints (2.9) that can be written as

$$E_t \sum_{s=0}^{\infty} \beta^s u_C(t+s) C_{t+s} \left(\frac{R_{t+s} - 1}{R_{t+s}} \right) = u_C(t) \frac{\mathbb{W}_t}{P_t} + E_t \sum_{s=0}^{\infty} \beta^s u_C(t+s) \frac{[G_{t+s} - T_{t+s}]}{R_{t+s}} \quad (4.4)$$

using (2.7).

These conditions define a set of equilibrium allocations, prices and policy variables. There are many equilibria. We want to determine conditions on the exogeneity of the policy variables such that there is a unique equilibrium in the

allocation and prices. We first consider the case in which a policy are sequences of numbers for money supplies and interest rates.

From the resources constraints, (2.10), the intratemporal conditions (2.11), and the cash in advance constraints, (3.4), we obtain the functions $C_t = C(R_t)$ and $L_t = L(R_t)$ and $P_t = \frac{M_t}{C(R_t)}$, $t \geq 0$. As long as $u_C(C_t, L_t)C_t$ depends on C_t or L_t , excluding therefore preferences that are additively separable and logarithmic in consumption, the system of equations can be summarized by the following dynamic equations:

$$\frac{u_C(C(R_t), L(R_t))}{\frac{M_t}{C(R_t)}} = \beta R_t E_t \left[\frac{u_C(C(R_{t+1}), L(R_{t+1}))}{\frac{M_{t+1}}{C(R_{t+1})}} \right], t \geq 0 \quad (4.5)$$

together with the budget constraints, (4.4).

Suppose the path of money supply is set exogenously in every date and state. In addition, in period zero the interest rate, R_0 , is set exogenously and, for each $t \geq 1$, for each state s^{t-1} , the interest rates are set exogenously in $\#S_t - 1$ states. In this case there is a single solution for the allocations and prices. Similarly, there is also a unique equilibrium if the nominal interest rate is set exogenously in every date and state, and so is the money supply in period 0, M_0 , as well as, for each $t \geq 1$, and for state s^{t-1} , the money supply in $\#S_t - 1$ states. The budget constraints restrict, not uniquely, the taxes and debt levels.

The proposition follows

Proposition 4.3. *Suppose policy are constant functions. In general, if money supply is determined exogenously in every date and state, and if interest rates are also determined exogenously in the initial period, as well as in $\Phi_t - \Phi_{t-1}$ states for each $t \geq 1$, then the allocations and prices can be determined uniquely. Similarly, if the exogenous policy instruments are the interest rates in every state, the initial money supply and the money supply, in $\Phi_t - \Phi_{t-1}$ states, for $t \geq 1$, then there is in general a unique equilibrium.*

The proposition states a general result. In the particular case where the preferences are additively separable and logarithmic in consumption, and money supply is set exogenously in every state, there is a unique equilibrium in the allocations and prices. There is no need to set exogenously the interest rates as well. This example is helpful in understanding the main point of the paper, that the degrees of freedom in conducting policy depend on how policy is conducted and on other characteristics of the environment.

4.2. Current or backward interest rate feedback rules .

We have shown Proposition 3.3. assuming that policy was conducted with constant functions for the policy variables. However, the use of interest rate rules that depend on current or past variables clearly preserves the same degrees of freedom in the determination of policy, as identified in that proposition. When fiscal policy is endogenous, it is still necessary to determine exogenously the levels of money supply in some but not all states. The corollary follows

Corollary 4.4. *When policy is conducted with current or backward interest rate feedback rules and fiscal policy is endogenous, there is a unique equilibrium if the money supply is set exogenously in $\#S_t - 1$ states, for each state s^{t-1} , $t \geq 1$, as well M_0 .*

5. Robustness: Finite horizon.

We have shown in the previous section that there are interest rate rules that implement a unique equilibrium but that current or backward feedback rules do not. This means that even if the same number of instruments is set exogenously, the remaining degrees of freedom in determining policy depend on how those degrees of freedom are filled. This happens because the model economy has an infinite horizon.

If the economy had a finite horizon it would be characterized by a finite number of equations and unknowns. In that case the number of degrees of freedom in conducting policy is a finite number that does not depend on whether policy is conducted with constant functions, functions of future, current or past variables, as long as these functions are truly exogenous, i.e. independent from the remaining equilibrium conditions.

To determine the degrees of freedom in the case of a finite horizon economy amounts to simply counting the number of equations and unknowns. We proceed to considering the case where the economy lasts for a finite number of periods $T + 1$, from period 0 to period T . After T , there is a subperiod for the clearing of debts, where money can be used to pay debts, so that

$$\mathbb{W}_{T+1} = M_T + R_TB_T + P_TG_T - P_TT_T = 0$$

The first order conditions in the finite horizon economy are the intratemporal conditions, (2.11) for $t = 0, \dots, T$, the cash in advance constraints, (3.4) also for

$t = 0, \dots, T$, the intertemporal conditions

$$\begin{aligned} \frac{u_C(t)}{P_t} &= R_t E_t \left[\frac{\beta u_C(t+1)}{P_{t+1}} \right], \quad t = 0, \dots, T-1 \\ Q_{t,t+1} &= \beta \frac{u_C(t+1)}{u_C(t)} \frac{P_t}{P_{t+1}}, \quad t = 0, \dots, T-1 \end{aligned} \quad (5.1)$$

and, for any $0 \leq t \leq T$, and state s^t , the budget constraints

$$\sum_{s=0}^{T-t} E_t Q_{t,t+s+1} M_{t+s} (R_{t+s} - 1) = \mathbb{W}_t + \sum_{s=0}^{T-t} E_t Q_{t,t+s+1} P_{t+s} [G_{t+s} - T_{t+s}]$$

where $E_0 Q_{T+1} \equiv \frac{E_0 Q_T}{R_T}$.

The budget constraints restrict, not uniquely, the levels of state noncontingent debts and taxes. Assuming these policy variables are not set exogenously we can ignore this restriction. The equilibrium can then be summarized by

$$\frac{u_C(C(R_t), L(R_t))}{\frac{M_t}{C(R_t)}} = \beta R_t E_t \left[\frac{u_C(C(R_{t+1}), L(R_{t+1}))}{\frac{M_{t+1}}{C(R_{t+1})}} \right], \quad t = 0, \dots, T-1 \quad (5.2)$$

Note that the total number of money supplies and interest rates is the same. There are $\Phi_0 + \Phi_1 + \dots + \Phi_T$ of each monetary policy variable. The number of equations is $\Phi_0 + \Phi_1 + \dots + \Phi_{T-1}$. In order for there to be a unique equilibrium need to add to the system $\Phi_0 + \Phi_1 + \dots + 2\Phi_T$ independent restrictions. One possibility is to set exogenously the interest rates in every state and in addition the money supply in every terminal node. Similarly there is a unique equilibrium if the money supply is set exogenously in every state and the interest rates are set in every terminal node. In this sense, the two monetary instruments are equivalent in this economy.

When policy is conducted with the forward looking feedback rule in Section 2, the policy for the interest rate in the terminal period R_T , cannot be a function of variables in period $T+1$. If these rates are exogenous constants, it still remains to determine the money supply in every state at T .

In this finite horizon economy there is an exact measure for the degrees of freedom in conducting policy. In an economy that lasts from $t = 0$ to $t = T$,

these are $\Phi_0 + \Phi_1 + \dots + 2\Phi_T$. This measure does not depend on how policy is conducted, whether with constant functions or functions of endogenous variables, and it also does not depend on price setting restrictions. The price setting restrictions introduce as many variables as number of restrictions.

6. Robustness: Price setting restrictions

In this section we show that the results derived above extend to an environment with prices set in advance. We modify the environment to consider price setting restrictions. There is a continuum of goods, indexed by $i \in [0, 1]$. Each good i is produced by a different firm. The firms are monopolistic competitive and set prices in advance with different lags.

The households have preferences described by (2.5) where C_t is now the composite consumption

$$C_t = \left[\int_0^1 c_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}}, \theta > 1.$$

Households have a demand function for each good given by

$$c_t(i) = \left(\frac{p_t(i)}{P_t} \right)^{-\theta} C_t.$$

where P_t is the price level,

$$P_t = \left[\int p_t(i)^{1-\theta} di \right]^{\frac{1}{1-\theta}}. \quad (6.1)$$

The households' intertemporal and intratemporal conditions are as before, (2.5), (2.6) and (2.7).

The government must finance an exogenous path of government purchases $\{G_t\}_{t=0}^{\infty}$, such that

$$G_t = \left[\int_0^1 g_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}}, \theta > 0 \quad (6.2)$$

Given the prices on each good i in units of money, $P_t(i)$, the government minimizes expenditure on government purchases by deciding according to

$$\frac{g_t(i)}{G_t} = \left(\frac{p_t(i)}{P_t} \right)^{-\theta} \quad (6.3)$$

The resource constraints can be written as

$$(C_t + G_t) \int_0^1 \left(\frac{p_t(i)}{P_t} \right)^{-\theta} di = A_t N_t. \quad (6.4)$$

We consider now that firms set prices in advance. A fraction α_j firms set prices j periods in advance with $j = 0, \dots, J-1$. Firms decide the price for period t with the information up to period $t-j$ to maximize:

$$E_{t-j} [Q_{t-j,t+1} (p_t(i) y_t(i) - W_t n_t(i))]$$

subject to the production function

$$y_t(i) \leq A_t n_t(i)$$

and the demand function

$$y_t(i) = \left(\frac{p_t(i)}{P_t} \right)^{-\theta} Y_t \quad (6.5)$$

where $y_t(i) = c_t(i) + g_t(i)$

The optimal price is

$$p_t(i) = p_{t,j} = \frac{\theta}{(\theta - 1)} E_{t-j} \left[\eta_{t,j} \frac{W_t}{A_t} \right]$$

where

$$\eta_{t,j} = \frac{Q_{t-j,t+1} P_t^\theta Y_t}{E_{t-j} [Q_{t-j,t+1} P_t^\theta Y_t]}.$$

The price level at date t can be written as

$$P_t = \left[\sum_{j=0}^{J-1} \alpha_j (p_{t,j})^{1-\theta} \right]^{\frac{1}{1-\theta}} \quad (6.6)$$

When we compare the two sets of equilibrium conditions, under flexible and prices set in advance, here we are adding more variables, the prices of the differently restricted firms, but we also add the same number of equations. This argument works in this case, because we can write the new equations as functions of current and past variables.

7. Concluding Remarks

The problem of multiplicity of equilibria under an interest rate policy has been addressed, after Sargent and Wallace (1975) and McCallum (1981), by an extensive literature on determinacy under interest rate rules. Interest rate feedback rules on endogenous variables such as the inflation rate can, with appropriately chosen coefficients, deliver determinate equilibria. There are still multiple equilibria but only one of those equilibria stays in the proximity of a steady state.

In this paper we show that in a simple monetary model with flexible prices or prices set in advance there are interest rate feedback rules, and also money supply feedback rules, that implement unique equilibria. The interest rate feedback rules are forward rules that resemble the policy rules that central banks follow.

The results are not robust to the following change in the theoretical environment. The model economy has an infinite horizon. Suppose that we considered instead the analogous finite horizon economy. In that economy, for an arbitrarily large horizon, single instrument feedback rules would not implement unique equilibria.

8. Appendix

Here we consider the example of section 3 in a stochastic environment and analyze the constant interest rate policy and the interest rate feedback rule. The equilibrium conditions for the stochastic case are (3.1), (3.3), (3.4) and

$$\frac{1}{P_t} = R_t \beta E_t \frac{1}{P_{t+1}}. \quad (8.1)$$

From the deterministic to the stochastic framework only the intertemporal condition changes, all the remaining conditions remain the same.

8.1. Constant interest rate

Here we assume that the central bank chooses to maintain a constant interest rate equal to $\mathbb{R} \geq 1$. As in the deterministic case C_t and L_t are pin down by (3.1) and (3.3). Now it is the expected inflation, π_t , that is pin down by (3.2), $E_t \pi_{t+1} = \mathbb{R} \beta$. There are many distributions of realized inflation that are compatible with that expected inflation and all can be part of an equilibrium. There is as well a multiplicity of equilibrium price sequences and as a consequence from (3.4) a multiplicity of equilibrium money sequences.

If there were frictions in the economy, like sticky wages, prices or portfolios, a constant interest rate policy would not determine the real allocation. In this case there would be a real indeterminacy, see Adão, Correia and Teles (2003). The intratemporal condition of the households would be different. For instance, if prices were set in advance the marginal rate of substitution between leisure and consumption would be equal to the real wage adjusted for the interest rate, but in general the real wage would not be a linear function of the technological shock. Since prices are predetermined the allocation would be determined by the money supply. It would be necessary to choose the money supply in some but not all states of nature to determine the allocation.

8.2. Interest rate feedback rule

The introduction of the concept of the time-invariant equilibrium is necessary to study local determinacy. This concept of equilibrium is the equivalent in the stochastic environment to the steady state equilibrium in the deterministic environment. In order to proceed an assumption is made, for each state s^t , the shocks (A_t) have an identical and independent distribution. The time-invariant equilibrium is a competitive equilibrium with the property that it is just a function of the shock. Formally, the time-invariant equilibrium is a tuple for consumption, leisure, interest rate, money growth and inflation, $\left\{ \mathbb{C}(s_t), \mathbb{L}(s_t), \mathbb{R}(s_t) > 1, \frac{\mathbb{M}(s_{t+1})}{\mathbb{M}(s_t)}, \Pi \right\}$, that satisfies the relevant competitive equilibrium conditions. These conditions are given by,

$$\begin{aligned} \Pi &= \frac{\mathbb{C}(s_t)}{\mathbb{C}(s_{t+1})} \frac{\mathbb{M}(s_{t+1})}{\mathbb{M}(s_t)}, \\ \mathbb{C}(s_t) + G_t &= A_t(1 - \mathbb{L}(s_t)), \\ \frac{u_C(s_t)}{u_L(s_t)} &= \frac{\mathbb{R}(s_t)}{A_t}, \\ u_C(s_t) &= \frac{\beta}{\Pi} \mathbb{R}(s_t) E_t[u_C(s_{t+1})]. \end{aligned} \tag{8.2}$$

For a given $\mathbb{R}(s_t)$ the two middle equations determine $\mathbb{C}(s_t)$ and $\mathbb{L}(s_t)$. Given Π the first equation determines the growth rate of money between a state and any of its subsequent states. Finally (8.2) determines $\mathbb{R}(s_t)$. For the particular utility function we are using (8.2) can be written as

$$\mathbb{R} = \frac{\Pi}{\beta}.$$

That is the time-invariant nominal interest rate does not depend on the shocks.

Suppose that the central bank conducts a pure current Taylor rule:

$$R_t = \mathbb{R} \left(\frac{\pi_t}{\Pi} \right)^{\tau\beta}, \quad (8.3)$$

where $\tau\beta \geq 1$ (the Taylor principle), and $\pi_t = \frac{P_t}{P_{t-1}}$.

After substituting (8.3) in the households' intertemporal condition, we get

$$E_t [z_{t+1}^{-1}] = (z_t^{-1})^{\tau\beta}, \quad (8.4)$$

where $z_t = \frac{\pi_t}{\Pi}$. By recursive substitution we get

$$\left\{ E_t \left\{ \left\{ E_{t+1} \left[\dots \left(E_{t+k-1} z_{t+k}^{-1} \right)^{\frac{1}{\tau\beta}} \dots \right] \right\}^{\frac{1}{\tau\beta}} \right\} \right\}^{\frac{1}{\tau\beta}} = z_t^{-1}, \text{ for all } k, t \quad (8.5)$$

In the following paragraph we supply an heuristic proof that the only equilibria are the time-invariant equilibrium and an infinity of other equilibria which have the characteristic that in some states of nature either inflation is going to infinity or is going to zero.

Since $\tau\beta > 1$, if $z_t^{-1} > 1$ then $z_t^{-1} \rightarrow \infty$ with positive probability. The proof is by contradiction. Assume it was not converging to infinity with positive probability, then it would be bounded with probability one, which means that no matter how arbitrary in the future you take the z_{t+s}^{-1} its expected value would be bounded with probability one. But since the exponent is a constant smaller than one by taking s sufficiently large will get the left hand side of (8.5) smaller than the right hand side. By a similar argument if $z_t^{-1} < 1$, have $z_t^{-1} \rightarrow 0$ with positive probability.

Thus, when the central bank follows a Taylor rule that obeys the Taylor principle it is able to get local determinacy. In a neighborhood of the time-invariant equilibrium with inflation Π there is no other equilibrium. We have just seen that the other equilibria which are infinite in number are either associated with inflation converging with probability bounded from zero to infinity or to zero.

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