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Aggregation, Persistence and Volatility in a Macromodel

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ABSTRACT: Starting from microeconomic foundations, we derive a general formula for the aggregation of outputs of heterogeneous firms (or sectors), and we solve explicitly for the fundamental intertemporal equilibrium path of the aggregate economy. The firms are subject to temporary technology shocks, but the aggregate output has radically different dynamical properties, and a special form of long memory and nonlinearity never used hitherto. We study, analytically, the implied time series properties of the new process characterizing aggregate GDP per capita. This process is more persistent than any dynamically-stable linear process (e.g. autoregressions) and yet is mean-reverting (unlike unit-root processes), and its volatility is of a greater order of magnitude than that of any of its components. This amplification of volatility means that even small shocks at the micro level can lead to large fluctuations at the macro level. The process is also characterized by long cycles which have random lengths and which are asymmetric. Increased monopoly power will tend to reduce the amplitude and increase the persistence of business cycles. Strikingly, we find that the nonlinear aggregate process has an S-shaped decay of memory, similar to the data but unlike linear time series models such as the widely-used Auto-Regressive Integrated Moving-Average (ARIMA) processes and their special cases (including fractional Integration).

KEYWORDS: Auto-Regressive Integrated Moving-Average (ARIMA) processes, Autocovariance functions, Autocorrelation functions (ACFs), Heterogeneous (non-representative) firms, Long memory processes, Monopolistic Competition, Real Business Cycle (RBC).

JEL Classification: C32, E1, E32.

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1 Introduction

Starting from microeconomic foundations, we derive a formula for the aggregation of outputs of heterogeneous firms (or sectors). These firms are subject to temporary technology shocks within an otherwise conventional model of Real Business Cycles (RBCs), but the aggregate output has a special form of long memory and nonlinearity, not discovered hitherto. One of the implications of our results is that two of the most counterfactual aspects of current RBC models, i.e. the lack of persistence and the need for large aggregate shocks,[1] are successfully dealt with *inter alia* solely through going beyond the assumption of a representative firm. Earlier work on the effect on time-series memory of aggregating over heterogeneous entities includes Robinson (1978), Granger (1980), Lewbel (1994), Chambers (1998), Forni and Reichlin (1998), Linden (1998), Pesaran (1999), Lippi and Zaffaroni (2000). Furthermore, dissatisfaction with the simple unit-root hypothesis in Auto-Regressive (AR) macroeconomic models, and the need for alternatives such as long memory models, has been highlighted in the work of Diebold and Rudebusch (1989), Haubrich and Lo (1991), Rudebusch (1993), Diebold and Senhadji (1996). In AR models, memory decays exponentially (Integrated of order 0) or is infinite (Integrated of order 1), but nothing in between. Long memory models fill this gap, and our work will give rise to a new nonlinear type of such a model, from microeconomic foundations. Strikingly, we find that the analytic nonlinear aggregate process for GDP has an S-shaped decay of memory, similar to the data but unlike linear time series models such as the widely-used Autoregressive Integrated Moving-Average (ARIMA) processes and their special cases, including fractional Integration.

One of our contributions is to develop a general equilibrium model, based on the monopolistic competition framework of Dixit and Stiglitz (1977), that deals with the heterogeneity of the productive sector while preserving analytic tractability. Previous studies of the effect of aggregation on persistence started by modelling the economy as divided across N sectors, each one following its own growth path, then showing that the aggregate output over these N sectors is more persistent than an individual sector's output. While making a useful contribution as a starting point, this early approach did not incorporate the feedback mechanism which is at the heart of general equilibrium theory.

There is an additional problem with the existing linear aggregation methods. In the aggregation literature obtaining explicit time series results, the *logarithms* of components of GDP have been modelled as (say) AR processes, then added up. This geometric averaging is not compatible with the way GDP data are compiled, for example by adding the *levels* of GDP components whose *logarithms* may follow

[1] The internal propagation mechanism of RBCs has been characterised as weak, and they require unrealistically large economy-wide technology shocks to account for the variations in the Solow residuals. See for instance Rotemberg and Woodford (1996) and Muellbauer (1997).

a linear process.[2] Another contribution of our paper is to solve analytically this aggregation problem explicitly, within an even more general aggregation framework motivated by economic theory.

In spite of the very general setting of our paper, we are able to obtain an explicit closed form solution for the equilibrium. This solution holds for general distributions of firms. It is highly nonlinear, yet it is sufficiently tractable that its time-series properties can be obtained by analytical methods. As expected, it is much richer than in the previous aggregation approach and features several effects that were masked by the straightforward arithmetic averaging. We find that the result is an unconventional nonlinear time series process for GDP. It is radically different from the process followed by the firms' productivities, which are conventional 'stationary' ARMA processes. This new process is nonlinear, more persistent than any dynamically-stable ARMA and yet is mean-reverting (unlike unit-root processes). Its volatility is of a greater order of magnitude than that of any of its components. This amplification of volatility means that even small shocks at the micro level can lead to large fluctuations at the macro level. The process is also characterized by long cycles which have random lengths and which are asymmetric. Our explicit formulae allow us to trace and quantify the effect of economically-meaningful parameters. For example, increased monopoly power will tend to reduce the amplitude and increase the persistence of business cycles. Our explicit formulae also allow us to estimate, in a simple way, the time series behaviour of GDP from the data without having to resort to simulations or calibrations. We find that the data do not support the autocorrelation function (ACF) specifications of ARIMA models, but support the S-shaped ACF which is implied by our economic model. Relative to linear (e.g. ARIMA) models, the ACF of our aggregate process implies a slower decay of memory initially, followed by a steep drop.

The plan is as follows. In Section 2, we modify a standard RBC model of monopolistic competition by assuming that the productivity of firms are subject to idiosyncratic and to common (economy-wide) shocks. We then derive the fundamental intertemporal equilibrium path of the economy and we derive analytically the dynamic process followed by GDP per capita. In Section 3, we introduce the statistical setup for the new aggregation procedure which followed from solving the economic model of Section 2. We then work out the time series properties of GDP per capita. The derivations are collected in an Appendix where the novel econometric results on aggregation are derived, and which also contains solutions to technical problems that are of independent interest, including a lemma of potential applicability in statistics and applied mathematics.

[2] It is standard practice in economics to model linearly the process driving *logarithms* of positive variables. If levels (not logarithms) were modelled as a linear process, then the changes would be in absolute (not relative or percentage) terms. Furthermore, if the errors were not restricted to be positive, then the level of the series (e.g. component of GDP) could become negative in the model.

2 Framework

In an environment with technological shocks, firms whose technology progresses at an above-average rate will operate at a below-average cost. In turn, these firms will have an incentive to lower their price so as to expand their market share. Consequently, the weight of these firms in the economy will increase relative to the weight of firms which experience an adverse shock. Additionally, to the extent that the output of one firm can be used as the input of another, a favourable technological shock to one firm will ultimately benefit the whole economy. Simply aggregating over N growth paths, as the literature on aggregation has done so far, would have amounted to assuming there is no substitutability across sectors and no inter-industry trade. One of the most appealing features of general equilibrium theory is that agents do not use ad-hoc rules, but adjust their behaviour to other agents' actions; and this is the approach we adopt in our model.

Modelling a heterogeneous productive sector under uncertainty requires us to make choices. Should firms operate under constant returns to scale, the firms with the lowest production cost could undercut their competitors and, under perfect competition, drive them out of business. Some mechanism must be invoked that will yield an optimal size for the firm. Our own choice is to obtain an optimal firm's size by using a framework of monopolistic competition of Dixit and Stiglitz (1977). Such a monopolistically competitive setup has recently been used by Devereux, Head and Lapham (1993, 1996) and by Galí (1999) in RBC models, showing that it was more successful than perfect competition. Our results are therefore directly comparable to theirs and guarantee that the effects obtained are indeed due to aggregation and not to some other feature.

Subsection 2.1 outlines our baseline model, a discrete-time RBC model with a large fixed number of heterogeneous monopolistically competitive firms *à la* Dixit-Stiglitz and a representative infinitely-lived agent *à la* Ramsey, and where accounting is performed in terms of some numéraire. Subsection 2.2 solves for the model's equilibrium at any given period, while Subsection 2.3 derive the optimal dynamic link between the static equilibria.

2.1 Outline of the model

2.1.1 Intermediate input sector

Technology There is a fixed number N of infinitely-lived monopolistically competitive firms, each one producing a non-storable differentiated product. Firm n produces according to a standard Cobb-Douglas production function which takes capital and labour as inputs

$$q_{n,t} = \theta_{n,t}\, k_{n,t}^{\gamma} l_{n,t}^{1-\gamma}, \qquad \gamma \in (0,1) \tag{1}$$

where $q_{n,t}$ is its output at time t, $\theta_{n,t}$ is its technical efficiency and $k_{n,t}$, $l_{n,t}$ are the inputs of labour and capital used up by the firm. Each firms' productivity

$\theta_{n,t}$ is subject to technology shocks that follow processes which will be detailed in (23) of Section 3.

Producer optimization The monopoly firms do not own any capital. Instead, they hire, in each period, physical capital and labour on competitive factor markets. Because the Modigliani-Miller theorem applies to our setting, this assumption entails no loss of generality. The nominal profit of firm n in period t is

$$\tilde{\pi}_{n,t} \equiv \tilde{p}_{n,t} q_{n,t} - \left(\tilde{\imath}_t \; k_{n,t} + \tilde{w}_t l_{n,t}\right)$$

where, in period t, $\tilde{p}_{n,t}$ is the price of the product, $\tilde{w}_t$ is the nominal wage rate and $\tilde{\imath}_t$ is the nominal rental. The firm is facing a demand curve which we will derive in (5) of Subsection 2.2.1. Hence, it is facing a sequence of unrelated profit maximizing problems and its objective in period t is

$$\max_{k_{n,t}, l_{n,t}} \tilde{p}_{n,t} q_{n,t} - \left(\tilde{\imath}_t \; k_{n,t} + \tilde{w}_t l_{n,t}\right), \quad \text{subject to } \theta_{n,t} \, k_{n,t}^{\gamma} \, l_{n,t}^{1-\gamma} \; = q_{n,t}. \tag{2}$$

Ownership Each firm will make a positive profit in each period, making a firm's ownership valuable. Firm ownership is traded in a stock market as shares of stocks and a firm's profits are immediately distributed to its shareholders as dividends. The representative agent initially holds all the shares and share prices will adjust to make him willing to do so in every period.

2.1.2 Final good sector

The final good industry, operating under perfect competition, uses the specialised inputs to produce a final good according to the standard CES aggregation function, see for instance Bénassy (1996),

$$y_t \equiv \left[\frac{1}{N^{1-\rho}} \sum_{n=1}^{N} q_{n,t}^{\rho} \right]^{1/\rho}, \qquad \rho \in (0,1), \tag{3}$$

where y_t is the aggregate output of the final good industry. Since this production function exhibits constant returns to scale in the inputs $q_{\bullet}$, and since this sector is perfectly competitive, final good firms make zero profits, and their number and the distribution of their ownership is immaterial. Notice that the elasticity of substitution between any two products is $1/(\rho-1)$. The final good industry generates a derived demand for the differentiated products, which, we will see, displays a constant elasticity of demand $1/(\rho-1)$.

The aggregate output y_t can be used either for consumption or for physical investment purposes. Investment in period t increases the capital stock of period $t+1$, i.e. with a one period lag. In order to be able to derive a closed form solution for the intertemporal equilibrium of our economy, we need to assume, as in Devereux et al. (1993, 1996), a 100% depreciation rate on capital. Hence, the stock of capital in period $t+1$ is equal to the investment of period t. All capital is purchased by the representative consumer. The assumption of full depreciation will be relaxed later in Corollary 2 of Section 3.

2.1.3 Representative agent

There is one infinitely-lived representative agent in this economy. He inelastically supplies labour, and we normalize labour supply at $l_t = 1$. He also owns all the share of stock and all of the outstanding capital.

Asset markets The consumer can invest in two types of financial investments, both risky: physical capital and shares of stocks. Let $\{vf_{n,t}\}_{t=0}^{\infty}$ be the (endogenous) sequence of ex-dividend capitalized real values of firm n and let $\{\pi_{n,t}\}_{t=0}^{\infty}$ be its stream of real profits and/or dividends. The ex-dividend real value $\overline{vf}_t$ of this portfolio, the gross real return on the market portfolio $\bar{\imath}_t$, and the aggregate real profits $\bar{\pi}_{t+1}$ are

$$\overline{vf}_t \equiv \sum_{n=1}^{N} vf_{n,t}, \quad \bar{\imath}_{t+1} \equiv \frac{\overline{vf}_{t+1} + \bar{\pi}_{t+1}}{\overline{vf}_t}, \qquad \text{where } \bar{\pi}_{t+1} \equiv \sum_{n=1}^{N} \pi_{n,t+1}.$$

Let i_{t+1} be the gross real return on investing in physical capital, which is equal to the rental, by the 100% depreciation assumption. At an equilibrium, aggregate real profits $\bar{\pi}_{t+1}$ are fixed percentage of aggregate output y_{t+1} and the same is true for i_{t+1}. Therefore, the stream of dividends $\{\bar{\pi}_t\}_{t=0}^{\infty}$ can be duplicated by investing in physical capital which means that these two risky assets are collinear.[3] Hence, the gross return on the market portfolio must be equal to the gross return on physical capital

$$\bar{\imath}_t = i_t \equiv \bar{\bar{\imath}}_t,$$

where $\bar{\bar{\imath}}_t$ is this common gross real return. Let a_t be the real value of the financial wealth of the consumer at the end of period t. At an equilibrium, the consumer must hold all physical capital k_{t+1} and the market portfolio $\overline{vf}_t$. His equilibrium financial wealth must be

$$a_t = k_{t+1} + \overline{vf}_t, \qquad \forall t \in \mathbb{N} \cup \{0\}.$$

[3] Note that, because of our representative consumer assumption, the equilibrium consumption allocation determines all the marginal utilities of every contingent consumption plan, i.e. they define a complete set of Arrow-Debreu state-contingent securities, see for instance Talmain (1999). This set of shadow securities can be used to price (by arbitrage) any arbitrary (stochastic or deterministic) stream of payoffs.

Since both components of $\bar{a}_t$ have the same gross return, $\bar{\bar{\imath}}_{t+1}$ is also the gross return on the consumer's portfolio. Let the consumer's consumption (of final goods) be c_t, his dynamic budget constraint can be expressed as

$$a_t = \bar{\bar{\imath}}_t a_{t-1} + w_t - c_t,$$

where w_t is the real wage in period t, and we recall that the consumer supplies one unit of labour per period. Given the (endogenous) initial market capitalization of the firms $\{vf_{n,-1}\}_{n=1}^{N}$ and the initial stock of capital in the economy k_0, his initial financial wealth is a_{-1} at the end of $t = -1$, and is $\bar{\bar{\imath}}_0 a_{-1}$ at the beginning of $t = 0$, where

$$a_{-1} = k_0 + \overline{vf}_{-1} \quad \text{and} \quad \bar{\bar{\imath}}_0 a_{-1} = \bar{\bar{\imath}}_0 \left(k_0 + \overline{vf}_{-1}\right) \ .$$

We are now in a position to state the consumer optimization.

Consumer optimization The consumer wants to maximise the present value of its expected utility subject to his budget constraint:

$$\max_{c_t, \{p_\bullet a_\bullet\}_t} \mathrm{E}_{\bullet|0} \left[\sum_{t=0}^{\infty} \delta^t \log c_t \right], \text{ s.t. } a_t = \bar{\bar{\imath}}_t a_{t-1} + w_t - c_t, \text{ and } a_{-1} = k_0 + \overline{vf}_{-1}, \tag{4}$$

and subject to the transversality condition implied by the intertemporal budget constraint, where $\{p_\bullet a_\bullet\}_t$ refers to the optimal portfolio allocation at time t, i.e. the weights of each asset in the consumer's portfolio. Here $\mathrm{E}_{\bullet|\tau}[\bullet]$ is the expectation operator taken conditionally on the information available at time τ, δ is the rate of time preference and the utility of consumption is logarithmic.

2.2 Equilibrium

2.2.1 Producers' optimum

The producers' optimization problems, both in the final good sector and in the intermediate input sector, are static: they maximize period by period as their decision in one period does not affect their problem in any other period.

Final good industry Cost minimization of (3) gives rise to the derived demand for good n, $q_{n,t}$, as a function of its own price $\tilde{p}_{n,t}$, of the price of the final good p_t and of total output of final good y_t

$$q_{n,t} \equiv \left[\frac{p_t}{\tilde{p}_{n,t}}\right]^{1/(1-\rho)} \frac{y_t}{N}, \qquad \text{with } \frac{1}{p_t^{\nu}} \equiv \frac{1}{N} \sum_{n=1}^{N} \frac{1}{\tilde{p}_{n,t}^{\nu}}, \tag{5}$$

with $\nu \equiv \rho / (1-\rho) \in \mathbb{R}_+$.[4]

[4]Equation (5) expresses the derived demand for intermediate input n. The elasticity of this demand is $1/(\rho - 1)$ which implies that the optimal pricing strategy of the individual producer

Intermediate input industry We now turn to deriving the minimal unit cost of the monopoly firm. Solving the cost minimization problem associated with (2), we find that every monopoly firm in the economy will choose to operate with the same capital/labour ratio, which consequently will be equal to the economy-wide capital/labour ratio k_t

$$\frac{k_{n,t}}{l_{n,t}} = \frac{\gamma}{1-\gamma}\frac{w_t}{i_t} = k_t, \qquad n = 1, 2, \ldots, N. \tag{6}$$

The derived demand for labour and capital by the intermediate input firm n producing $q_{n,t}$ is

$$k_{n,t} = \frac{k_t^{1-\gamma}\, q_{n,t}}{\theta_{n,t}} \quad \text{and} \quad l_{n,t} = \frac{q_{n,t}}{\theta_{n,t}\, k_t^{\gamma}}. \tag{7}$$

Optimal prices The static nature of the various firms' optimization problem enables us to calculate the various prices and outputs as a function of the current state variables (k_t,$\{\theta_{n,t}\}_{n=1}^{N}$) without reference to other periods. The profit maximizing policy of firm n is a fixed mark-up $(1/\rho)$ over cost. Using the optimal capital/labour ratio (6) and the derived demand for the two factors (7), the optimal price charged is

$$\tilde{p}_{n,t} = \frac{w_t}{(1-\gamma)\rho\, k_t^{\gamma}}\frac{1}{\theta_{n,t}}p_t. \tag{8}$$

Using this and the aggregate price level of (5), we find that the ratio of prices is inversely proportional to the ratio of productivities

$$\frac{p_t}{\tilde{p}_{n,t}} = \frac{\theta_{n,t}}{\theta_t}, \tag{9}$$

where the aggregate productivity θ_t is obtained as

$$\theta_t^{\nu} \equiv \frac{1}{N}\sum_{n=1}^{N}\theta_{n,t}^{\nu}. \tag{10}$$

Substituting (9) into (8) gives us w_t which can then be plugged into (6), and we get

$$w_t = (1-\gamma)\rho\,\theta_t\, k_t^{\gamma} \quad \text{and} \quad i_t = \frac{\gamma\,\rho\,\theta_t}{k_t^{1-\gamma}}. \tag{11}$$

is to set a constant mark-up of $(1-\rho)/\rho = \nu^{-1}$ over its unit cost. The variable ν^{-1} turns out to perform two functions in our set-up: (*i*) it measures monopoly power, and (*ii*) it measures the degree of complementarity across the N monopolistically-competitive firms of the economy. We will not attempt to disentangle these two aspects.

Factor allocation and final output Substituting for the relative price from (9) into the demand function for good n given by (5), we obtain the output of good n as

$$q_{n,t} = \left[\frac{\theta_{n,t}}{\theta_t}\right]^{1/(1-\rho)} \frac{y_t}{N}. \tag{12}$$

Substituting into the firms' demand for labour (7), we find

$$l_{n,t} = \frac{\theta_{n,t}^{\nu}}{\theta_t^{1/(1-\rho)}} \frac{1}{k_t^{\gamma}} \frac{y_t}{N} .$$

Aggregating over all monopoly firms, and recalling that labour is only used in the production of intermediate inputs and that the consumer inelastically supplies one unit of labour in each period ($l_t = 1$), we have the per-capita production

$$1 = \sum_{n=1}^{N} l_{n,t} = \frac{y_t}{\theta_t\, k_t^{\gamma}} \iff y_t = \theta_t\, k_t^{\gamma} . \tag{13}$$

Interpretation In each period, only the final good has a final use (as the intermediate inputs do not contribute directly to utility and cannot be used for investment). Hence, the question of resource allocation in period t consists in finding how to allocate the two factors (k_t which is predetermined and $l_t = 1$ which is fixed) so as to maximize the output of the final good y_t. The optimal allocation we found is that firm n will hire labour $l_{n,t}$ and capital $k_{n,t}$ in quantities

$$l_{n,t} = \frac{1}{N}\frac{\theta_{n,t}^{\nu}}{\theta_t^{\nu}} \quad \text{and} \quad k_{n,t} = \frac{1}{N}\frac{\theta_{n,t}^{\nu}}{\theta_t^{\nu}} k_t,$$

leading to a final output production per capita of $y_t = \theta_t\, k_t^{\gamma}$. Equation (13) can be used as a black-box to translate the stock of capital k_t and the productivity of the economy θ_t into final output y_t. In fact, if a central planner were to be put in charge of the distribution of factors at time t with the goal of maximizing final output y_t, he would choose exactly the same resource allocation: because firms all use the same mark-up, they distort factor allocation symmetrically resulting in no static relative distortion of resources. However, absolute factor income is distorted as profits claim a share of national income.

2.2.2 Consumer's equilibrium

Any solution to the consumer's problem (4) must be an Euler path. In particular, for any asset x with real gross return $i_{x,t+1}$ in period $t+1$, the consumer will invest in this asset at time t up to the point where

$$\frac{1}{c_t} = \delta \mathrm{E}_{\bullet|t}\left[\frac{i_{x,t+1}}{c_{t+1}}\right]. \tag{14}$$

2.3 Dynamic equilibrium

A dynamic equilibrium is a process for prices and for the allocation of resources and output such that: (*i*) the producers are optimizing in every period (their static problem), (*ii*) the consumer is optimizing (his intertemporal problem), (*iii*) the goods markets and the asset markets clear in every period. Setting prices and quantities as in Subsection 2.2.1 ensures that the producers are optimizing and that the goods market are in equilibrium. The asset markets consists of the market for shares of stocks and of the market for physical capital. The market for shares is in equilibrium if the representative agent wants to hold 100% of the existing shares, i.e. if (14) holds for all $\left\{\{i_{n,t}\}_{t=0}^{\infty}\right\}_{n=1}^{N}$ when the agent owns all the firms. Clearly, physical capital is the only outlet for aggregate accumulation. The market for physical capital is in equilibrium when the aggregate output not used for immediate consumption is equal to the amount of capital required in the next period, i.e.

$$k_{t+1} = s_t y_t, \quad s_t \equiv 1 - \frac{c_t}{y_t}, \tag{15}$$

where s_t is the national savings rate. Equilibrium in the physical capital market obtains when the consumer can lease, at time $t+1$, all the capital k_{t+1} he accumulated in period t, at a rental that validates his expectations for that period.

Finally, the consumer must be optimizing; i.e. the Euler equations (14), his dynamic budget constraint in (4) and the intertemporal budget constraint must all be satisfied. In particular, the Euler equation must be satisfied for the real gross return on physical capital i_{t+1}. Substituting for i_{t+1} from (11), we find

$$\frac{1}{c_t} = \mathrm{E}_{\bullet|t}\left[\frac{\delta\, i_{t+1}}{c_{t+1}}\right] \Longrightarrow \frac{1}{(1-s_t)\,\theta_t\, k_t^{\gamma}} = \mathrm{E}_{\bullet|t}\left[\frac{\delta\,\gamma\,\rho}{\theta_t\,(1-s_{t+1})\, s_t\, k_t^{\gamma}}\right]. \tag{16}$$

Using the method of repeated substitutions, we find that

$$\begin{aligned}\frac{1}{1-s_t} &= 1 + \delta\,\gamma\,\rho\,\mathrm{E}_{\bullet|t}\left[\frac{1}{1-s_{t+1}}\right] \\ &= \left[1 + s^* + \cdots + (s^*)^{\tau}\right] + (s^*)^{\tau+1}\,\mathrm{E}_{\bullet|t}\left[\frac{1}{1-s_{t+\tau+1}}\right], \text{ where } s^* \equiv \delta\,\gamma\rho.\end{aligned}$$

One solution to this difference stochastic equation corresponds to the transversality condition $\lim_{\tau\to\infty}(s^*)^{\tau+1}\,\mathrm{E}_{\bullet|t}\left[1/(1-s_{t+\tau+1})\right] = 0$, which will hold if the national savings rate is bounded away from 100%. It can be shown that this condition implies that the intertemporal budget constraint holds. This solution corresponds to

$$s_t = s^* = \delta\,\gamma\rho, \qquad \forall t \in \mathbb{N} \cup \{0\}, \tag{17}$$

and is our equilibrium solution. The other Euler equations (14) for the gross returns on equity $\{i_{n,t}\}_{n=1}^{N}$ determine the valuation of the firms $\{vf_{n,t}\}_{n=1}^{N}$.

Substituting from (17) for s_t into (13), we obtain the dynamic equation for GDP per capita, characterizing our fundamental path, as

$$y_t = \theta_t \ (s^*)^{\gamma} \ y_{t-1}^{\gamma}. \tag{18}$$

The dynamic properties of the process $\{y_t\}$ of (18) will be determined once we derive those of $\{\theta_t\}$, where θ_t is defined in (10). We now turn to the derivation of the implied time series properties of GDP.

3 The effect of heterogeneity on the Time Series properties

In standard RBC models which give rise to ARs, memory decays exponentially (dynamically-stable AR roots) or is infinite (unit root), but nothing in between, as the AR root of largest magnitude increases to 1. Macroeconomic processes require a more elaborate characterization, and we shall show that the economic model outlined in the previous section gives rise to a process which is very different from ARs.

Why are the published results in the time series literature on aggregation not applicable here? There are a number of complications that our economic model gives rise to:

1. The aggregation is not linear (arithmetic mean) and is not log-linear (geometric mean), either of which could have been handled by current techniques. Here, we have a hybrid made of the arithmetic sum of geometric ARs. This results in a highly nonlinear process, for which linear models (e.g. ARIMA; see Abadir and Taylor (1999) for precise definitions) will not capture all the salient features.

2. Dependence between the various firms (hence AR components of the aggregate) complicates the setup, and the time series properties have not been worked out in the general case, except in Granger (1980) where the aggregation is log-linear and the dependence structure is simplified.

3. The results of Abadir and Talmain (1998) make it now tractable to characterize explicitly the effect of the nonlinear transformation of time series. However, they do not consider any of the aggregation issues raised here.

All these complications are not avoided by non-CES aggregation. Any other method which sums geometric ARs (i.e. which sums the *levels* of technology shocks whose *logarithms* follow a linear process) will run into similar nonlinearities. This aggregation is essential in economics, where one often aggregates levels of variates which are strictly positive (hence not representable by Normal ARs in levels which can become negative). The main distinction of our aggregation approach is encapsulated in the following simplified example of two different aggregations:

$$\sum_{n=1}^{N} \log q_{n,t} \neq \log \left(\sum_{n=1}^{N} q_{n,t} \right)$$

where the $q_{n,t}$'s follow some multiplicative process such that $q_{n,t} > 0$ and their changes are in percentage terms. The left hand side is the aggregation studied in the time series literature, but it is not the one that arises in economics: components of, say, GDP are not log-transformed before being added up. However, the right hand side is what arises from a proper aggregation of positive components of GDP, and this gives rise to a *sum of multiplicative* processes which is a nonlinear function *even when components are added linearly* (special case of our CES in (3)). The distinction may seem subtle, but it introduces an important statistical nonlinearity, as we will demonstrate in this section. Some of the resulting conclusions will be strikingly different from those in the existing literature, and the main highlights are:

1. through the heterogeneity of firms, temporary shocks are magnified into a more persistent process, and this furthers our understanding of the propagation, magnitude and persistence of such shocks;

2. we are able to trace and quantify the effect of economically-meaningful parameters (e.g. extent of monopolistic power in the economy) on these time series patterns;

3. the resulting process is highly nonlinear, behaving as mean-reverting in some cases but like a random walk in other cases, all within the same model (unlike existing statistical long-memory models);

4. the slowly-decaying S-shaped autocorrelation function (measuring the persistence of business cycles) implied by our economic model turns out to be close to the one observed from the data, but very different from the one implied by parsimonious ARIMA models.

We now introduce in Subsection 3.1 the setup for solving the dynamics in (18) for GDP per capita, which follow from the microeconomic foundations. In Subsection 3.2, the time series properties are derived and described, and the setup is generalized to allow for any nonzero depreciation rate. Finally, Subsection 3.3 considers further properties and empirical implications of the dynamic process.

3.1 Statistical setup

Let the individual sequences $\{\theta_{n,t}\}_{t=1}^{\infty}$ be generated by the geometric (multiplicative) AR process

$$\log\theta_{n,t} = \mu_n + \alpha_n \log\theta_{n,t-1} + \varepsilon_{n,t}, \tag{19}$$

where $|\alpha_n| < 1$ and $\{\varepsilon_{n,t}\}_{t=1}^{\infty} \sim \mathrm{IN}(0, \sigma_n^2)$ is a sequence of Independent Normal variates with zero mean and variance σ_n^2. After Theorem 1, we give a condition (likely to be satisfied in practice) which extends our results to more general ARMA processes, so the full complexity of an ARMA is not yet needed at this stage of derivations.

In (19), we condition upon $\theta_{n,0} = 1$, since the model is unaffected by the value of $\theta_{n,0}$. To see this, rewrite the process as

$$\log\tilde{\theta}_{n,t} = \tilde{\mu}_n + \alpha_n \log\tilde{\theta}_{n,t-1} + \varepsilon_{n,t},$$

where $\tilde{\theta}_{n,t} \equiv \theta_{n,t}/\theta_{n,0}$ (giving $\tilde{\theta}_{n,0} = 1$) and $\tilde{\mu}_n \equiv \mu_n + (\alpha_n - 1)\log\theta_{n,0}$, so that any other value of $\theta_{n,0}$ can be absorbed into a redefined μ_n without changing the dynamic structure of (19). Furthermore, in all the derivations of the previous section, expectations are taken conditionally on the information available at time $t = 0$, so that randomness begins with $t = 1$.

In (19), the unconditional mean of $\log\theta_{n,t}$ is $\mu_n/(1-\alpha_n)$, which varies over n even when μ is fixed (unless $\mu = 0$). We are interested in having a varying unconditional mean, modelled parsimoniously. We therefore simplify the setup by assuming henceforth that $\mu_n = \mu$ for all n.

Let $\tau \in \mathbb{N} \cup \{0\}$ and $\nu \in \mathbb{R}_+$. Then, the autocorrelation function of $\{\theta_{n,t}^{\nu}\}_{t=1}^{\infty}$ for any given n is

$$\begin{aligned} r_{t,t+\tau} &\equiv \frac{v_{t,t+\tau}}{\sqrt{v_{t,t}v_{t+\tau,t+\tau}}} \\ &= \frac{\exp\left(\frac{\nu^2\sigma_n^2\alpha_n^{\tau}\left(1-\alpha_n^{2t}\right)}{1-\alpha_n^2}\right) - 1}{\sqrt{\exp\left(\frac{\nu^2\sigma_n^2(1-\alpha_n^{2t})}{1-\alpha_n^2}\right) - 1}\sqrt{\exp\left(\frac{\nu^2\sigma_n^2\left(1-\alpha_n^{2(t+\tau)}\right)}{1-\alpha_n^2}\right) - 1}}, \end{aligned} \tag{20}$$

where $v_{t,t+\tau}$ is the autocovariance function of $\{\theta_{n,t}^{\nu}\}_{t=1}^{\infty}$; see the Appendix for details.[5] For large t (or small α_n^t), this function behaves as

$$r_{t,t+\tau} \simeq \frac{\exp\left(\frac{\nu^2\sigma_n^2}{1-\alpha_n^2}\alpha_n^{\tau}\right) - 1}{\exp\left(\frac{\nu^2\sigma_n^2}{1-\alpha_n^2}\right) - 1} \tag{21}$$

[5] We define the autocovariance by $v_{t,t+\tau} \equiv \mathrm{E}\left[\left(x_t^{\nu} - \mathrm{E}\left[x_t^{\nu}\right]\right)\left(x_{t+\tau}^{\nu} - \mathrm{E}\left[x_{t+\tau}^{\nu}\right]\right)\right]$ instead of the usual $\mathrm{E}\left[\left(x_t^{\nu} - \mathrm{E}\left[x_t^{\nu}\right]\right)\left(x_{t+\tau}^{\nu} - \mathrm{E}\left[x_t^{\nu}\right]\right)\right]$ which is designed only for (asymptotically) stationary series. Also, our definition of the autocorrelation function differs from the common one, $v_{t,t+\tau}/v_{t,t}$, which can exceed 1 for nonstationary series.

which does not depend on t, because the process is strictly stationary *only asymptotically* (as $t \to \infty$). It is however (dynamically-)stable for *all* $t \in \mathbb{N}$, because $|\alpha| < 1$.

At least three comments are in order for $r_{t,t+\tau}$, comparing it to the well-known asymptotic autocorrelation function α_n^τ of $\{\log \theta_{n,t}\}_{t=1}^\infty$ (i.e. the autocorrelation function of an arithmetic AR):

1. As $\tau \to \infty$, the function (21) declines exponentially as α_n^τ (by expanding the exponential in the numerator).

2. More generally, a small-σ_n expansion (expanding $\exp[\bullet]$ for $\sigma_n \to 0$) of the function will reproduce the asymptotic autocorrelation function α_n^τ of $\{\log \theta_{n,t}\}_{t=1}^\infty$. The same is true for $\nu \to 0$. The general rule that gives rise to such results is derived in Abadir and Talmain (1998).

3. For non-trivial σ_n, the autocorrelation function depends on σ_n, which actually carries the same weight as ν, in contrast to the usual arithmetic AR where σ_n has no effect at all on autocorrelations. This will have important implications for the effect of σ_n on autocorrelation functions when analysing the nonlinear aggregation. It is a feature that does not arise in the traditional literature on aggregation in time series.

Now, recall that the aggregate θ_t is given by (10). Let

$$X_t \equiv \frac{1}{N} \sum_{n=1}^{N} \theta_{n,t}^{\nu}, \tag{22}$$

where $\{\theta_{n,t}\}_{t=1}^\infty$ are generated by (19). Clearly, the aggregate X_t is a highly nonlinear function of the impulses (shocks) $\varepsilon_{n,t}$ of (19), which are driving the stochastics in this system; see the Appendix (espec. the beginning of the proof of Theorem 1) for more details. These shocks $\varepsilon_{n,t}$ can be decomposed into orthogonal components, so that we write

$$\log \theta_{n,t} = \mu + \alpha_n \log \theta_{n,t-1} + u_{n,t} + \tilde{\beta}_n \tilde{e}_t, \tag{23}$$

with

$$\begin{aligned} \{u_{n,t}\}_{t=1}^\infty &\sim \mathrm{IN}\left(0, \omega_n^2\right) \quad \text{and} \quad \mathrm{E}\left[u_{n,t} u_{k,s}\right] = 0, \ \forall k \neq n, \\ \{\tilde{e}_t\}_{t=1}^\infty &\sim \mathrm{IN}\left(0, \psi^2\right) \quad \text{and} \quad \mathrm{E}\left[u_{n,t} \tilde{e}_s\right] = 0. \end{aligned}$$

The $u_{n,t}$ are idiosyncratic shocks, whereas the $\tilde{e}_t$ are common shocks whose impact is transmitted to individual series via the scaling parameters given by the $\tilde{\beta}_n$. By definition and without loss of generality, the two types of shocks are independent of each other. Furthermore, there is no loss of generality in using $\{e_t\}_{t=1}^{\infty} \sim$ IN$(0, 1)$ and

$$\tilde{\beta}_n \tilde{e}_t = \tilde{\beta}_n \psi e_t \equiv \beta_n e_t \tag{24}$$

in order to replace $\tilde{\beta}_n \tilde{e}_t$ by $\beta_n e_t$ in (23), which we shall do henceforth. It is also assumed that $\mu \in \mathbb{R}_+$ for simplicity.

The underlying probability measure is defined over time (t) and space (n), and in the latter case, it is the one from which the individual parameters α_n, ω_n^2 and β_n are drawn. One may specify a joint density for α, ω^2 and β, but there is no reason to believe that they interact in a systematic way: α_n is independent of the errors, with the idiosyncratic and common errors being mutually independent by definition. It is therefore enough to specify the marginal densities of α, ω^2 and β. For simplicity, we will assume that $\alpha \in (0, 1)$, $\omega \in \mathbb{R}_+$ and $\beta \in \mathbb{R}_+$ are continuous variates with density functions $f_\alpha(\alpha)$, $f_\omega(\omega)$ and $f_\beta(\beta)$, respectively. We will further assume that:

1. the variate $\alpha \in (0, 1)$ is distributed according to the Beta density $f_\alpha(\alpha) = \alpha^{g_\alpha - 1}(1-\alpha)^{h_\alpha - 1}/\mathrm{B}(g_\alpha, h_\alpha)$;
2. the variate $\omega \in \mathbb{R}_+$ is distributed according to the Generalized Gamma density $f_\omega(\omega) = \lambda h_\omega^{g_\omega} \omega^{\lambda g_\omega - 1} \exp\left[-h_\omega \omega^\lambda\right]/\Gamma(g_\omega)$; and
3. the variate $\beta \in \mathbb{R}_+$ is distributed according to the Generalized Gamma density $f_\beta(\beta) = \lambda h_\beta^{g_\beta} \beta^{\lambda g_\beta - 1} \exp\left[-h_\beta \beta^\lambda\right]/\Gamma(g_\beta)$;

where $\Gamma(\bullet)$ is the Gamma (generalized factorial) function, $\mathrm{B}(g, h) \equiv \Gamma(g)\Gamma(h)/\Gamma(g+h)$ is the Beta function, the parameters $g_\bullet, h_\bullet$ are all positive and further:

- $h_\alpha \in (0, 1]$, such that we exclude the unrealistic case of $h_\alpha > 1$ where AR roots close to unity are almost ruled out;
- $\lambda \in (2, \infty)$ and $g_\omega, g_\beta \in \left(\frac{1}{2}, \infty\right)$, implying the exclusion of the unrealistic case where $\omega = 0$ and $\beta = 0$ are the most 'likely' values (mode of the density).

The first assumed density is typical in the literature on aggregation; e.g. see Granger (1980). The next two assumptions are required because of the relevance of the variance in our geometric AR setting (unlike in the literature on arithmetic ARs). They are reasonable, in practice, because of two reasons. First, one would expect the average variance of idiosyncratic shocks (ω_n^2) and/or of the amplification of common shocks (β_n) to be finite. Secondly, the likelihood of survival (existence) of firms should decline rapidly (e.g. exponentially) as the size of the risk they are exposed to becomes larger; both after a possible mode of the density near (but not at) zero. The Generalized Gamma is a very rich class incorporating many known densities as special cases; for example, the χ^2, exponential, Weibull and one-sided Normal are all nested within it.

To illustrate that a Generalized Gamma is not an unreasonable assumption, we include Figures 1 and 2 where we plot $f_\omega(\omega)$ and $f_{\tilde{\beta}}(\tilde{\beta})$ for some parameter values and compare them to the histograms for the published data. The source is the Risk Measurement Services of LBS (1988), where a listing of the 2,150 UK corporations' beta and specific risk for 1988 is found. The standard deviation of the idiosyncratic risk is simply the product of the published specific risk (given as a percentage of common risk) and of the standard deviation of the common risk.

The way we have defined β_n in (23)-(24) shows that the variance ψ^2 of the economy-wide shock need not be forced to take unrealistically large levels in practice. Representative-firm models force $\beta_n = \beta$ for all firms n (i.e. no variance), so that the variance ψ has to do all the work of accounting for the variance in βe_t (or $\tilde{\beta}\tilde{e}_t$), while in reality the existing heterogeneity of firms will already contribute to the variance of our factor $\beta_n e_t$ (or $\tilde{\beta}_n\tilde{e}_t$).[6]

3.2 Time series properties of GDP per capita

Now, the question of interest is whether the autocorrelation function of the aggregate series $\{X_t\} \equiv \{N^{-1}\sum_{n=1}^{N}\theta_{n,t}^{\nu}\}$ decays more slowly than the rate in (20) belonging to any of its typical components $\theta_{n,t}^{\nu}$, and also how it compares to the everlasting memory of random walks. The common assumption here and in the literature on aggregation is that N is large. However, we depart from the main line of proofs used in the literature on the aggregation of time series. The standard approach has so far been to derive the spectrum of $\{\theta_{n,t}^{\nu}\}_{t=1}^{\infty}$ for the special case $\nu \to 0$, then approximating the spectrum of the aggregate series, from which one finally approximates its autocorrelation function. Instead, we take the more direct approach of deriving the autocorrelation function of $\{X_t\}$ from those of $\{\theta_{n,t}^{\nu}\}_{t=1}^{\infty}$, and this for any $\nu \in \mathbb{R}_+$.[7]

[6] It can be shown that the variance of β_n is inversely proportional to $h_\beta^{2/\lambda}$, namely $h_\beta^{2/\lambda}\mathrm{Var}(\beta_n) = \left[\Gamma\left(g_\beta + \frac{2}{\lambda}\right)/\Gamma\left(g_\beta\right)\right] - \left[\Gamma\left(g_\beta + \frac{1}{\lambda}\right)/\Gamma\left(g_\beta\right)\right]^2$.

[7] We needed a general result for any such ν. The cost of such a generalization is that we assumed Normality of $\varepsilon_{n,t}$ for any given n (but not as n varies), as opposed to a general distribution with finite first four moments which would be required for the standard analysis of spectral estimation to go through.

Theorem 1 *The autocovariance function of* $\{X_t\}$ *is*

$$V_{t,t+\tau} \equiv E[X_t X_{t+\tau}] - E[X_t]\, E[X_{t+\tau}]$$

$$\simeq \frac{\pi^2 (16)^{2-g_\beta+g_\omega} \left(\nu^4 t (t+\tau)/4\right)^{\frac{\lambda}{2}\left(g_\beta+g_\omega-1\right)}}{\left(\nu^2\mu^2 (t-1)\, t\, (t+\tau-1)\, (t+\tau)\right)^{h_\alpha} h_\beta^{2g_\beta-1} h_\omega^{2g_\omega-1}} \left(\frac{\Gamma(g_\alpha+h_\alpha)}{\Gamma(g_\alpha)\,\Gamma(g_\beta)\,\Gamma(g_\omega)}\right)^2$$

$$\exp\left[\nu\mu(2t+\tau) + \frac{\nu^4\left[t^2+(t+\tau)^2\right]}{2^{\frac{4}{\lambda}+2}}\left(\frac{1}{h_\beta^{4/\lambda}}+\frac{1}{h_\omega^{4/\lambda}}\right) - \frac{\nu^\lambda\left[t^{\frac{\lambda}{2}}+(t+\tau)^{\frac{\lambda}{2}}\right]}{2^{\frac{\lambda}{2}+2}}\left(\frac{1}{h_\beta}+\frac{1}{h_\omega}\right)\right]$$

$$\left(\exp\left[\frac{\nu^4\sqrt{t^3(t+\tau)}}{2\left(4h_\beta^2\right)^{\frac{2}{\lambda}}}\right]-1\right),$$

and its corresponding autocorrelation function is

$$R_{t,t+\tau} \equiv \frac{V_{t,t+\tau}}{\sqrt{V_{t,t}V_{t+\tau,t+\tau}}} \simeq \frac{\exp\left[\frac{\nu^4\sqrt{t^3(t+\tau)}}{2\left(4h_\beta^2\right)^{\frac{2}{\lambda}}}\right]-1}{\sqrt{\exp\left[\frac{\nu^4 t^2}{2\left(4h_\beta^2\right)^{\frac{2}{\lambda}}}\right]-1}\sqrt{\exp\left[\frac{\nu^4 (t+\tau)^2}{2\left(4h_\beta^2\right)^{\frac{2}{\lambda}}}\right]-1}}.$$

The approximations that are reported in this theorem are known as leading-term approximations. They give the dominant term in the expansions of functions. For more details on such issues, see Abadir (1999). The other terms, which are alternating in sign, can be obtained as infinite series from the Appendix, but they would add expository complications without affecting the message of this paper.

By a standard lag-polynomial factorization in time series analysis, the leading-term approximations given in the theorem are unaffected if the microeconomic processes followed by the technology shocks are dynamically-stable ARMA instead of AR(1) processes, as long as the AR root with largest modulus is not a repeated or conjugate root.[8] This is the case if the density function of these other roots is stochastically dominated by the density $f_\alpha(\alpha)$ as $\alpha \to 1$.

It is possible to use spectral analysis, which is in a one-to-one relation with autocovariances, subject to extra caution since the spectra of nonstationary series are not time-invariant. Here, we simply infer the amplitude of cycles from the autocovariance function, and the frequency of cycles from the autocorrelation function.[9] We must stress that our measures of time-dependence are formulated

[8] This extension is very general. By the Wold decomposition, *any* stationary process has an MA representation with time-invariant coefficients in its lag-polynomial. Furthermore, this polynomial can be approximated arbitrarily closely by Padé approximants, namely the ratio of two low-order polynomials, which is precisely the parsimonious ARMA representation.

[9] Non-Normal processes could contain more information than is revealed by their first two moments. However, we focus on them here, because they answer our questions about amplitude (volatility) and persistence (memory) of business cycles.

in terms of moments, rather than with reference to a particular statistical model which may or may not be correct. For example, we do not use AR parameters as measures of the extent of time-dependence, since these would be inadequate in the current context.

Recall that $X_t \equiv \theta_t^{\nu}$, with $\nu \equiv \rho/(1-\rho) \in \mathbb{R}_+$ as the parameter measuring the degree of substitutability and/or competition between the components $q_{n,t}$ of GDP. We can now state the following remarks on the theorem relating to $\{X_t\}$:

1. Clearly, the (autoco)variance of the series changes over time, indicating that the process which we have for $\{X_t\}$ is not strictly stationary. As we shall see in Remarks 6-8 below, it is a long-memory process and there are cases where it can even behave like a random walk. One can only talk about 'the' (autoco)variance of the series with reference to some point in time. The remarks to follow will either presume conditioning on a fixed point in time, or tracing the evolution as time passes. Furthermore, (31) of our Appendix gives the mean of the stochastic trend of $\{X_t\}$, which turns out to be time-varying and not symmetric around any value in $\mathbb{R}_+$.

2. An increase of λ implies that fewer large values of β and ω are likely to be observed (as implied by the density function of the Generalized Gamma), which should dampen the magnitude of shocks to θ_t. As one would expect, the exponential term containing $(t+\tau)^{\lambda/2}$ in $V_{t,t+\tau}$ of our theorem shows that this is indeed the case: the amplitude of the business cycle declines as λ increases beyond 4. As for the persistence of the cycle, the effect is less pronounced and generally shortens the cycle (see Remarks 6-8 below for more detail).

3. What are the effects of ν, $g_\bullet$ and $h_\bullet$ on the amplitude of the business cycle? The parameters ν^{-1} (mark-up, or degree of complementarity and/or monopoly power), h_β^{-1} (extent of common shock and heterogeneity of firms) and h_ω^{-1} (extent of idiosyncratic shock) have an important influence. The effect of changing h_β^{-1} and h_ω^{-1} on the diffusion (amplification) of common shocks is similar to changing ν^λ by varying ν. Letting either of h_β^{-1}, h_ω^{-1} or ν tend to zero dampens the amplitude of the business cycles, while increasing them will have a dampening effect only if $\lambda > 4$ (i.e. if the decay of the tail of the density of β and ω is sufficiently fast). Finally, of the two shocks in our model, the common shock is the more potent one, as is seen from the last exponential term of $V_{t,t+\tau}$ in the theorem. The spread of the two shocks is also positively related to g_β and g_ω, with both increasing the autocovariance of the series over time, albeit in a weaker way than the scaling parameters h_β^{-1} and h_ω^{-1}. Notice that the parameters of the distribution of α_n have little relative impact, so long as roots close to unity are not excluded (i.e. given our earlier assumption of $h_\alpha \in (0,1]$), with the impact of roots close

to zero (i.e. effect of g_α) being virtually non-existent.[10] The parameters of the Beta density for α_n are not as important as the other heterogeneity parameters λ and $h_\bullet$, once these are accounted for (which is not the case in the time series literature). Some technical aspects of this phenomenon are discussed after (31). This is another manifestation of the importance of scaling in nonlinear models; see the discussion in Remark 3 after (21).

4. What are the effects of ν, $g_\bullet$ and $h_\bullet$ on the persistence of the business cycle? Here, ν and h_β^{-1} have an important influence, but h_ω^{-1} does not. The effect of h_β^{-1} on the persistence of common shocks is similar to ν^λ as ν varies, which we shall analyse more fully in Remarks 6-8 below. Other parameters have a lesser effect on memory, if at all.

5. We have talked about the effect of parameters in our setting of heterogeneous firms. Now we need to compare our theorem's result with representative-firm models. In the latter, when a firm is hit by dynamically-stable technology shocks, we can use (20). When it faces a geometric random walk (unit root)

$$\log x_t = \log x_{t-1} + \varepsilon_t, \qquad \{\varepsilon_t\}_{t=1}^{\infty} \sim \mathrm{IN}\left(0, \sigma^2\right),$$

the procedure leading to (20) and given in the first part of the Appendix would yield

$$\begin{aligned} v_{t,t+\tau}^{(1)} &= \exp\left[\frac{\nu^2\sigma^2}{2}(2t+\tau)\right]\left(\exp\left[\nu^2\sigma^2 t\right] - 1\right), \\ r_{t,t+\tau}^{(1)} &= \sqrt{\frac{\exp\left[\nu^2\sigma^2 t\right] - 1}{\exp\left[\nu^2\sigma^2\left(t+\tau\right)\right] - 1}}, \end{aligned}$$

where the superscript $\bullet^{(1)}$ refers to the unit root case, and lowercase $v_\bullet$ and $r_\bullet$ refer to the case of a single AR series (no aggregation). Note the linearity of the exponentials in ν^2 and t, and the absence of a $t+\tau$ term in the numerator's exponential. Exponential terms appear, because we are dealing with a geometric random walk instead of an arithmetic one. We are now able to compare our theorem to representative-firm models.

6. As $t \to \infty$ or $\nu \to \infty$, the binomial expansion gives

$$\sqrt{1+\frac{\tau}{t}} \simeq 1 + \frac{\tau}{2t} - \frac{\tau^2}{8t^2},$$

[10] Unlike the result for h_α, the one about the ineffectual parameter g_α is usual in the time series literature. It has allowed the generalization of the density of α_n to a semiparametric specification to be introduced with no additional complications; see for example Lippi and Zaffaroni (2000).

so that

$$
\begin{aligned}
R_{t,t+\tau} &\simeq \exp\left[-\frac{\nu^4\left(t+\frac{5\tau}{4}\right)\tau}{4\left(4h_\beta^2\right)^{\frac{2}{\lambda}}}\right] \\
r_{t,t+\tau}^{(1)} &\simeq \exp\left[-\frac{\nu^2\sigma^2}{2}\tau\right],
\end{aligned}
$$

where the decay rate of our theorem's $R_{t,t+\tau}$ is faster than the one for the unit-root process, $r_{t,t+\tau}^{(1)}$, as either t, τ or ν increase.[11]

7. As $\tau \to \infty$, one may analyse the memory features of the process as we consider points that are further apart in time, and we have

$$
\begin{aligned}
R_{t,t+\tau} &\simeq \left(\exp\left[\frac{\nu^4 t^2}{2\left(4h_\beta^2\right)^{\frac{2}{\lambda}}}\right]-1\right)^{-\frac{1}{2}} \exp\left[-\frac{\nu^4\left(\left(t+\tau\right)^2-2\sqrt{t^3\left(t+\tau\right)}\right)}{4\left(4h_\beta^2\right)^{\frac{2}{\lambda}}}\right] \\
r_{t,t+\tau}^{(1)} &\simeq \sqrt{1-\exp\left[-\nu^2\sigma^2 t\right]}\exp\left[-\frac{\nu^2\sigma^2}{2}\tau\right]
\end{aligned}
$$

where the decay rate of $R_{t,t+\tau}$ in terms of τ is again faster than for $r_{t,t+\tau}^{(1)}$.

8. As $\nu \to 0$, the rate of decay of the memory of $\{X_t\}$ is slower than the corresponding α_n^τ of the stable ($|\alpha_n| < 1$) AR of (20), whatever measure of autocorrelation is adopted, as we have

$$
\begin{aligned}
R_{t,t+\tau} &\simeq \sqrt{\frac{t}{t+\tau}} \\
r_{t,t+\tau}^{(1)} &\simeq \sqrt{\frac{t}{t+\tau}}.
\end{aligned}
$$

The result for $R_{t,t+\tau}$ is striking: we recover the behaviour of the random walk for our process, even though there are no unit roots in *any* of the components of the aggregate X_t. The resulting effect is that our model will generate *seemingly* random-walk behaviour as $\nu \to 0$, though the model has generally less memory than a random walk. This may help interpret the findings of a near-unit-root in some macroeconomic series when simple (log-)linear models are fitted to the data.

[11] Direct comparison of the nonstationary's $R_{t,t+\tau}$ with the stationary's $r_{t,t+\tau}$ (not the unit-root's $r_{t,t+\tau}^{(1)}$) of (20) for extreme parameter values would require writing $|\alpha|^\tau = \exp\left[\tau \log|\alpha|\right]$ then doing some manipulations. Long-memory processes like $\{X_t\} \equiv \{N^{-1}\sum_{n=1}^{N}\theta_{n,t}^\nu\}$ will have more memory than its stable components $\left\{\theta_{n,t}^\nu\right\}$, and looking at autocovariances (amplitudes) or $V_{t,t+\tau}/V_{t,t}$ would be more instructive if comparing processes with different memory characteristics.

9. The parameter $\nu \equiv \rho/(1-\rho) \in \mathbb{R}_+$ is the inverse of the mark-up, and is inversely related to complementarity and to monopoly power. The case $\nu \to 0$ where complementarity and/or monopoly power increase leads to the aggregate series having very long memory, to the point of being confounded with a unit root (Remark 8). This contrasts with perfect substitutability and/or competition where $\nu \to \infty$ (Remark 6) leading to less memory in the aggregate series, and a clear distinction from the permanent memory of a unit root. Unlike (log-)linear models, our nonlinear model can generate both types of behaviour, depending on ν, and provides a rich framework for analysing economies with different characteristics.

10. The interpretation of the special case of extreme complementarity and/or monopoly power is worth exploring in depth. When $\nu \to 0$, CES aggregation collapses to Cobb-Douglas geometric averaging, as studied in the time series literature albeit in a less general context which typically omits dependent (economy-wide) shocks there. Granger (1980) seminal paper being an exception to this omission of the dependent case, why are his results different (e.g. compared to Remarks 3 and 8 here)? In a paragraph starting at the end of his p.235, Granger (1980) talks about approximating the spectrum of the aggregate of stochastically dependent AR series in terms of the spectra of *independent* series, dropping the interaction terms as a simplifying approximation. Whereas we approximate the target autocovariance of $\{X_t\}$ directly, he approximates the aggregate (X_t in our context) by its mean, then uses this for a further approximation of the spectrum (hence ACF). The ACF of $\{X_t\}$ and the ACF of its mean are very different. The latter disregards interactions over time between the terms in the components of the aggregate X_t, and the heterogeneity of the β's is unfortunately cancelled out by the approximation of X_t by its mean: only $\mathrm{E}[\beta_n]$ (but not $\mathrm{Var}(\beta_n)$ etc.) remains in Granger's approximation. The difference between the two approaches is highlighted by the more recent warnings of the econometric literature on functional central limit theorems, where correlations build up instead of cancelling out. It also explains why we end up with unit-root like behaviour as $\nu \to 0$, which is a stronger I(1) form of memory than Granger's $\mathrm{I}(1-q/2)$, $q > 0$, in his notation (his d_W and d_y are 0 here), this stronger memory being due to the interaction of the dependent terms.

11. The autocovariance function $V_{t,t+\tau}$ is not level-invariant; that is, it naturally depends on μ. However, the leading term of the autocorrelation function $R_{t,t+\tau}$ in our theorem does not depend on μ.

We now need a result linking the memory of $\{X_t\} \equiv \{N^{-1}\sum_{n=1}^{N}\theta_{n,t}^{\nu}\}$ to the required one for the per capita GDP $\{y_t\}$, by means of $X_t \equiv \theta_t^{\nu}$ and (18).

Corollary 1 *The leading terms of the autocorrelation functions of* $\{\log X_t\}$ *and* $\{\log y_t\}$ *coincide.*

The Cobb-Douglas parameter γ has almost no impact on the memory of y_t, as it is swamped by the long-memory impact of aggregation, and all the previous remarks following our theorem on $\{X_t\}$ apply to $\{y_t\}$ too. The series $\{X_t\}$ can be thought of as a nonlinear "common stochastic trend" which drives the series of interest; e.g. GDP per capita $\{y_t\}$.

We have given the autocorrelation function of $\{X_t\}$ in our theorem. For large t or τ, it is possible to obtain an explicit expression for the ACF of $\{\log X_t\}$. This is done by the method in Abadir and Talmain (1998). Here, it implies doing a large-h_β, h_ω (i.e. small-variance) expansion, which downplays higher order terms of the exponential ($\log^{-1}$) expansion. Applied to either of the preceding Remarks 6 or 7, the result is a slow decay of memory. For a given ν, the main interest is in the formula of Remark 7, which yields the memory features of the process as we consider points that are further apart in time; i.e. large τ. For $\{\log y_t\}$, letting a denote a positive arbitrary constant which is not a function of $h_\bullet$, we get

$$R_{t,t+\tau} \simeq \frac{\sqrt{2}\left(4h_\beta^2\right)^{\frac{1}{\lambda}}}{\nu^2 t\left(1+\left(4h_\beta^2\right)^{-\frac{2}{\lambda}} a\tau^2\right)^{\frac{\nu^4}{4a}}} \tag{25}$$

which shows that memory increases with large h_β (low sectoral heterogeneity, i.e. little diversity) and small ν (large complementarity and/or monopoly power). In our setting, the effect of h_α is not as powerful as in the available time series models of linear aggregation of arithmetic ARs. Nonlinear aggregation of dependent series downplays the effect of h_α. In contrast, the effects of ν and h_β are much more important.

The assumption, in Section 2, of a 100% depreciation rate is unrealistic but it allows us to have a closed form solution for the dynamics of $\{y_t\}$. How robust are the derived time series properties to this assumption? Let $d \in (0,1]$ be the depreciation rate of the capital stock in the economy, and assume that capital accumulation occurs according to a Lucas-Prescott (1971) type of specification (cf Abadir and Talmain (2001)) as

$$k_t = k_{t-1}^{1-d}\left(s_{t-1}y_{t-1}\right)^d, \tag{26}$$

e.g. due to adjustment costs, and $(s_{t-1}y_{t-1})$ is investment in physical capital for the next period. Notice that we have not solved for the process $\{s_t\}$ explicitly in this equation. However, this turns out to be inessential, as the following result shows.

Corollary 2 *The leading term of the autocorrelation function of* $\{\log y_t\}$ *is invariant to* $d \in (0,1]$*, when* $s_t \in [s_\ell, 1]$ *where* $s_\ell > 0$.

The main effect of depreciation on the autocovariance function of $\{\log y_t\}$ is to scale down its amplitude by a factor of d^2. As for correlations, their invariance to scaling implies that the leading term of the autocorrelation function is unchanged by the value of d, and our previous remarks about the ACF go through without modification.

3.3 Further properties and empirical implications

Our paper has characterized the time series process that arises from the economic model, and has explored its properties and implications. However, it has not derived the optimal (if any) identification, estimation and inference procedures for dealing with its parameters, as this goes beyond the scope of this paper. For an illustration of some of the problems that can arise and the magnitude of the task, see Stoker (1984) and Trivedi (1985). In the remainder of this section, we shall presume that the parameters of the densities $f_\bullet(\bullet)$ have been identified by estimation from the micro data; cf. Figures 1-2. As it turned out in the theorem, only a subset of these parameters really matters for the dynamics of GDP per capita, $\{y_t\}$. We now outline a simple procedure to summarize the time series behaviour of $\{\log y_t\}$ which does *not* involve estimating the parameters of the underlying economy. The procedure fits a simplified autocorrelation function derived from our theorem and corollary, thus describing how $\{\log y_t\}$ evolves over time without having to identify the underlying parameters.

The procedure is as follows. Normalize the scaling factor in (25) such that $R_{t,t} = 1$, and estimate from the data by (say) nonlinear least squares the resulting autocorrelation function

$$\left(1 + \left(4h_\beta^2\right)^{-\frac{2}{\lambda}} a\tau^2\right)^{-\frac{\nu^4}{4a}}.$$

The formula of $V_{t,t+\tau}$ in our theorem also suggests that the other measure of autocorrelation, $V_{t,t+\tau}/V_{t,t}$, will decay at the rate

$$\left(1 + \left(\frac{1}{h_\beta} + \frac{1}{h_\omega}\right) a \left(\frac{\tau}{2}\right)^{\frac{\lambda}{2}}\right)^{-\frac{\nu\lambda}{4a}}$$

when $\lambda > 4$. One may nest both functional forms into the autocorrelation function

$$\left(1 + b_1 |\tau|^{b_2}\right)^{-b_3} \tag{27}$$

for $\tau \in \mathbb{Z}$, and where b_1, b_2, b_3 are all positive. Figure 3 illustrates its rate of decay compared to a fractionally-integrated I(1/3) series, AR(1) processes with a root very close to 1, and an AR(2) with *both* roots very close to 1. Not only is our aggregate process more persistent than these ARs (as demonstrated also algebraically in the remarks following the theorem), but it also has a hyperbolically-decaying S-shape which is very different from the exponential decay implied by dynamically-stable AR models and from the immediate hyperbolic decay of fractionally integrated processes. More generally, relative to linear (e.g. ARIMA) models, the ACF of our aggregate process implies a slower decay of memory initially, followed by a steep drop.

Why could ARIMA models not reproduce this S-shape? The AR contributes exponentially-decaying components, while every MA lag contributes a single impulse (jump) in the ACF. The Integrated component provides a hyperbolic decay when fractional orders are allowed; but, again, no concave parts that could look anything like our ACF's S-shape. See also the figures in Granger and Newbold (1986, pp. 16, 20, 28). The only way an ARIMA(a_1, a_2, a_3) could generate an S-shape is by having a_3 equal to a very large fraction of the data: every point of concavity of the ACF curve near the origin (about 1/3 of the 'data' in Figure 3) would be reproduced by as many MA lags. Such an unparsimonious model would not satisfy any Information Criterion (e.g. AIC or BIC) anyway. Furthermore, standard RBC models have indicated that $\{\log y_t\}$ would follow autoregressions of at most order 2 (implying either the permanent memory of a unit root, or exponential decays of ACFs as in our Figure 3), but not general ARIMA; e.g. see McCallum (1989, pp. 23, 43).

As an example to illustrate how our functional form (27) fits the data compared to AR models, we use UK and US data on the logarithm of GDP per capita. The data are annual 1948-99 and 1927-98, respectively, and they give us estimates of the autocorrelations over 1958-99 and 1937-98, respectively. Estimating autocorrelations for earlier periods within this dataset would have given unreliable results, since means and covariances would have been estimated with less than 10 observations. Using the nonlinear least squares routine of SPSS v.10, we estimate our (27) and compare it to the estimated autocorrelation of an AR(2). Figures 4 and 5 present the results. For the AR(2), the estimated roots are 0.999 and -0.058 for the UK, 0.995 and -0.037 for the US. The R^2 based on mean-corrected sums of squares from these nonlinear regressions are 0.56 and 0.50, respectively. The fit is worse than the one based on our (27) whose corrected R^2 are 0.82 and 0.83, respectively. Furthermore, our functional form gives a shape that is much closer to the observed data than the one implied by the AR. The latter implies a functional form which is almost linear, given how close the estimates are to a unit root.

An important point arises again from this limited empirical exercise. If the data were generated by our nonlinear economic process, erroneously fitting an AR model would give near unit roots such that the fitted ACF is almost linear and as close as possible to the data's true S-shaped ACF. This is another reason, in addition to Remark 8 above (following Theorem 1), why unit roots may seem excessively common in macroeconomic datasets.

Another important and related point comes out of comparing AR models of the same dataset, when estimated in two different ways: nonlinear fitting of the implied ACF (as above) versus linear estimation in the time domain. The latter estimated roots become 0.999 and +0.058 for the UK, and 0.974 and 0.633 for the US. The change in the latter estimate is particularly striking, and reveals the extent to which AR models are inadequate (misspecified) for the US data: fitting ARs in the time domain gives very different answers from fitting them in the ACF domain.

Finally, after our theorem we have mentioned a possible refinement to our functions, by taking the smaller-order terms from the series expansion. These terms are oscillating, and could help represent some small cyclical deviations around our S-shaped curve. This could add some improvement to the fit of our functions but is beyond the main purpose of our paper.

4 Concluding remarks

Our findings can be summarized as follows:

1. We provided a fully specified microeconomically-founded model and derive explicitly its fundamental equilibrium. This step enabled us to study analytically the time-series property of the solution. The economic model gives rise to long-memory, and to a nonlinear process. A single-sector model with one AR would have had exponential decay of memory and would not have given rise to long memory (except for a unit root, which leads to infinite memory). Aggregation means that linear ARIMA macromodels will not pick up the nonlinearity. We have shown how the autocorrelation function of log-linear models would be much less affected by the variance of the shocks than our nonlinear model, and how different the time series properties of these two models are. This means that the usual assumption of linearity is not innocuous, not even as a first-order approximation. We have illustrated how the slowly-decaying S-shaped autocorrelation function implied by our economic model turns out to be close to the one observed from the data, but very different from the one implied by standard linear models.

2. Persistence, endogenous cycles and overreaction: small temporary shocks in our model lead to long memory, without requiring unit roots. There is no need for large shocks at the microeconomic level in order to generate large macroeconomic fluctuations. Both the common and idiosyncratic shocks matter and are magnified exponentially at the macro level, though the former is more potent than the latter. Their size is positively related to the amplitude of the aggregate shock, but negatively related to the aggregate memory (temporary large shocks are more frequent, hence more easily

reversed). The decay of memory is however much slower than the usual exponential rate for stable ARs.

3. The effect of a higher degree of complementarity across sectors and/or monopolistic power in the economy is longer memory, but the amplitude of the cycle is reduced. With higher monopoly power, firms do not adjust their output so much after shocks, so there are less fluctuations but more persistence. This complements the findings of Blundell, Griffith and Van Reenen (1993). Furthermore, as monopoly power increases, our nonlinear model will generate behaviour that *seems* increasingly like a random walk, even though there are no unit roots in any of the components of the aggregate, and though the model has generally less memory than a random walk.

4. There is eventual mean-reversion in the cycles generated by our model, unlike in infinite-memory unit-root models. The length of the cycle is random, and the process is not periodic.

5. Mean-reversion, coupled with long memory, implies that an economy can get 'stuck' in a mode for a while, and have asymmetric business cycles for that duration (e.g. post-war expansion, subsequent recessions). Mean-reversion acts as an attractor: there is a slow tendency to the long-term trend, with occasional bursts away from it.

6. An implication of our result is that the individual components of GDP will *not* be co-moving with aggregate GDP, because of the latter's long-memory property. This analytical observation is supported by the empirical findings of Gregoir and Lenglart (1999).

7. The aggregation over heterogeneous units does not reduce variability: the standard intuition from the Laws of Large Numbers and the Central Limit Theorems (CLTs) would be misleading in our time-series context, and one should think of Functional CLTs (FCLTs) instead. In the latter, correlations build up over time, instead of cancelling out over different units. The autocorrelation of sums of variates is greater than the autocorrelation of the components, thus producing long memory. Intuitively, this arises because of the increased likelihood of correlation of one of the components with another (possibly different) component at a later date.

APPENDIX

Derivation of (20). The variate $x_t \equiv \theta_{n,t}$ of (19) can be written as

$$x_t = \exp\left[\sum_{j=0}^{t-1} \alpha^j \varepsilon_{t-j}\right],$$

where we use as shorthand α for α_n, and $\varepsilon_t \sim \mathrm{IN}(\mu, \sigma^2)$ for $\mu + \varepsilon_{n,t} \sim \mathrm{IN}(\mu, \sigma_n^2)$, in the first proof given in this Appendix. Then,

$$\mathrm{E}\left[(x_t x_{t+\tau})^\nu\right] = \mathrm{E}\left[\exp\left[\nu\left((1+\alpha^\tau)\sum_{j=0}^{t-1}\alpha^j\varepsilon_{t-j} + \sum_{j=0}^{\tau-1}\alpha^j\varepsilon_{t+\tau-j}\right)\right]\right],$$

where empty sums are taken to be zero, by convention. By the independence of the sequence $\{\varepsilon_t\}$, and by the moment generating function (MGF) of the distribution $\mathrm{N}(\mu, \sigma^2)$

$$\mathrm{E}\left[\exp\left[b\varepsilon_t\right]\right] = \exp\left[b\mu + \frac{b^2\sigma^2}{2}\right],$$

we get

$$\begin{aligned}
&\mathrm{E}\left[(x_t x_{t+\tau})^\nu\right] \\
= \; & \exp\left[\nu\mu\left((1+\alpha^\tau)\sum_{j=0}^{t-1}\alpha^j + \sum_{j=0}^{\tau-1}\alpha^j\right) + \frac{\nu^2\sigma^2}{2}\left((1+\alpha^\tau)^2\sum_{j=0}^{t-1}\alpha^{2j} + \sum_{j=0}^{\tau-1}\alpha^{2j}\right)\right] \\
= \; & \exp\left[\nu\mu\frac{(1+\alpha^\tau)(1-\alpha^t) + 1 - \alpha^\tau}{1-\alpha} + \frac{\nu^2\sigma^2}{2}\frac{(1+\alpha^\tau)^2(1-\alpha^{2t}) + 1 - \alpha^{2\tau}}{1-\alpha^2}\right] \\
= \; & \exp\left[\frac{\nu\mu\left(2-\alpha^t(1+\alpha^\tau)\right)}{1-\alpha} + \frac{\nu^2\sigma^2(1+\alpha^\tau)\left(2-\alpha^{2t}(1+\alpha^\tau)\right)}{2(1-\alpha^2)}\right].
\end{aligned}$$

Finally, by a similar method,

$$\begin{aligned}
\mathrm{E}\left[x_t^\nu\right] &= \mathrm{E}\left[\exp\left[\nu\sum_{j=0}^{t-1}\alpha^j\varepsilon_{t-j}\right]\right] \\
&= \exp\left[\nu\mu\sum_{j=0}^{t-1}\alpha^j + \frac{\nu^2\sigma^2}{2}\sum_{j=0}^{t-1}\alpha^{2j}\right] \\
&= \exp\left[\frac{\nu\mu(1-\alpha^t)}{1-\alpha} + \frac{\nu^2\sigma^2(1-\alpha^{2t})}{2(1-\alpha^2)}\right],
\end{aligned}$$

so that the autocovariance function of x_t^ν is

$$\begin{aligned} v_{t,t+\tau} &\equiv \mathrm{E}\left[(x_t x_{t+\tau})^\nu\right] - \mathrm{E}\left[x_t^\nu\right]\mathrm{E}\left[x_{t+\tau}^\nu\right] \\ &= \exp\left[\frac{\nu\mu\left(2-\alpha^t\left(1+\alpha^\tau\right)\right)}{1-\alpha} + \frac{\nu^2\sigma^2\left(1+\alpha^\tau\right)\left(2-\alpha^{2t}\left(1+\alpha^\tau\right)\right)}{2\left(1-\alpha^2\right)}\right] \\ &\quad - \exp\left[\frac{\nu\mu\left(2-\alpha^t\left(1+\alpha^\tau\right)\right)}{1-\alpha} + \frac{\nu^2\sigma^2\left(2-\alpha^{2t}\left(1+\alpha^{2\tau}\right)\right)}{2\left(1-\alpha^2\right)}\right] \\ &= \exp\left[\frac{\nu\mu\left(2-\alpha^t\left(1+\alpha^\tau\right)\right)}{1-\alpha} + \frac{\nu^2\sigma^2\left(2-\alpha^{2t}\left(1+\alpha^{2\tau}\right)\right)}{2\left(1-\alpha^2\right)}\right] \\ &\quad \times \left(\exp\left[\frac{\nu^2\sigma^2\alpha^\tau\left(1-\alpha^{2t}\right)}{1-\alpha^2}\right] - 1\right) \end{aligned}$$

and its autocorrelation function is as stated in (20). ||

The following lemma is required for the proof of our theorem.

Lemma 1 *For ξ a Generalized-Gamma variate with density function*

$$\frac{\lambda h_\xi^{g_\xi}\xi^{\lambda g_\xi - 1}}{\Gamma\left(g_\xi\right)}\exp\left[-h_\xi\xi^\lambda\right]$$

where $\lambda \in (2,\infty)$ and $g_\xi \in \mathbb{R}_+$, we have

$$E\left[\xi^\zeta \exp\left(b\xi^2\right)\right] \simeq \frac{2^{\frac{3}{2}-g_\xi}\sqrt{\pi}}{\Gamma\left(g_\xi\right)b^{\frac{\zeta}{2}}}\left(\frac{b^{\frac{\lambda}{2}}}{2h_\xi}\right)^{\frac{2\zeta}{\lambda}+g_\xi-\frac{1}{2}}\exp\left(\frac{b^2}{\left(2h_\xi\right)^{\frac{4}{\lambda}}} - \frac{b^{\frac{\lambda}{2}}}{4h_\xi}\right)$$

for $\zeta + \lambda g_\xi \in \mathbb{R}_+$ and large $b \in \mathbb{R}_+$.

Proof. By a change of variable followed by expanding and taking the leading term for large $b \in \mathbb{R}_+$ [e.g. see Abadir's (1999) fractional Hermite polynomials],

$$\begin{aligned}
& \mathrm{E}\left[\xi^{\zeta} \exp\left(b\xi^2\right)\right] \\
\equiv\; & \frac{\lambda h_\xi^{g_\xi}}{\Gamma(g_\xi)} \int_0^\infty \xi^{\zeta+\lambda g_\xi - 1} \exp\left(b\xi^2 - h_\xi \xi^\lambda\right) \mathrm{d}\xi \\
=\; & \frac{h_\xi^{g_\xi}\left(b^{-\frac{\lambda}{2}}\right)^{\frac{\zeta}{\lambda}+g_\xi}}{\Gamma(g_\xi)} \int_0^\infty \xi^{\frac{\zeta}{\lambda}+g_\xi-1} \exp\left(\xi^{\frac{2}{\lambda}} - \frac{h_\xi}{b^{\frac{\lambda}{2}}}\xi\right) \mathrm{d}\xi \\
=\; & \frac{h_\xi^{g_\xi}\left(b^{-\frac{\lambda}{2}}\right)^{\frac{\zeta}{\lambda}+g_\xi}}{\Gamma(g_\xi)} \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{\ell=0}^{j} \binom{j}{\ell} (-1)^\ell \int_0^\infty \xi^{\frac{\zeta}{\lambda}+g_\xi-1+\frac{2}{\lambda}(j-\ell)+\frac{1}{2}\ell} \exp\left(\sqrt{\xi} - \frac{h_\xi}{b^{\frac{\lambda}{2}}}\xi\right) \mathrm{d}\xi \\
\simeq\; & \frac{2\sqrt{2\pi} h_\xi^{g_\xi}\left(b^{-\frac{\lambda}{2}}\right)^{\frac{\zeta}{\lambda}+g_\xi}}{\Gamma(g_\xi)} \exp\left(\frac{b^{\frac{\lambda}{2}}}{4h_\xi}\right) \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{\ell=0}^{j} \binom{j}{\ell} (-1)^\ell \left(2h_\xi b^{-\frac{\lambda}{2}}\right)^{-2\left(\frac{\zeta}{\lambda}+g_\xi+\frac{2}{\lambda}(j-\ell)+\frac{1}{2}\ell\right)+\frac{1}{2}} \\
=\; & \frac{2^{\frac{3}{2}-g_\xi}\sqrt{\pi}}{\Gamma(g_\xi)\, b^{\frac{\zeta}{2}}} \left(\frac{b^{\frac{\lambda}{2}}}{2h_\xi}\right)^{\frac{2\zeta}{\lambda}+g_\xi-\frac{1}{2}} \exp\left(\frac{b^2}{(2h_\xi)^{\frac{4}{\lambda}}} - \frac{b^{\frac{\lambda}{2}}}{4h_\xi}\right)
\end{aligned}$$

where $\binom{a}{\ell} \equiv \Gamma(a+1)/[\Gamma(a-\ell+1)\,\ell!]$ are the binomial coefficients. Incidentally, the derivations provide an asymptotic expansion for the Meijer G function which includes hypergeometric functions as a special case, and is of intrinsic mathematical interest. It is also of statistical interest because it can be reformulated to give the leading term of the MGF of a Generalized-Gamma variate (since ξ^2 is also another Generalized Gamma variate). $\|$

Proof of Theorem 1. It can be seen that the results displayed earlier in this Appendix go through for any $\theta_{n,t}$, with the parameter equivalencies

$$\begin{aligned}
\alpha &\leftrightarrow \alpha_n, \\
\{\varepsilon_t\} &\leftrightarrow \{u_{n,t} + \beta_n e_t\}, \\
\sigma^2 &\leftrightarrow \omega_n^2 + \beta_n^2,
\end{aligned}$$

so that

$$\begin{aligned}
\theta_{n,t} &= \exp\left[\sum_{j=0}^{t-1} \alpha_n^j \left(u_{n,t-j} + \beta_n e_{t-j}\right)\right], \\
\mathrm{E}_{\bullet|n}\left[(\theta_{n,t}\theta_{n,t+\tau})^\nu\right] &= \exp\left[\frac{\nu\mu\left(2-\alpha_n^t\left(1+\alpha_n^\tau\right)\right)}{1-\alpha_n} + \frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1+\alpha_n^\tau\right)\left(2-\alpha_n^{2t}\left(1+\alpha_n^\tau\right)\right)}{2\left(1-\alpha_n^2\right)}\right], \\
\mathrm{E}_{\bullet|n}\left[\theta_{n,t}^\nu\right] &= \exp\left[\frac{\nu\mu\left(1-\alpha_n^t\right)}{1-\alpha_n} + \frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1-\alpha_n^{2t}\right)}{2\left(1-\alpha_n^2\right)}\right].
\end{aligned}$$

The penultimate expression is readily generalizable to

$$
\begin{aligned}
\mathrm{E}_{\bullet|n}\left[(\theta_{n,t}\theta_{k,t+\tau})^{\nu}\right] &= \mathrm{E}_{\bullet|n}\left[\exp\left[\nu\sum_{j=0}^{t-1}\alpha_n^j\left(u_{n,t-j}+\beta_n e_{t-j}\right)+\nu\sum_{j=0}^{t+\tau-1}\alpha_k^j\left(u_{k,t+\tau-j}+\beta_k e_{t+\tau-j}\right)\right]\right] \\
&= \mathrm{E}_{\bullet|n}\left[\exp\left[\nu\sum_{j=0}^{t-1}\alpha_n^j u_{n,t-j}+\nu\sum_{j=0}^{t+\tau-1}\alpha_k^j u_{k,t+\tau-j}\right.\right. \\
&\qquad \left.\left.+\nu\sum_{j=0}^{t-1}\left(\beta_n\alpha_n^j+\beta_k\alpha_k^{j+\tau}\right)e_{t-j}+\nu\beta_k\sum_{j=0}^{\tau-1}\alpha_k^j e_{t+\tau-j}\right]\right] \\
&= \exp\left[\nu\mu\left(\sum_{j=0}^{t-1}\alpha_n^j+\sum_{j=0}^{t+\tau-1}\alpha_k^j\right)+\frac{\nu^2\omega_n^2}{2}\sum_{j=0}^{t-1}\alpha_n^{2j}+\frac{\nu^2\omega_k^2}{2}\sum_{j=0}^{t+\tau-1}\alpha_k^{2j}\right. \\
&\qquad \left.+\frac{\nu^2}{2}\sum_{j=0}^{t-1}\left(\beta_n\alpha_n^j+\beta_k\alpha_k^{j+\tau}\right)^2+\frac{\nu^2\beta_k^2}{2}\sum_{j=0}^{\tau-1}\alpha_k^{2j}\right] \\
&= \exp\left[\nu\mu\left(\frac{1-\alpha_n^t}{1-\alpha_n}+\frac{1-\alpha_k^{t+\tau}}{1-\alpha_k}\right)+\frac{\nu^2\omega_n^2\left(1-\alpha_n^{2t}\right)}{2\left(1-\alpha_n^2\right)}+\frac{\nu^2\omega_k^2\left(1-\alpha_k^{2t+2\tau}\right)}{2\left(1-\alpha_k^2\right)}\right. \\
&\qquad +\frac{\nu^2}{2}\left(\frac{\beta_n^2\left(1-\alpha_n^{2t}\right)}{1-\alpha_n^2}+\frac{\beta_k^2\alpha_k^{2\tau}\left(1-\alpha_k^{2t}\right)}{1-\alpha_k^2}+\frac{2\beta_n\beta_k\alpha_k^{\tau}\left(1-\alpha_n^t\alpha_k^t\right)}{1-\alpha_n\alpha_k}\right) \\
&\qquad \left.+\frac{\nu^2\beta_k^2\left(1-\alpha_k^{2\tau}\right)}{2\left(1-\alpha_k^2\right)}\right] \\
&= \exp\left[\nu\mu\left(\frac{1-\alpha_n^t}{1-\alpha_n}+\frac{1-\alpha_k^{t+\tau}}{1-\alpha_k}\right)+\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1-\alpha_n^{2t}\right)}{2\left(1-\alpha_n^2\right)}\right. \\
&\qquad \left.+\frac{\nu^2\left(\omega_k^2+\beta_k^2\right)\left(1-\alpha_k^{2t+2\tau}\right)}{2\left(1-\alpha_k^2\right)}+\frac{\nu^2\beta_n\beta_k\alpha_k^{\tau}\left(1-\alpha_n^t\alpha_k^t\right)}{1-\alpha_n\alpha_k}\right],
\end{aligned}
$$

when $k \neq n$, and where it is understood that the conditioning of the expectation is with respect to both n, k defined on the same space $\{1, 2, \ldots N\}$ and generically referred to by $\mathrm{E}_{\bullet|n}$.

We need

$$\begin{aligned}
& \mathrm{E}_{\bullet|n}\left[X_t X_{t+\tau}\right] \\
= & \mathrm{E}_{\bullet|n}\left[\left(\frac{1}{N}\sum_{n=1}^{N}\theta^{\nu}_{n,t}\right)\left(\frac{1}{N}\sum_{n=1}^{N}\theta^{\nu}_{n,t+\tau}\right)\right] \\
= & \frac{1}{N^2}\sum_{n=1}^{N}\mathrm{E}_{\bullet|n}\left[\theta^{\nu}_{n,t}\theta^{\nu}_{n,t+\tau}+\sum_{k\neq n}\theta^{\nu}_{n,t}\theta^{\nu}_{k,t+\tau}\right] \\
= & \frac{1}{N^2}\sum_{n=1}^{N}\left(\exp\left[\frac{\nu\mu\left(2-\alpha_n^t\left(1+\alpha_n^{\tau}\right)\right)}{1-\alpha_n}+\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1+\alpha_n^{\tau}\right)\left(2-\alpha_n^{2t}\left(1+\alpha_n^{\tau}\right)\right)}{2\left(1-\alpha_n^2\right)}\right]\right. \\
& \quad +\sum_{k\neq n}\exp\left[\nu\mu\left(\frac{1-\alpha_n^t}{1-\alpha_n}+\frac{1-\alpha_k^{t+\tau}}{1-\alpha_k}\right)+\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1-\alpha_n^{2t}\right)}{2\left(1-\alpha_n^2\right)}\right. \\
& \qquad \left.\left.+\frac{\nu^2\left(\omega_k^2+\beta_k^2\right)\left(1-\alpha_k^{2t+2\tau}\right)}{2\left(1-\alpha_k^2\right)}+\frac{\nu^2\beta_n\beta_k\alpha_k^{\tau}\left(1-\alpha_n^t\alpha_k^t\right)}{1-\alpha_n\alpha_k}\right]\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}_{\bullet|n}\left[X_t\right] &= \frac{1}{N}\sum_{n=1}^{N}\mathrm{E}_{\bullet|n}\left[\theta^{\nu}_{n,t}\right] \\
&= \frac{1}{N}\sum_{n=1}^{N}\exp\left[\frac{\nu\mu\left(1-\alpha_n^t\right)}{1-\alpha_n}+\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1-\alpha_n^{2t}\right)}{2\left(1-\alpha_n^2\right)}\right]
\end{aligned}$$

for the autocovariance function of $\{X_t\}$ to be derived as

$$V_{t,t+\tau}\equiv\mathrm{E}\left[X_t X_{t+\tau}\right]-\mathrm{E}\left[X_t\right]\mathrm{E}\left[X_{t+\tau}\right]. \tag{28}$$

For large N, the operators

$$\frac{1}{N}\sum_{n=1}^{N}\simeq\mathrm{E}_n$$

are exchangeable (by the Law of Large Numbers), with E_n denoting the expectation taken with respect to the distribution of parameters of the individual series subscripted by n (or k). By the law of iterated expectations, $\mathrm{E}_n\left[\mathrm{E}_{t|n}\left[\bullet\right]\right]=$

$E_{n,t}[\bullet] \equiv E[\bullet]$, the latter being the required expectation for (28), and we have

$$
\begin{aligned}
& V_{t,t+\tau} && (29)\\
\simeq\; & \frac{1}{N} E_n \exp\left[\frac{\nu\mu\left(2-\alpha_n^t\left(1+\alpha_n^\tau\right)\right)}{1-\alpha_n} \right.\\
& \qquad \left. + \frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1+\alpha_n^\tau\right)\left(2-\alpha_n^{2t}\left(1+\alpha_n^\tau\right)\right)}{2\left(1-\alpha_n^2\right)}\right]\\
& + E_n E_k \exp\left[\nu\mu\left(\frac{1-\alpha_n^t}{1-\alpha_n}+\frac{1-\alpha_k^{t+\tau}}{1-\alpha_k}\right)+\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1-\alpha_n^{2t}\right)}{2\left(1-\alpha_n^2\right)}\right.\\
& \qquad \left. + \frac{\nu^2\left(\omega_k^2+\beta_k^2\right)\left(1-\alpha_k^{2t+2\tau}\right)}{2\left(1-\alpha_k^2\right)} + \frac{\nu^2\beta_n\beta_k\alpha_k^\tau\left(1-\alpha_n^t\alpha_k^t\right)}{1-\alpha_n\alpha_k}\right]\\
& - E_n \exp\left[\frac{\nu\mu\left(1-\alpha_n^t\right)}{1-\alpha_n}+\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1-\alpha_n^{2t}\right)}{2\left(1-\alpha_n^2\right)}\right]\\
& \quad E_n \exp\left[\frac{\nu\mu\left(1-\alpha_n^{t+\tau}\right)}{1-\alpha_n}+\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1-\alpha_n^{2t+2\tau}\right)}{2\left(1-\alpha_n^2\right)}\right]\\
\simeq\; & E_n\left[\exp\left(\frac{\nu\mu\left(1-\alpha_n^t\right)}{1-\alpha_n}+\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1-\alpha_n^{2t}\right)}{2\left(1-\alpha_n^2\right)}\right) E_k\left[\exp\left(\frac{\nu\mu\left(1-\alpha_k^{t+\tau}\right)}{1-\alpha_k}\right.\right.\right.\\
& \qquad \left.\left.\left. + \frac{\nu^2\left(\omega_k^2+\beta_k^2\right)\left(1-\alpha_k^{2t+2\tau}\right)}{2\left(1-\alpha_k^2\right)} + \frac{\nu^2\beta_n\beta_k\alpha_k^\tau\left(1-\alpha_n^t\alpha_k^t\right)}{1-\alpha_n\alpha_k}\right)\right]\right]\\
& - E_n\left[\exp\left(\frac{\nu\mu\left(1-\alpha_n^t\right)}{1-\alpha_n}+\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1-\alpha_n^{2t}\right)}{2\left(1-\alpha_n^2\right)}\right)\right]\\
& \quad E_n\left[\exp\left(\frac{\nu\mu\left(1-\alpha_n^{t+\tau}\right)}{1-\alpha_n}+\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)\left(1-\alpha_n^{2t+2\tau}\right)}{2\left(1-\alpha_n^2\right)}\right)\right]\\
\equiv\; & \widetilde{V}_{t,t+\tau} \equiv G_{t,t+\tau} - H_t H_{t+\tau},
\end{aligned}
$$

for $\exists\beta_n \neq 0$. Notice that the drawings from the densities of α, ω, β are done for firm n independently from firm k, when $k \neq n$, so that the joint expectation can be written in terms of the iterated marginals as $E_{n,k}[\bullet] = E_n[E_k[\bullet]]$. It is seen in the resulting formula for $V_{t,t+\tau}$ that a non-trivial common stochastic shock ($\exists\beta_n \neq 0$) introduces a lot more persistence in the aggregate series than when all $\beta_n = 0$ (in which case $V_{t,t+\tau} \simeq 0$ above). For β's small relative to ω's, one may even get frequent negative autocovariances as opposed to long cyclical behaviour. The latter is what we need to analyse now.

Returning to formula (29) for $\widetilde{V}_{t,t+\tau}$, one may remark the following two aspects. First, by the mutual independence of the distributions of $\alpha_n, \omega_n, \beta_n$, one can substitute

$$\mathrm{E}_n[\bullet] = \mathrm{E}_{\omega_n}\left[\mathrm{E}_{\beta_n}[\mathrm{E}_{\alpha_n}[\bullet]]\right] = \mathrm{E}_{\omega_n,\beta_n}[\mathrm{E}_{\alpha_n}[\bullet]]$$

and similarly for $\mathrm{E}_k[\bullet]$. Second, the main value of the integrals (expectations) comes from $\alpha \to 1$, and one may exploit this property to solve an otherwise intractable multiple integral. Let us take one component at a time from $\widetilde{V}_{t,t+\tau}$ of (29), starting with the easiest (latter) one for expository purposes:

$$\begin{aligned}
H_t &\equiv \mathrm{E}_n\left[\exp\left(\frac{\nu\mu(1-\alpha_n^t)}{1-\alpha_n} + \frac{\nu^2\left(\omega_n^2+\beta_n^2\right)(1-\alpha_n^{2t})}{2(1-\alpha_n^2)}\right)\right] \\
&= \mathrm{E}_{\omega_n,\beta_n}\left[\int_0^1 \alpha^{g_\alpha-1}(1-\alpha)^{h_\alpha-1}\exp\left(\frac{\nu\mu(1-\alpha^t)}{1-\alpha} + \frac{\nu^2\left(\omega_n^2+\beta_n^2\right)(1-\alpha^{2t})}{2(1-\alpha^2)}\right)\frac{\mathrm{d}\alpha}{\mathrm{B}(g_\alpha,h_\alpha)}\right] \\
&\simeq \mathrm{E}_{\omega_n,\beta_n}\left[\int_0^1 \alpha^{g_\alpha-1}(1-\alpha)^{h_\alpha-1}\exp\left(\nu\mu t\alpha^{t-1} + \frac{\nu^2\left(\omega_n^2+\beta_n^2\right)t}{2}\alpha^{2(t-1)}\right)\frac{\mathrm{d}\alpha}{\mathrm{B}(g_\alpha,h_\alpha)}\right] \\
&= \int_0^1 \alpha^{g_\alpha-1}(1-\alpha)^{h_\alpha-1}\exp\left(\nu\mu t\alpha^{t-1}\right)\mathrm{E}_{\omega_n,\beta_n}\left[\exp\left(\frac{\nu^2\left(\omega_n^2+\beta_n^2\right)t}{2}\alpha^{2(t-1)}\right)\right]\frac{\mathrm{d}\alpha}{\mathrm{B}(g_\alpha,h_\alpha)}
\end{aligned}$$

where the approximation makes use of l'Hôpital's rule. With the help of our lemma, we can take the required expectations with respect to ω and β as

$$\begin{aligned}
H_t \simeq\ & \frac{4\pi}{\mathrm{B}(g_\alpha,h_\alpha)\Gamma(g_\beta)\Gamma(g_\omega)}\left(\frac{(\nu^2t/2)^{\frac{\lambda}{2}}}{4h_\beta}\right)^{g_\beta-\frac{1}{2}}\left(\frac{(\nu^2t/2)^{\frac{\lambda}{2}}}{4h_\omega}\right)^{g_\omega-\frac{1}{2}} \\
& \int_0^1 \alpha^{\lambda(t-1)\left(g_\omega+g_\beta-1\right)+g_\alpha-1}(1-\alpha)^{h_\alpha-1} \\
& \exp\left[\nu\mu t\alpha^{t-1} + \frac{\nu^4t^2}{4}\left(\frac{1}{(2h_\beta)^{\frac{4}{\lambda}}} + \frac{1}{(2h_\omega)^{\frac{4}{\lambda}}}\right)\alpha^{4(t-1)} - \left(\frac{\nu^2t}{2}\right)^{\frac{\lambda}{2}}\left(\frac{1}{4h_\beta}+\frac{1}{4h_\omega}\right)\alpha^{\lambda(t-1)}\right]\mathrm{d}\alpha.
\end{aligned}$$

By a change of variable replacing α by $\alpha^{1/(t-1)}$, then approximating for large t, we get

$$\begin{aligned}
H_t \;\simeq\; & \frac{4\pi}{\mathrm{B}\left(g_\alpha,h_\alpha\right)\Gamma\left(g_\beta\right)\Gamma\left(g_\omega\right)(t-1)}\left(\frac{\left(\nu^2t/2\right)^{\frac{\lambda}{2}}}{4h_\beta}\right)^{g_\beta-\frac{1}{2}}\left(\frac{\left(\nu^2t/2\right)^{\frac{\lambda}{2}}}{4h_\omega}\right)^{g_\omega-\frac{1}{2}} \\
& \int_0^1 \alpha^{\lambda\left(g_\omega+g_\beta-1\right)+\frac{g_\alpha}{t-1}-1}\left(1-\alpha^{\frac{1}{t-1}}\right)^{h_\alpha-1} \\
& \exp\left[\nu\mu t\alpha+\frac{\nu^4t^2}{4}\left(\frac{1}{\left(2h_\beta\right)^{\frac{4}{\lambda}}}+\frac{1}{\left(2h_\omega\right)^{\frac{4}{\lambda}}}\right)\alpha^4-\left(\frac{\nu^2t}{2}\right)^{\frac{\lambda}{2}}\left(\frac{1}{4h_\beta}+\frac{1}{4h_\omega}\right)\alpha^\lambda\right]\mathrm{d}\alpha \\
\simeq\; & \frac{4\pi}{\mathrm{B}\left(g_\alpha,h_\alpha\right)\Gamma\left(g_\beta\right)\Gamma\left(g_\omega\right)(t-1)^{h_\alpha}}\left(\frac{\left(\nu^2t/2\right)^{\frac{\lambda}{2}}}{4h_\beta}\right)^{g_\beta-\frac{1}{2}}\left(\frac{\left(\nu^2t/2\right)^{\frac{\lambda}{2}}}{4h_\omega}\right)^{g_\omega-\frac{1}{2}} \\
& \int_0^1 \alpha^{\lambda\left(g_\omega+g_\beta-1\right)-1}(1-\alpha)^{h_\alpha-1} \\
& \exp\left[\nu\mu t\alpha+\frac{\nu^4t^2}{4}\left(\frac{1}{\left(2h_\beta\right)^{\frac{4}{\lambda}}}+\frac{1}{\left(2h_\omega\right)^{\frac{4}{\lambda}}}\right)\alpha^4-\left(\frac{\nu^2t}{2}\right)^{\frac{\lambda}{2}}\left(\frac{1}{4h_\beta}+\frac{1}{4h_\omega}\right)\alpha^\lambda\right]\mathrm{d}\alpha \\
=\; & \frac{4\pi}{\mathrm{B}\left(g_\alpha,h_\alpha\right)\Gamma\left(g_\beta\right)\Gamma\left(g_\omega\right)(t-1)^{h_\alpha}}\left(\frac{\left(\nu^2t/2\right)^{\frac{\lambda}{2}}}{4h_\beta}\right)^{g_\beta-\frac{1}{2}}\left(\frac{\left(\nu^2t/2\right)^{\frac{\lambda}{2}}}{4h_\omega}\right)^{g_\omega-\frac{1}{2}} \\
& \sum_{j=0}^{\infty}\frac{\left(\nu^4t^2/4\right)^j}{j!}\left(\frac{1}{\left(2h_\beta\right)^{\frac{4}{\lambda}}}+\frac{1}{\left(2h_\omega\right)^{\frac{4}{\lambda}}}\right)^j\sum_{\ell=0}^{\infty}\frac{\left(-\left(\nu^2t/2\right)^{\frac{\lambda}{2}}\right)^\ell}{\ell!}\left(\frac{1}{4h_\beta}+\frac{1}{4h_\omega}\right)^\ell \\
& \int_0^1 \alpha^{4j+\lambda\left(\ell+g_\omega+g_\beta-1\right)-1}(1-\alpha)^{h_\alpha-1}\exp\left(\nu\mu t\alpha\right)\mathrm{d}\alpha,
\end{aligned}$$

the middle step having been obtained by

$$1-\alpha^{\frac{1}{t-1}}=1-\left(1-(1-\alpha)\right)^{\frac{1}{t-1}}\simeq\frac{1-\alpha}{t-1}.$$

Noting that

$$\int_0^1 \alpha^{b-1}(1-\alpha)^{h_\alpha-1}\exp\left(\nu\mu t\alpha\right)\mathrm{d}\alpha\simeq\frac{\Gamma\left(h_\alpha\right)}{\left(\nu\mu t\right)^{h_\alpha}}\exp\left(\nu\mu t\right) \qquad (30)$$

for large t [e.g. see Kummer's function in Abadir (1999)] with $\nu\mu \in \mathbb{R}_{+}$, we can write

$$\begin{aligned}
H_t &\simeq \frac{4\pi\Gamma(g_\alpha+h_\alpha)\exp(\nu\mu t)}{\Gamma(g_\alpha)\Gamma(g_\beta)\Gamma(g_\omega)(\nu\mu(t-1)t)^{h_\alpha}}\left(\frac{(\nu^2 t/2)^{\frac{\lambda}{2}}}{4h_\beta}\right)^{g_\beta-\frac{1}{2}}\left(\frac{(\nu^2 t/2)^{\frac{\lambda}{2}}}{4h_\omega}\right)^{g_\omega-\frac{1}{2}} \qquad (31)\\
&\quad \sum_{j=0}^{\infty}\frac{(\nu^4 t^2/4)^j}{j!}\left(\frac{1}{(2h_\beta)^{\frac{4}{\lambda}}}+\frac{1}{(2h_\omega)^{\frac{4}{\lambda}}}\right)^j \sum_{\ell=0}^{\infty}\frac{\left(-(\nu^2 t/2)^{\frac{\lambda}{2}}\right)^\ell}{\ell!}\left(\frac{1}{4h_\beta}+\frac{1}{4h_\omega}\right)^\ell\\
&= \frac{4\pi\Gamma(g_\alpha+h_\alpha)}{\Gamma(g_\alpha)\Gamma(g_\beta)\Gamma(g_\omega)(\nu\mu(t-1)t)^{h_\alpha}}\left(\frac{(\nu^2 t/2)^{\frac{\lambda}{2}}}{4h_\beta}\right)^{g_\beta-\frac{1}{2}}\left(\frac{(\nu^2 t/2)^{\frac{\lambda}{2}}}{4h_\omega}\right)^{g_\omega-\frac{1}{2}}\\
&\quad \exp\left[\nu\mu t+\frac{\nu^4 t^2}{2^{\frac{4}{\lambda}+2}}\left(\frac{1}{h_\beta^{4/\lambda}}+\frac{1}{h_\omega^{4/\lambda}}\right)-\left(\frac{\nu^2 t}{2}\right)^{\frac{\lambda}{2}}\left(\frac{1}{4h_\beta}+\frac{1}{4h_\omega}\right)\right].
\end{aligned}$$

As in Granger (1980), and in spite of the different setting here (geometric AR), we find that the Beta parameter b of (30) is relatively unimportant in determining the time-series features. This is no wonder, since AR roots near 0 have little impact on aggregate memory, while roots near unity are more critical in this respect. In terms of the original first Beta parameter, g_α acts just as a 'scaling' for H_t rather than an important parameter (e.g. power of t).

Going back to the required $\widetilde{V}_{t,t+\tau}$ of (29), we have now derived an approximation for H_t, and accordingly $H_{t+\tau}$. We need to do the same for $G_{t,t+\tau}$. We start with the same approximation of the integrals (expectations) near $\alpha = 1$ by l'Hôpital's rule, and

$$\begin{aligned}
&G_{t,t+\tau}\\
&\simeq \mathrm{E}_n\left[\exp\left(\nu\mu t\alpha_n^{t-1}+\frac{\nu^2 t\left(\omega_n^2+\beta_n^2\right)}{2}\alpha_n^{2(t-1)}\right)\right.\\
&\quad \left.\mathrm{E}_k\left[\exp\left(\nu\mu(t+\tau)\alpha_k^{t+\tau-1}+\frac{\nu^2(t+\tau)\left(\omega_k^2+\beta_k^2\right)}{2}\alpha_k^{2(t+\tau-1)}+\nu^2 t\beta_n\beta_k\alpha_k^{t+\tau-1}\alpha_n^{t-1}\right)\right]\right]\\
&= \frac{1}{(\mathrm{B}(g_\alpha,h_\alpha))^2}\mathrm{E}_{\omega_n,\beta_n}\left[\int_0^1 \alpha_n^{g_\alpha-1}(1-\alpha_n)^{h_\alpha-1}\exp\left(\nu\mu t\alpha_n^{t-1}+\frac{\nu^2 t\left(\omega_n^2+\beta_n^2\right)}{2}\alpha_n^{2(t-1)}\right)\right.\\
&\quad \mathrm{E}_{\omega_k,\beta_k}\left[\int_0^1 \alpha_k^{g_\alpha-1}(1-\alpha_k)^{h_\alpha-1}\right.\\
&\quad \left.\left.\exp\left(\nu\left(\mu(t+\tau)+\nu t\beta_n\beta_k\alpha_n^{t-1}\right)\alpha_k^{t+\tau-1}+\frac{\nu^2(t+\tau)\left(\omega_k^2+\beta_k^2\right)}{2}\alpha_k^{2(t+\tau-1)}\right)\mathrm{d}\alpha_k\right]\mathrm{d}\alpha_n\right].
\end{aligned}$$

By the transformations replacing α_k by $\alpha_k^{1/(t+\tau-1)}$ and α_n by $\alpha_n^{1/(t-1)}$, followed by the same large-t expansion as before,

$$G_{t,t+\tau} \simeq \frac{1}{\left(\mathrm{B}\left(g_\alpha, h_\alpha\right)\right)^2 \left(\left(t-1\right)\left(t+\tau-1\right)\right)^{h_\alpha}}$$
$$\mathrm{E}_{\omega_n,\beta_n}\left[\int_0^1 \frac{\left(1-\alpha_n\right)^{h_\alpha-1}}{\alpha_n} \exp\left(\nu\mu t\alpha_n + \frac{\nu^2 t\left(\omega_n^2+\beta_n^2\right)}{2}\alpha_n^2\right) \mathrm{E}_{\omega_k,\beta_k}\left[\int_0^1 \frac{\left(1-\alpha_k\right)^{h_\alpha-1}}{\alpha_k}\right.\right.$$
$$\left.\left.\exp\left(\nu\left(\mu\left(t+\tau\right)+\nu t\beta_n\beta_k\alpha_n\right)\alpha_k + \frac{\nu^2\left(t+\tau\right)\left(\omega_k^2+\beta_k^2\right)}{2}\alpha_k^2\right)\mathrm{d}\alpha_k\right]\mathrm{d}\alpha_n\right].$$

As before, taking expectations with respect to ω and integrating α_k out,

$$G_{t,t+\tau}$$
$$\simeq \frac{4\pi}{\left(\Gamma\left(g_\omega\right)\mathrm{B}\left(g_\alpha,h_\alpha\right)\right)^2\left(\left(t-1\right)\left(t+\tau-1\right)\right)^{h_\alpha}}\left(\frac{\left(\nu^4 t\left(t+\tau\right)/4\right)^{\frac{\lambda}{2}}}{16h_\omega^2}\right)^{g_\omega-\frac{1}{2}}$$
$$\mathrm{E}_{\beta_n}\left[\mathrm{E}_{\beta_k}\left[\int_0^1 \alpha_n^{\lambda\left(g_\omega-\frac{1}{2}\right)-1}\left(1-\alpha_n\right)^{h_\alpha-1}\exp\left(\nu\mu t\alpha_n + \frac{\nu^2 t\beta_n^2}{2}\alpha_n^2 + \frac{\nu^4 t^2}{4\left(2h_\omega\right)^{\frac{4}{\lambda}}}\alpha_n^4 - \frac{\left(\nu^2 t/2\right)^{\frac{\lambda}{2}}}{4h_\omega}\alpha_n^\lambda\right)\right.\right.$$
$$\int_0^1 \alpha_k^{\lambda\left(g_\omega-\frac{1}{2}\right)-1}\left(1-\alpha_k\right)^{h_\alpha-1}\exp\left(\nu\left(\mu\left(t+\tau\right)+\nu t\beta_n\beta_k\alpha_n\right)\alpha_k + \frac{\nu^2\left(t+\tau\right)\beta_k^2}{2}\alpha_k^2\right.$$
$$\left.\left.\left.+\frac{\nu^4\left(t+\tau\right)^2}{4\left(2h_\omega\right)^{\frac{4}{\lambda}}}\alpha_k^4 - \frac{\left(\nu^2\left(t+\tau\right)/2\right)^{\frac{\lambda}{2}}}{4h_\omega}\alpha_k^\lambda\right)\mathrm{d}\alpha_k\mathrm{d}\alpha_n\right]\right]$$
$$\simeq \frac{4\pi\Gamma\left(h_\alpha\right)}{\left(\mathrm{B}\left(g_\alpha,h_\alpha\right)\Gamma\left(g_\omega\right)\right)^2\left(\nu\left(t-1\right)\left(t+\tau-1\right)\right)^{h_\alpha}}\left(\frac{\left(\nu^4 t\left(t+\tau\right)/4\right)^{\frac{\lambda}{2}}}{16h_\omega^2}\right)^{g_\omega-\frac{1}{2}}$$
$$\exp\left(\nu\mu\left(t+\tau\right) + \frac{\nu^4\left(t+\tau\right)^2}{4\left(2h_\omega\right)^{\frac{4}{\lambda}}} - \frac{\left(\nu^2\left(t+\tau\right)/2\right)^{\frac{\lambda}{2}}}{4h_\omega}\right)$$
$$\mathrm{E}_{\beta_n}\left[\mathrm{E}_{\beta_k}\left[\exp\left(\frac{\nu^2\left(t+\tau\right)\beta_k^2}{2}\right)\int_0^1 \frac{\alpha_n^{\lambda\left(g_\omega-\frac{1}{2}\right)-1}\left(1-\alpha_n\right)^{h_\alpha-1}}{\left(\mu\left(t+\tau\right)+\nu t\beta_n\beta_k\alpha_n\right)^{h_\alpha}}\right.\right.$$
$$\left.\left.\exp\left(\nu\left(\mu t+\nu t\beta_n\beta_k\right)\alpha_n + \frac{\nu^2 t\beta_n^2}{2}\alpha_n^2 + \frac{\nu^4 t^2}{4\left(2h_\omega\right)^{\frac{4}{\lambda}}}\alpha_n^4 - \frac{\left(\nu^2 t/2\right)^{\frac{\lambda}{2}}}{4h_\omega}\alpha_n^\lambda\right)\mathrm{d}\alpha_n\right]\right].$$

Using the binomial expansion, integrating α_n out, then collecting the expansion again,

$$
\begin{aligned}
& G_{t,t+\tau} \qquad (32) \\
\simeq\ & \frac{4\pi\Gamma(h_\alpha)}{\left(\mathrm{B}(g_\alpha,h_\alpha)\Gamma(g_\omega)\right)^2\left(\nu\mu(t-1)(t+\tau-1)(t+\tau)\right)^{h_\alpha}}\left(\frac{\left(\nu^4 t(t+\tau)/4\right)^{\frac{\lambda}{2}}}{16h_\omega^2}\right)^{g_\omega-\frac{1}{2}} \\
& \exp\left(\nu\mu(t+\tau)+\frac{\nu^4(t+\tau)^2}{4(2h_\omega)^{\frac{4}{\lambda}}}-\frac{\left(\nu^2(t+\tau)/2\right)^{\frac{\lambda}{2}}}{4h_\omega}\right) \\
& \mathrm{E}_{\beta_n}\left[\mathrm{E}_{\beta_k}\left[\exp\left(\frac{\nu^2(t+\tau)\beta_k^2}{2}\right)\sum_{j=0}^{\infty}\binom{-h_\alpha}{j}\left(\frac{\nu t\beta_n\beta_k}{\mu(t+\tau)}\right)^j\int_0^1\alpha_n^{j+\lambda\left(g_\omega-\frac{1}{2}\right)-1}(1-\alpha_n)^{h_\alpha-1}\right.\right. \\
& \left.\left.\exp\left(\nu(\mu t+\nu t\beta_n\beta_k)\alpha_n+\frac{\nu^2 t\beta_n^2}{2}\alpha_n^2+\frac{\nu^4 t^2}{4(2h_\omega)^{\frac{4}{\lambda}}}\alpha_n^4-\frac{(\nu^2 t/2)^{\frac{\lambda}{2}}}{4h_\omega}\alpha_n^\lambda\right)\mathrm{d}\alpha_n\right]\right] \\
\simeq\ & \frac{4\pi}{\left(\nu^2\mu(t-1)t(t+\tau-1)(t+\tau)\right)^{h_\alpha}}\left(\frac{\Gamma(h_\alpha)}{\mathrm{B}(g_\alpha,h_\alpha)\Gamma(g_\omega)}\right)^2\left(\frac{\left(\nu^4 t(t+\tau)/4\right)^{\frac{\lambda}{2}}}{16h_\omega^2}\right)^{g_\omega-\frac{1}{2}} \\
& \exp\left(\nu\mu(2t+\tau)+\frac{\nu^4\left[t^2+(t+\tau)^2\right]}{4(2h_\omega)^{\frac{4}{\lambda}}}-\frac{(\nu^2/2)^{\frac{\lambda}{2}}\left[t^{\frac{\lambda}{2}}+(t+\tau)^{\frac{\lambda}{2}}\right]}{4h_\omega}\right) \\
& \mathrm{E}_{\beta_n}\left[\exp\left(\frac{\nu^2 t\beta_n^2}{2}\right)\mathrm{E}_{\beta_k}\left[\frac{\exp\left(\nu^2 t\beta_n\beta_k+\nu^2(t+\tau)\beta_k^2/2\right)}{(\mu+\nu\beta_n\beta_k)^{h_\alpha}}\sum_{j=0}^{\infty}\binom{-h_\alpha}{j}\left(\frac{\nu t\beta_n\beta_k}{\mu(t+\tau)}\right)^j\right]\right] \\
=\ & \frac{4\pi}{\left(\nu^2(t-1)t(t+\tau-1)\right)^{h_\alpha}}\left(\frac{\Gamma(g_\alpha+h_\alpha)}{\Gamma(g_\alpha)\Gamma(g_\omega)}\right)^2\left(\frac{\left(\nu^4 t(t+\tau)/4\right)^{\frac{\lambda}{2}}}{16h_\omega^2}\right)^{g_\omega-\frac{1}{2}} \\
& \exp\left(\nu\mu(2t+\tau)+\frac{\nu^4\left[t^2+(t+\tau)^2\right]}{4(2h_\omega)^{\frac{4}{\lambda}}}-\frac{(\nu^2/2)^{\frac{\lambda}{2}}\left[t^{\frac{\lambda}{2}}+(t+\tau)^{\frac{\lambda}{2}}\right]}{4h_\omega}\right) \\
& \mathrm{E}_{\beta_n}\left[\exp\left(\frac{\nu^2 t\beta_n^2}{2}\right)\mathrm{E}_{\beta_k}\left[\frac{\exp\left(\nu^2 t\beta_n\beta_k+\nu^2(t+\tau)\beta_k^2/2\right)}{(\mu+\nu\beta_n\beta_k)^{h_\alpha}\left(\mu(t+\tau)+\nu t\beta_n\beta_k\right)^{h_\alpha}}\right]\right].
\end{aligned}
$$

We can use Watson's lemma to approximate

$$(\mu+\nu\beta_n\beta_k)\left(\mu(t+\tau)+\nu t\beta_n\beta_k\right)$$

by $\mu^2 (t+\tau)$ when integrating for the expectations with respect to β_k and β_n. Our lemma gives these expectations as

$$\begin{aligned}
& \mathrm{E}_{\beta_n}\left[\exp\left(\frac{\nu^2 t\beta_n^2}{2}\right)\mathrm{E}_{\beta_k}\left[\exp\left(\nu^2 t\beta_n\beta_k+\frac{\nu^2(t+\tau)\beta_k^2}{2}\right)\right]\right] \\
= \;& \sum_{j=0}^{\infty}\frac{\left(\nu^2 t\right)^j}{j!}\mathrm{E}_{\beta_n}\left[\beta_n^j\exp\left(\frac{\nu^2 t\beta_n^2}{2}\right)\right]\mathrm{E}_{\beta_k}\left[\beta_k^j\exp\left(\frac{\nu^2(t+\tau)\beta_k^2}{2}\right)\right] \\
\simeq \;& \frac{4\pi}{\left[\Gamma(g_\beta)\right]^2}\left(\frac{\left(\nu^4 t(t+\tau)/4\right)^{\frac{\lambda}{2}}}{16h_\beta^2}\right)^{g_\beta-\frac{1}{2}}\exp\left(\frac{\nu^4\left[t^2+(t+\tau)^2\right]}{4\left(2h_\beta\right)^{\frac{4}{\lambda}}}-\frac{\left(\nu^2/2\right)^{\frac{\lambda}{2}}\left[t^{\frac{\lambda}{2}}+(t+\tau)^{\frac{\lambda}{2}}\right]}{4h_\beta}\right) \\
& \sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{\nu^4\sqrt{t^3(t+\tau)}}{2\left(4h_\beta^2\right)^{\frac{2}{\lambda}}}\right)^j
\end{aligned}$$

where the sum can be collected as an exponential. Plugging this and the approximation of Watson's lemma into (32),

$$\begin{aligned}
& G_{t,t+\tau} \\
\simeq \;& \frac{\pi^2(16)^{2-g_\beta+g_\omega}\left(\nu^4 t(t+\tau)/4\right)^{\frac{\lambda}{2}(g_\beta+g_\omega-1)}}{\left(\nu^2\mu^2(t-1)t(t+\tau-1)(t+\tau)\right)^{h_\alpha}h_\beta^{2g_\beta-1}h_\omega^{2g_\omega-1}}\left(\frac{\Gamma(g_\alpha+h_\alpha)}{\Gamma(g_\alpha)\Gamma(g_\beta)\Gamma(g_\omega)}\right)^2\exp\left[\frac{\nu^4\sqrt{t^3(t+\tau)}}{2\left(4h_\beta^2\right)^{\frac{2}{\lambda}}}\right] \\
& \exp\left[\nu\mu(2t+\tau)+\frac{\nu^4\left[t^2+(t+\tau)^2\right]}{2^{\frac{4}{\lambda}+2}}\left(\frac{1}{h_\beta^{4/\lambda}}+\frac{1}{h_\omega^{4/\lambda}}\right)-\frac{\nu^\lambda\left[t^{\frac{\lambda}{2}}+(t+\tau)^{\frac{\lambda}{2}}\right]}{2^{\frac{\lambda}{2}+2}}\left(\frac{1}{h_\beta}+\frac{1}{h_\omega}\right)\right].
\end{aligned}$$

Together with (31), this gives the required $V_{t,t+\tau} \simeq \widetilde{V}_{t,t+\tau} \equiv G_{t,t+\tau} - H_t H_{t+\tau}$ and the corresponding autocorrelation function by

$$R_{t,t+\tau} \equiv \frac{V_{t,t+\tau}}{\sqrt{V_{t,t}V_{t+\tau,t+\tau}}} \simeq \frac{\widetilde{V}_{t,t+\tau}}{\sqrt{\widetilde{V}_{t,t}\widetilde{V}_{t+\tau,t+\tau}}},$$

which completes the proof. ||

Proof of Corollary 1. From (18), using the backshift (lag) operator B and letting $X_t \equiv \theta_t^\nu$,

$$\log y_t = \frac{\gamma}{1-\gamma}\log s^* + \frac{1}{\nu(1-\gamma\mathrm{B})}\log X_t,$$

apart from the inconsequential initial conditions (see the discussion following (19) in the text). The constants $(1-\gamma)^{-1}\gamma\log s^*$ and ν have no impact on the relation between the autocorrelation functions of $\{\log X_t\}$ and $\{\log y_t\}$. Furthermore, given that $|\gamma| < 1$, the term $(1 - \gamma \mathrm{B})$ has an exponentially-decaying (low-order) effect on the transfer function linking the long-memory process $\{\log X_t\}$ to $\{\log y_t\}$. $\|$

Proof of Corollary 2. The output per capita of the final product in (13) gives us $k_t = (y_t/\theta_t)^{1/\gamma}$, which we substitute into (26) to get

$$\left(\frac{y_t}{\theta_t}\right)^{\frac{1}{\gamma}} = \left(\frac{y_{t-1}}{\theta_{t-1}}\right)^{\frac{1-d}{\gamma}} (s_{t-1}y_{t-1})^d .$$

Upon taking logarithms and rearranging,

$$\begin{aligned} \log y_t &= \gamma d \log s_{t-1} + (1 - (1-\gamma)\, d) \log y_{t-1} + (1-d) \log \left(\frac{\theta_t}{\theta_{t-1}}\right) + d \log \theta_t \\ &= \frac{\gamma d \log s_{t-1} + (1-d) \log (\theta_t/\theta_{t-1}) + d \log \theta_t}{1 - (1-(1-\gamma)\, d)\, \mathrm{B}}. \end{aligned}$$

As in the proof of Corollary 1, we have $1-(1-\gamma)\, d \in (0,1)$ so that the denominator has an exponentially-decaying (low-order) effect on the transfer function. As for the numerator, $\{\log s_{t-1}\}$ is a bounded process and $\{\log \theta_t\}$ is a long memory process which dominates its difference $\{\log(\theta_t/\theta_{t-1})\}$; so that the leading term of the autocorrelations of $\{\log y_t\}$ and $\{\log X_t\} \equiv \{\nu \log \theta_t\}$ still coincide. Notice that the leading terms of the autocovariances differ from those in the proof of Corollary 1 by a factor of d^2. $\|$

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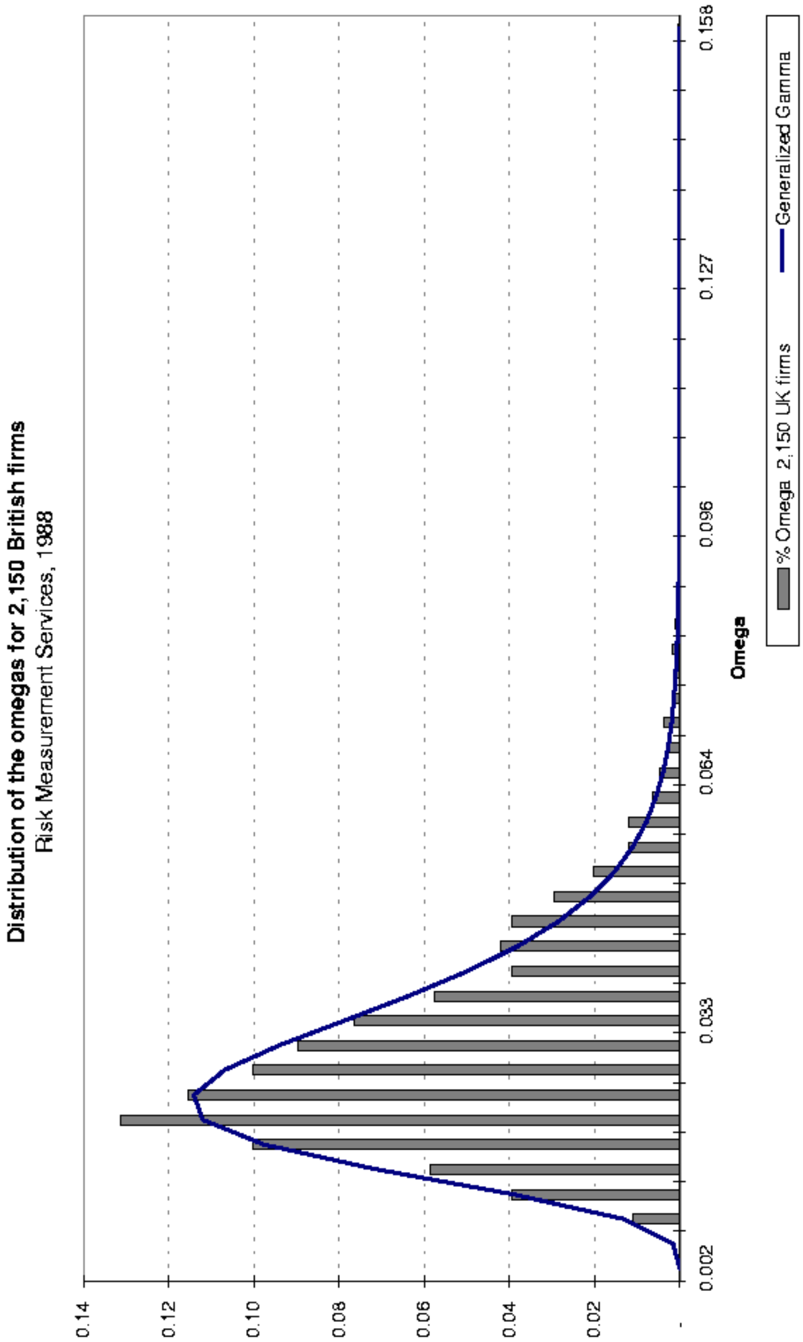


Distribution of the omegas for 2,150 British firms
Risk Measurement Services, 1988
0.14
0.12
0.10
0.08
0.06
0.04
0.02
-
0.002
0.033
0.064
0.096
0.127
0.158
Omega
% Omega 2,150 UK firms
Generalized Gamma

Figure 1:

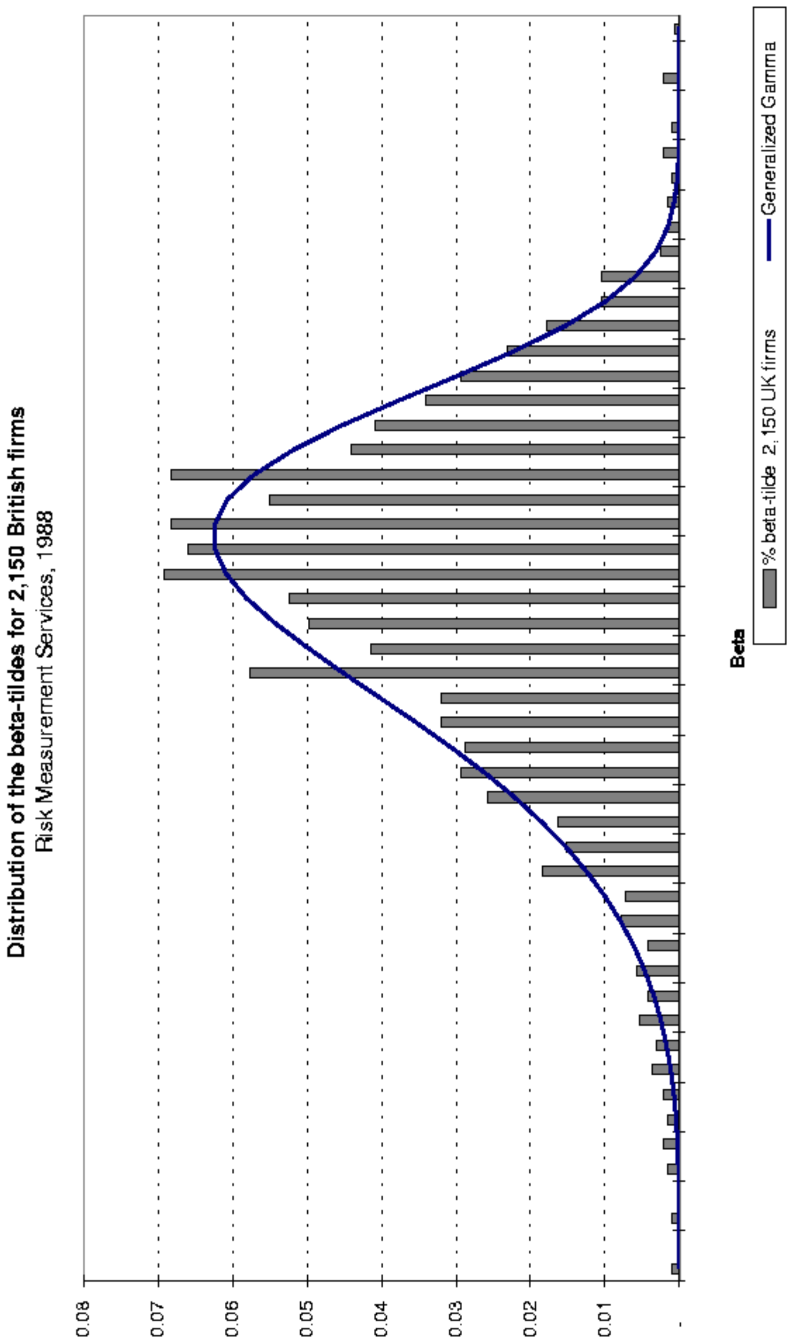

Distribution of the beta-tildes for 2,150 British firms
Risk Measurement Services, 1988
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
-
Beta
% beta-tilde 2,150 UK firms
Generalized Gamma

Figure 2:

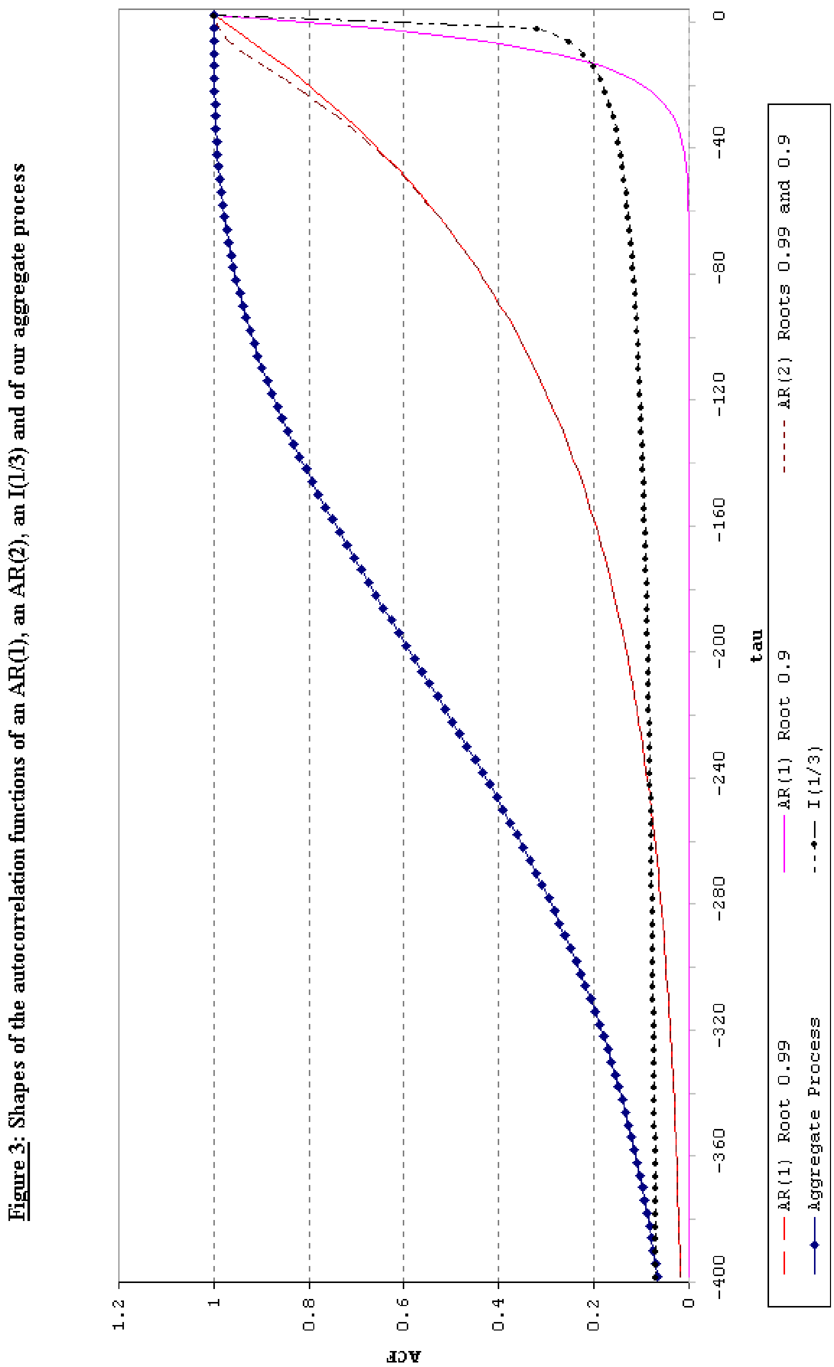
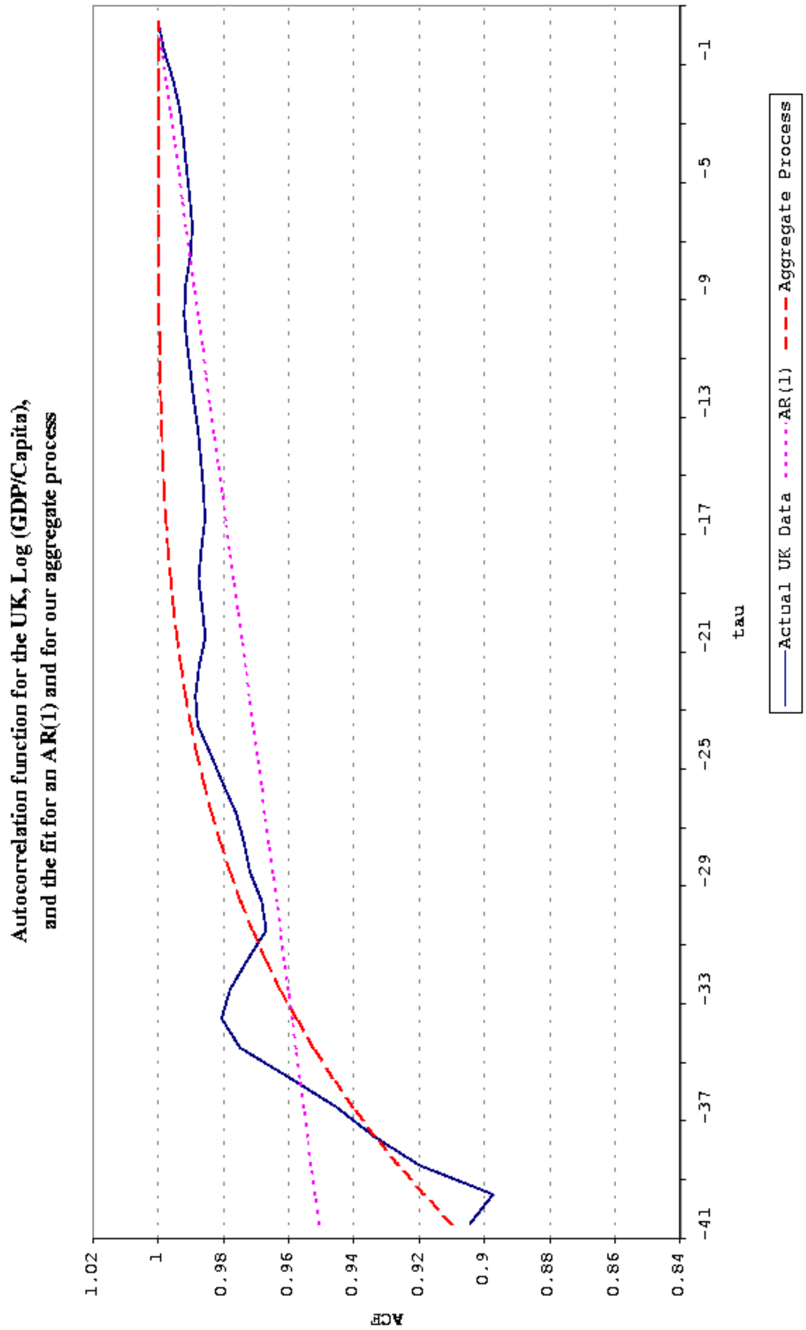


Figure 3: Shapes of the autocorrelation functions of an AR(1), an AR(2), an I(1/3) and of our aggregate process
ACF
1.2
1
0.8
0.6
0.4
0.2
0
tau
-400
-360
-320
-280
-240
-200
-160
-120
-80
-40
0
AR(1) Root 0.99
AR(1) Root 0.9
AR(2) Roots 0.99 and 0.9
Aggregate Process
I(1/3)

Figure 3:



Autocorrelation function for the UK, Log (GDP/Capita),
and the fit for an AR(1) and for our aggregate process
ACF
tau
Actual UK Data
AR(1)
Aggregate Process

Figure 4:

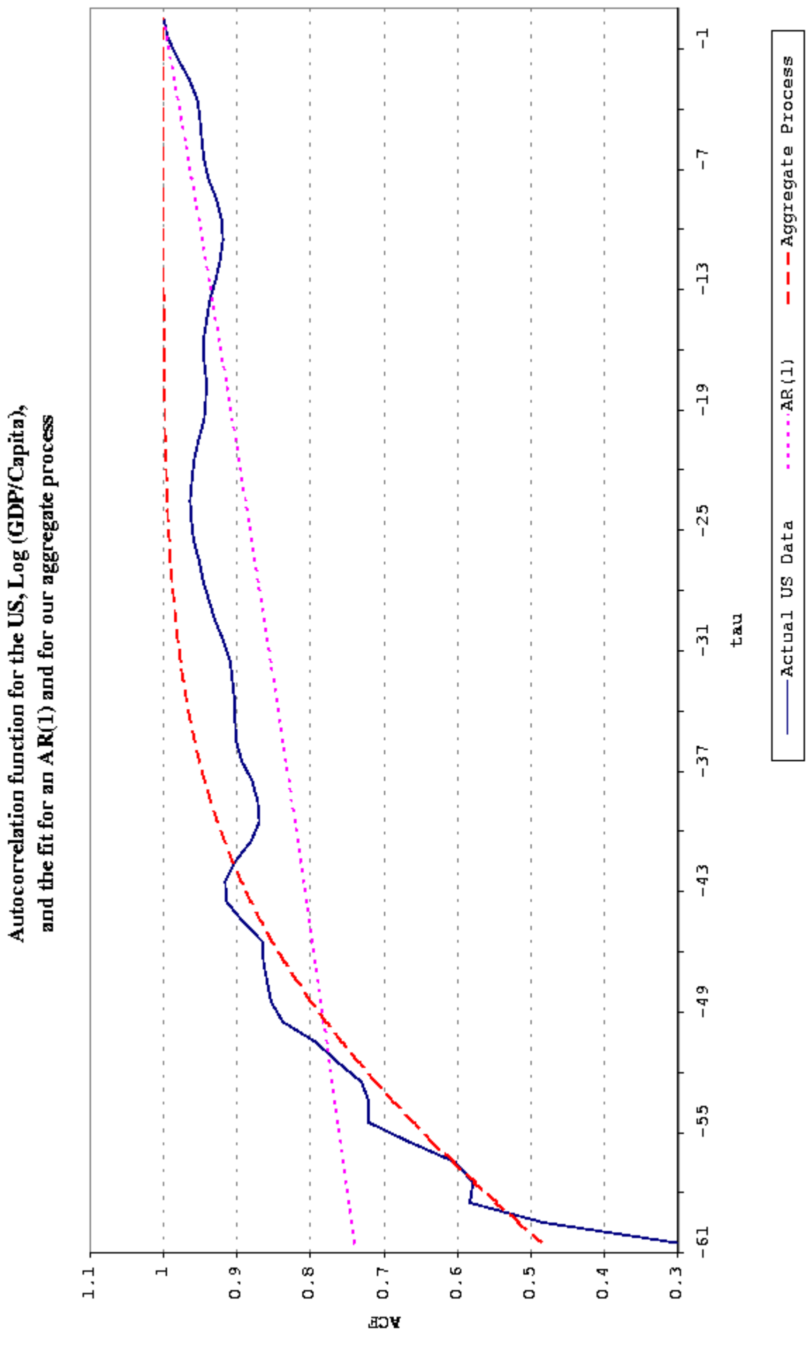

Autocorrelation function for the US, Log (GDP/Capita),
and the fit for an AR(1) and for our aggregate process
ACF
tau
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
-61
-55
-49
-43
-37
-31
-25
-19
-13
-7
-1
Actual US Data
AR(1)
Aggregate Process

Figure 5

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