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Implicit in Oil Prices**

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The estimation of risk premium implicit in oil prices

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Abstract:

Futures prices can be seen as a sum of the expected value of the underlying asset price with a risk premium. In order to disentangle those two components of the futures prices, one can try to model the relationship between spot and futures prices, in order to obtain a closed expression for the risk premium, or to use information from spot and option prices to estimate risk aversion functions. Given the high volatility of the ratios between futures and spot prices, we opted for the latter, estimating risk-neutral and subjective probability density functions, respectively from option and spot prices observed. Looking at the prices of Brent and West Texas Intermediate Light/Sweet Crude Oil options, evidence obtained suggests that the risk premium is typically very low for levels near the futures prices.

JEL Classification codes: G14.

Key words: Risk aversion function, Risk neutral density, Subjective density, Kernel estimation, Oil option pricing.

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1. Introduction

Oil prices are important determinants of inflation. Thus, the assessment of expectations on future values of oil prices is a relevant issue for macroeconomic modelling.

Typically, some scenarios about oil prices are considered within this framework, or oil futures prices are used as the best guess of future price values. However, it is known that futures prices, regardless the underlying asset, may also incorporate a risk premium component.

In this paper, we propose to estimate risk-aversion functions for Brent and West Texas Intermediate oil using option and spot prices. As it is well established (see, e.g. Söderlind and Svensson (1997) or Bahra (1996)), option prices are related to the risk-neutral probability density function (RND) of the underlying asset price in a future moment corresponding to the settlement date.

Another important theoretical result concerns the relationship between the subjective or “true” probability density function (SD) and the RND, corresponding the former to the product of the latter by the risk-aversion function. Therefore, if we are able to estimate SD, we get an estimate for the risk-aversion function.

Naturally, the proxy for the SD gives directly an estimate for the expected value. However, it doesn't provide any information on the investors' behaviour regarding risk-aversion. Thus, the estimation of risk-aversion functions is a relevant issue, in addition to the computation of the expected value under the SD measure.

Generally, a proxy for SD is used, based on the observed prices, though the estimation techniques may differ. For instance, in Ait-Sahalia and Lo (2000) and Jackwerth (1997) a kernel estimator of the past returns is used, while Coutant (1999) uses Hermite polynomials' expansions and Rosenberg and Engle (1997) use a GARCH model. The methods for the estimation of the RND functions are also numerous, but generally are based on the relationship between RND functions and the option prices, while the SD estimator has not that theoretical background.

In this paper, a mixture of two log-normal densities is used to estimate the RND, while a kernel estimator is the tool adopted for extracting the SD from observed prices. The paper is organised as following: in the next section, the concept of risk aversion is approached; the third section is devoted to the relationship between risk-aversion and density functions; the representative agent's preferences are analysed in the fourth section; in the fifth section the estimation methodology is detailed; data and results are presented in the sixth section and the main conclusions are stated at the end.

2. The concept of risk-aversion

The aversion to risk is one of the most important concepts in financial economics. As stated in Ingersoll (1987), it is generally said that a decision maker with a von Neumann-Morgenstern utility function¹ is risk averse at a given wealth level "if he is unwilling to accept every actuarially fair and immediately resolved gamble

¹ A von Neumann-Morgenstern utility function ($u(z)$) is a function on sure things: $u(z) = H(P_z)$, $\forall z \in Z$, being z a consumption plan defined on a collection of consumption plans X , P_z the sure consumption plan and H an utility function (for more details see, e.g., Huang and Litzenberger (1988), chapter 1).

with only wealth consequences, that is, those that leave consumption good prices unchanged".²

It can be shown that a necessary and sufficient condition for a decision maker being strictly risk averse corresponds to his utility function being strictly concave at all relevant wealth levels:³

$$(1) \quad E[U(W + \varepsilon)] < U(E[W + \varepsilon]) = U(W)$$

being W the wealth level, ε the outcome of a lottery and U a utility function.

As stated in Ingersoll (1987), a risk averse agent accepts to pay an insurance risk premium (Π_i) in order to avoid a lottery:⁴

$$(2) \quad E[U(W + \varepsilon)] = U(W - \Pi_i)$$

Using a Taylor expansion of both sides of (2), we will be able to determine the risk premium, as following:⁵

$$(3) \quad \begin{aligned} & E \left[U(W) + \varepsilon U'(W) + \frac{1}{2} \varepsilon^2 U''(W) + \frac{1}{6} \varepsilon^3 U'''(W + \alpha \varepsilon) \right] = \\ & = U(W) - \Pi_i U'(W) + \frac{1}{2} \Pi_i^2 U''(W - \beta \Pi_i) \end{aligned}$$

² Global risk aversion arises when the decision-maker is risk averse at all relevant wealth levels.

³ See appendix A. If the decision-maker is risk averse, though not strictly, the inequality signal becomes \leq .

⁴ Conversely, a risk averse agent would demand a compensatory premium (Π_c) in order to participate in a lottery: $E[U(W + \Pi_c + \varepsilon)] = U(W)$. This definition is more usual in finance literature, while the former is more representative in economic analysis. If the risk is small and the utility function is sufficiently smooth, the two risk premiums are roughly equal.

⁵ The coefficients α and β are such that the Taylor expansions of both sides of equation (2) may be represented by the third- and the second-order expansions presented in equation (3).

being α a coefficient of the expansion's remainder ($0 \leq \alpha \leq 1$).

Given that the utility function is assumed to be sufficiently smooth, the last term in the left-hand side (LHS) will approximately be nil. Besides, assuming that the risk premium is small, the last term of the right hand side (RHS) is approximately nil too. Thus, under the assumption of $E[\varepsilon] = 0$ we get the following expression from (3):

$$(4) \quad \frac{1}{2} \varepsilon^2 U''(W) \cong -\Pi_i U'(W)$$

i.e.

$$(5) \quad \Pi_i = \frac{1}{2} \text{var}(\varepsilon) \left[-\frac{U''(W)}{U'(W)} \right]$$

The term in brackets in equation (5) corresponds to the usually known Arrow-Pratt absolute risk-aversion function and it is a measure of local risk-aversion, independent from the scaling factor $\frac{1}{2} \text{var}(\varepsilon)$.⁶ If the representative agent is risk-averse, this function must always be positive, i.e., the utility function must be concave. As the Arrow-Pratt risk-aversion function is a measure of the relative change in the slope of the utility function at a given wealth level, it must be decreasing and convex in the wealth level under the risk-aversion assumption.

⁶ The variance in the scaling factor may be approached by the variance of the returns of a financial asset.

3. Risk-aversion functions

The risk-aversion functions can be derived in a context of a single-period or of an intertemporal model. Within the former framework, let us assume a complete market economy,⁷ where all investors take decisions on the quantity of state-contingent claims to purchase. The target is the maximisation at the current date t of their expected utility in T , subject to the constraint of the discounted expected value of wealth at T being equal to their initial wealth:⁸

$$(6) \quad \begin{aligned} & \text{Max}_{\zeta_{S_T}} \int_{-\infty}^{+\infty} p(S_T) U(\zeta_{S_T}, S_T) dS_T \\ & \text{subject to } e^{-r\tau} \int_{-\infty}^{+\infty} \zeta_{S_T} q(S_T) dS_T = W_t \end{aligned}$$

being

p = subjective or true density (in order to wealth states, henceforth denoted by SD), representing the probability measure P ;

q = RND (in order to wealth states), representing the probability measure Q , or the price vector of the state-contingent claims;

ζ_{S_T} = quantity of state-contingent claims purchased;

$U()$ = utility function, depending on the quantity of state-contingent claims purchased and the future state of nature;

S_T = state of nature or underlying asset price in T ;

W_t = endowed or initial wealth;

$r = \tau$ -maturity risk-free interest rate, with $\tau = T - t$;

⁷ A market in which each state is insurable, i.e., for each state one can find a portfolio of assets with a non-zero return only in that state (like Arrow-Debreu securities). As stated in Ingersoll (1987), a complete market guarantees that a representative investor exists, though not necessarily the utility function of the representative investor.

⁸ The constraint may also be read as the identity between the initial wealth and the value of the state-contingent claims purchased.

$W_T(S_T) = \zeta_{S_T}$, i.e., the wealth in T as a function of each state corresponds to the number of state-contingent claims purchased, as previously stated.⁹

As usually, the equilibrium will be given by the first order conditions, differentiating the Lagrangean of the optimisation problem (denoted by L) in order to the consumption:

$$(7a) \quad \frac{\partial L}{\partial \zeta_{S_T}} = p(S_T) U'_{\zeta}(\zeta_{S_T}, S_T) - \lambda e^{-r\tau} q(S_T) = 0$$

$$(7b) \quad \frac{\partial L}{\partial \lambda} = W_t - e^{-r\tau} \int_{-\infty}^{+\infty} \zeta_{S_T} q(S_T) dS_T = 0$$

From (7a), one can easily conclude that in equilibrium the following condition must hold:

$$(8) \quad U'_{\zeta}(\zeta_{S_T}, S_T) = \lambda e^{-r\tau} \frac{q(S_T)}{p(S_T)}$$

In order to estimate the Arrow-Pratt absolute risk aversion measure, the second derivative of the utility function will be required. From (8) this corresponds to:

$$(9) \quad U''_{\zeta}(\zeta_{S_T}, S_T) = \lambda e^{-r\tau} \frac{q'(S_T)p(S_T) - q(S_T)p'(S_T)}{p(S_T)^2}$$

Therefore, the Arrow-Pratt absolute risk aversion will be:

⁹ Notice that in the objective function it is considered the SD in order to weight the utility values, while in the budget constraint it is used the risk-neutral density, given that it corresponds to the price of the state-contingent claims, as previously stated.

$$(10) \quad RA(S_T) = -\frac{U''(\zeta_{S_T}, S_T)}{U'(\zeta_{S_T}, S_T)} = \frac{p'(S_T)}{p(S_T)} - \frac{q'(S_T)}{q(S_T)}$$

As referred at the beginning of the section, the risk-aversion function can also be derived from an intertemporal equilibrium model. In fact, let us consider a standard dynamic exchange economy with dynamically complete securities markets, a single consumption good, no exogenous income, where a representative agent maximises his expected utility at each date t in order to his consumption and saving decisions, subject to the usual budget constraints.¹⁰ This problem may be formalised as follows:

$$(11) \quad \text{Max } E_t \left[\sum_{j=0}^{\infty} \delta^j U(C_{t+j}) \right]$$

where δ is the time constant discount factor, C_{t+j} is the investor's consumption in the period $t+j$ and $U(C_{t+j})$ is the utility of consumption in that period. One of the Euler conditions of the problem stated in (11) consists in the solution of the same optimisation problem in a two-period setting:

$$(12) \quad \begin{aligned} & \text{Max}_{\zeta} \quad U(C_t) + \delta E_t[U(C_{t+1})] \\ & \text{s.t. } C_t = e_t - P_t^R \zeta \\ & \quad C_{t+1} = e_{t+1} + P_{t+1}^R \zeta \end{aligned}$$

being e the income, P^R the price of a financial asset providing real cash-flows and ζ is again the number of asset units bought. Inserting the constraints in the objective function, the following optimisation problem arises:

¹⁰ As stated in Ingersoll (1987), a complete market guarantees that a representative investor exists, though not necessarily the utility function of the representative investor.

$$(13) \quad \text{Max}_{\zeta} U(C_t, C_{t+1}) = U(e_t - P_t^R \zeta) + \delta E_t [U(e_{t+1} + P_{t+1}^R \zeta)]$$

One of the Euler conditions will then be:

$$(14) \quad \frac{\partial U(\cdot)}{\partial \zeta} = 0 \Leftrightarrow P_t^R U'(C_t) = \delta E_t [U'(C_{t+1}) P_{t+1}^R]$$

Therefore, in equilibrium the marginal utility of consuming one real monetary unit less at time t must be equal to the discounted expected value of the marginal utility of consuming at time $t+1$ the proceeds of an investment of P^R monetary units at time t in the financial asset.

Following equation (14), the basic pricing or the consumption CAPM (CCAPM) equation is easily obtained:

$$(15) \quad P_t^R = E_t [P_{t+1}^R D_{t+1}]$$

where $D_{t+1} = \delta \frac{U'(C_{t+1})}{U'(C_t)}$ is the intertemporal marginal rate of substitution, the stochastic discount factor (sdf) or the pricing kernel.

Given that real assets are usually scarce,¹¹ equation (2.15) is often adapted to nominal assets. Following Campbell *et al.* (1997), if the nominal price index at

¹¹ The most well known are inflation-indexed Government bonds and they exist only in a few countries. The UK inflation-indexed Government bond market is the most prominent and its information content has been studied in several papers (see, for instance, Deacon and Derry (1994) and Remolona *et al.* (1998)).

time t is denoted by Q_t , with $\Pi_{t+1} = \frac{Q_{t+1}}{Q_t}$ being the rate of inflation from t to $t+1$,

we get:

$$(16) \quad \frac{P_t}{Q_t} = E_t \left[\frac{P_{t+1}}{Q_{t+1}} D_{t+1} \right]$$

Multiplying (16) by Q_t , an equation similar to (15) is obtained for nominal bond prices, with the stochastic discount factor $M_{t+1} = \frac{D_{t+1}}{\Pi_{t+1}}$.

For call-option prices, equation (16) corresponds to:

$$(17) \quad C(X)_t = E_{P,t} [\max(S_T - X, 0) \cdot M_T]$$

where $C(X)_t$ is the price in t of an European call-option with expiry date T , S_T is the underlying asset price at the expiry date and the subscript P denotes that the expected value is computed using the true or original probability measure P , represented by a density function p .

As stated above, within the consumption-based CAPM, the stochastic discount factor is the nominal intertemporal marginal rate of substitution, denoted by $MRS_{t,T}$. Therefore, from equations (16) and (17) it is obtained:

$$(18) \quad \begin{aligned} C(X)_t &= E_t \left[\max(S_T - X, 0) \left[\delta \frac{U'(C_T)}{U'(C_t)} \frac{Q_T}{Q_t} \right] \right] \\ &= E_t [\max(S_T - X, 0) \cdot MRS_{t,T}] \end{aligned}$$

where $MRS_{t,T} = \delta \frac{U'(C_T)}{U'(C_t)} \bigg/ \frac{Q_T}{Q_t}$. In order to compute the expected value in (18), one has to use the density p_t related to the probability measure P :

$$\begin{aligned}
 C(X)_t &= \int_0^\infty \max(S_T - X, 0) \cdot MRS_{t,T} p_t(S_T) dS_T \\
 (19) \quad &= \int_0^\infty \max(S_T - X, 0) \cdot \frac{MRS_{t,T} p_t(S_T)}{\int_0^\infty MRS_{t,T} p_t(S_T)} e^{-r_{t,\tau}} dS_T \\
 &= e^{-r_{t,\tau}} \int_0^\infty \max(S_T - X, 0) q_t(S_T) dS_T \\
 &= e^{-r_{t,\tau}} E_{Q,t} \max(S_T - X, 0)
 \end{aligned}$$

where $r_{t,\tau}$ is the risk-free rate in t for maturity τ ($\tau = T - t$) and $q_t = \frac{MRS_{t,T} p_t(S_T)}{\int_0^\infty MRS_{t,T} p_t(S_T)}$ is alternatively known as the risk-neutral

probability density associated to the probability measure Q (see Cox and Ross (1976)), the equivalent martingale measure (see Harrison and Kreps (1979)) or the state-price density (SPD), being, as referred in Ait-Sahalia and Lo (2000), the continuous-state counterpart to the prices of Arrow-Debreu state-contingent claims.¹² It can be easily concluded that $q_t(S_T)$ is a probability density function, as it assumes values only in the interval between 0 and 1 and its integral is equal to 1. Moreover, as according to (19) the call-option price is the discounted expected value of its future pay-off, S_t is a martingale in the probability measure Q .

Differentiating (19) in order to the strike price, we obtain:

¹² These assets were introduced in economics by Arrow (1964) and Debreu (1959). They are characterised by paying one monetary unit in a given state and nothing in all other states. Probability density functions could be directly obtained from the prices of Arrow-Debreu securities if these were traded for every state.

$$\begin{aligned}
(20a) \quad \frac{\partial C}{\partial X} &= -e^{-r_t \tau} \int_X^\infty q(S_T) dS_T = \\
&= -e^{-r_t \tau} (1 - \int_{-\infty}^X q(S_T) dS_T)
\end{aligned}$$

i.e.,

$$(20b) \quad 1 + \frac{\partial C}{\partial X} e^{r_t \tau} = P_Q [S_T \leq X]$$

Thus, we have the cumulative probability distribution function. Obviously, the density function will be obtained by the differentiation of the LHS of (20b):¹³

$$(21) \quad q(X) = e^{r_t \tau} \cdot \frac{\partial^2 C(X)}{\partial X^2}$$

In order to get information on the risk-aversion of the representative agent, one has to compare the SD and the RND, for instance, computing the ratio between them:

$$(22) \quad \zeta_t(Y_T) = \frac{q_t(Y_T)}{p_t(Y_T)}$$

From (19) we see that this ratio is proportional to *MRS*, i.e.:

$$(23) \quad \zeta_t(Y_T) = \theta MRS_{t,T} = \theta \frac{U'_T(Y_T)}{U'_t(Y_t)}$$

¹³ Notice that for put option prices, the result is the same, as the only difference is in the sign of the first element of the LHS of (20b).

Instead of looking at ζ_T , we can extract information on risk-aversion using the Arrow-Pratt absolute risk aversion in (10), which can be easily computed from ζ_T and its first derivative, which corresponds to:

$$(24) \quad \zeta_t'(Y_T) = \theta \frac{U_T''(Y_T)}{U_t'(Y_t)} = \frac{q_t'(Y_T)p_t(Y_T) - q_t(Y_T)p_t'(Y_T)}{p_t(Y_T)^2}$$

Computing the symmetric of the ratio between ζ_T and ζ_T' , we get:

$$(25) \quad -\frac{\zeta_t'(Y_T)}{\zeta_t(Y_T)} = -\theta \frac{U_T''(Y_T)}{U_t'(Y_t)} \bigg/ \theta \frac{U_t'(Y_t)}{U_t'(Y_t)} = -\frac{U_T''(Y_T)}{U_t'(Y_t)} = RA(Y_T)$$

Consequently, using equations (22) and (24), we can estimate the risk-aversion function from the probability density functions p and q :

$$(26a) \quad RA(Y_T) = -\frac{\zeta_t'(Y_T)}{\zeta_t(Y_T)} = -\frac{q_t'(Y_T)p_t(Y_T) - q_t(Y_T)p_t'(Y_T)}{p_t(Y_T)^2} \bigg/ \frac{q_t(Y_T)}{p_t(Y_T)}$$

i.e.

$$(26b) \quad RA(Y_T) = \frac{p_t'(Y_T)}{p_t(Y_T)} - \frac{q_t'(Y_T)}{q_t(Y_T)}$$

4. The representative agent's preferences

Rubinstein (1994) shows that, in a general equilibrium model, any two of the following imply the third: (i) the preferences of the representative agent; (ii) the asset's stochastic process; and iii) the RND. Consequently, one can identify the asset's stochastic process in accordance with a given utility function (see, e.g., Bick (1990), Wang (1993) and He and Leland (1993)). Instead, the stochastic

process may be obtained if specific preferences are assumed (see, e.g., Derman and Kani (1994) and Dupire (1994)).

Therefore, a relevant question arising is the identification of the utility function compatible with the risk-aversion function to be estimated. For instance, if the RND shape is similar to a log-normal density, then there is evidence that Black-Scholes model holds. As shown in the literature (see, e.g., Bick (1987)), this implies that the representative agent has Constant Relative Risk Aversion (CRRA) preferences, resulting from the following utility function:

$$(27) \quad U(Y) = \begin{cases} \frac{Y^{1-\lambda}}{1-\lambda}, & \text{if } \lambda \neq 1 \\ \ln(Y), & \text{if } \lambda = 1 \end{cases}$$

being λ a nonnegative parameter representing the level of relative risk aversion, as can be easily seen from the resulting risk-aversion function:

$$(28) \quad RA(Y) = \frac{\lambda}{Y}$$

Other useful utility functions to consider are those included in the linear risk tolerance (LRT) or hyperbolic absolute risk aversion (HARA) class.¹⁴ These are defined as:

$$(29) \quad U(Y) = \frac{1-\lambda}{\lambda} \left(\frac{aY}{1-\lambda} + b \right)^\lambda, \text{ with } b > 0,$$

and the risk-aversion functions have the following specification:

$$(30) \quad RA(Y) = \frac{a(1-\gamma)}{aY + b(1-\gamma)}.$$

Another interesting class of utility functions is the negative exponential class:

$$(31) \quad U(Y) = -e^{-\lambda Y}$$

which gives a constant absolute risk-aversion λ .

5. The estimation methodology

In order to estimate the risk-aversion functions, the estimation of both probability densities above mentioned is required. Regarding the SD, a kernel estimation of the returns' density was performed.¹⁵ As referred in Rosenberg and Engle (1997), assuming a kernel smoothing of the past returns as a good proxy for future returns may be seen as a martingale assumption for pricing kernel. This technique is very useful as it permits to estimate the SD without avoiding to impose a parametric structure on the representative agent's utility function.

As we are interested in the density of prices, instead of returns, some transformations were required. Using the density function of the returns, denoted by $p_v(v)$, it is possible to compute the distribution function of Y_T :¹⁶

¹⁴ The risk-tolerance function is just the inverse of the risk-aversion function.

¹⁵ This method was also used in Jackwerth (1997) and Ait-Sahalia and Lo (2000).

¹⁶ For details on the kernel estimation of the returns' density, see appendix B.

$$(32) \quad \text{Prob}(Y_T < Y) = \text{Prob}(Y_t e^{u_t} \leq Y) = \text{Prob}(u_t \leq \log(Y/Y_t)) = \int_{-\infty}^{\log(Y/Y_t)} p_v(v_j) dv_j$$

Using the definition of probability density function and the Leibniz rule for differentiating an integral, we get:

$$(33) \quad p(Y) = \frac{\partial \text{Prob}(Y_T \leq Y)}{\partial Y} = \frac{\partial \left(\int_{-\infty}^{\log(Y/Y_t)} p_v(v_j) dv_j \right)}{\partial Y} \\ = p_v(v_j) \cdot \frac{\partial \log(Y/Y_t)}{\partial Y} = p_v(v_j) \frac{1}{Y_t} \Big/ Y/Y_t = \frac{p_v(v_j)}{Y}$$

Regarding the estimation of the RND, several methods may be found in the literature. Some estimate the RND using non-parametric techniques, i.e., avoiding to impose any specification on the stochastic process of the financial asset, the option premium function in order to the strike price, the implied volatility or the RND.

The most straightforward way to get the RND from option prices is using the prices of state-contingent claims or Arrow-Debreu securities. Though these securities are not usually available in financial markets, one can construct them from the option prices.¹⁷ Another non-parametric technique is presented in Jackwerth and Rubinstein (1996), where a discrete approximation to the fourth derivative of the option price function in order to the strike price is minimised. Though these methods provide more immediate results, the density functions obtained are frequently too irregular.¹⁸

¹⁷ See appendix C.

¹⁸ For more details see, e.g., Adão et al. (1997), Bahra (1996), Melick and Thomas (1997) or Söderlind and Svensson (1997).

Alternatively, one can estimate the RND imposing a parametric specification to some function related to the RND, such as the price or the volatility curve (see, e.g., Shimko (1993)). A very popular parametric technique, given its flexibility and promptness, has been the fitting of a mixture of two log-normal distributions, solving the following optimisation problem:

(34)

$$\begin{aligned} \text{Min}_{\alpha_1, \alpha_2, \beta_1, \beta_2, \theta} & \sum_{i=1}^N \left[\hat{C}(X_i, \tau) - C_i^0 \right]^2 + \sum_{i=1}^N \left[\hat{P}(X_i, \tau) - P_i^0 \right]^2 + \left[\theta e^{\alpha_1 + \frac{1}{2}\beta_1^2} + (1-\theta)e^{\alpha_2 + \frac{1}{2}\beta_2^2} - e^{rt}S \right]^2 \\ \text{s.t. } & \beta_1, \beta_2 > 0 \text{ e } 0 \leq \theta \leq 1. \end{aligned}$$

being

$$\begin{aligned} \hat{C}(X_i, \tau) &= e^{-r\tau} \int_{X_i}^{\infty} (\theta L(\alpha_1, \beta_1; S_T) + (1-\theta)L(\alpha_2, \beta_2; S_T))(S_T - X_i) dS_T \\ (35) \quad &= e^{-r\tau} \theta \left[e^{\alpha_1 + \frac{1}{2}\beta_1^2} N\left(\frac{-\ln(X_i) + (\alpha_1 + \beta_1^2)}{\beta_1}\right) - X_i N\left(\frac{-\ln(X_i) + \alpha_1}{\beta_1}\right) \right] + \\ &= e^{-r\tau} (1-\theta) \left[e^{\alpha_2 + \frac{1}{2}\beta_2^2} N\left(\frac{-\ln(X_i) + (\alpha_2 + \beta_2^2)}{\beta_2}\right) - X_i N\left(\frac{-\ln(X_i) + \alpha_2}{\beta_2}\right) \right] \end{aligned}$$

$$\begin{aligned} \hat{P}(X_i, \tau) &= e^{-r\tau} \int_{-\infty}^{X_i} (\theta L(\alpha_1, \beta_1; S_T) + (1-\theta)L(\alpha_2, \beta_2; S_T))(X_i - S_T) dS_T \\ (36) \quad &= e^{-r\tau} \theta \left[-e^{\alpha_1 + \frac{1}{2}\beta_1^2} N\left(\frac{\ln(X_i) - (\alpha_1 + \beta_1^2)}{\beta_1}\right) + X_i N\left(\frac{\ln(X_i) - \alpha_1}{\beta_1}\right) \right] + \\ &= e^{-r\tau} (1-\theta) \left[-e^{\alpha_2 + \frac{1}{2}\beta_2^2} N\left(\frac{\ln(X_i) - (\alpha_2 + \beta_2^2)}{\beta_2}\right) + X_i N\left(\frac{\ln(X_i) - \alpha_2}{\beta_2}\right) \right] \end{aligned}$$

and where $L(\alpha_i, \beta_i; S_T)$ is the log-normal density function i ($i = 1, 2$), the parameters α_1 and α_2 are the means of the respective normal distributions, β_1 and

β_2 are the standard-deviations of the latter and θ the weight attached to each distribution.

Melick and Thomas (1997) developed a similar method for the estimation of the RND using American option prices, based on a mixture of three log-normal densities and on bounds constructed upon European option values. The RND estimates presented in the following section were obtained with a linear combination of two log-normal distributions, both for European and American options, using call and put option prices for each market and maturity dates.¹⁹

6. Data and results

The SD functions were estimated using monthly average Brent and WTI oil prices collected by Datastream and Bank of Portugal, from January 1977 and February 1982, respectively, to each maturity date considered. Though there are some more complicated methods to determine the optimal bandwidth (h^*), we adopted a similar rule to Jackwerth (1997), i.e., $h^* = \sigma n^{-1/5}$, where σ is the standard deviation of the series to be smoothed and n the number of observations.²⁰

The RND functions for the Brent price were estimated using call and put option prices of the Brent Crude Options traded at International Petroleum Exchange, for the following dates of 1999: 12 April, 12 May, 12 July and 12 August.²¹ For

¹⁹ Some of the observed prices were eliminated from the database, whenever a concavity in the option price function appeared.

²⁰ As it is referred in Jackwerth (1997), solving an optimisation problem, in order to the bandwidth, to obtain the smoothest distribution consistent with the observed returns gives similar results but is significantly slower.

²¹ Though these options are American, they may be treated as European, given that the premium is paid only at expiry date and margins are adjusted daily on a market-to-market basis. Therefore,

WTI price, call and put option prices observed on 2 September 1999 for several maturities were used.

Given the restrictions on data availability, Brent oil data was considered for the analysis of time and maturity evolution of RND and risk-aversion functions, while WTI oil data is richer for getting information on longer maturities. Therefore, our goal was not to compare results obtained for both oil prices, but instead to use them as complementary.

First, we identified the evolution of risk aversion functions along time, uniquely using options with two months to maturity for the Brent, i.e., respectively the July, August, October and November contracts for each date referred.²² Consequently, two-month returns were used.

The Brent kernel densities estimated exhibit a very similar shape (see chart 1). Regarding the price densities, as we originally estimated the density of the returns and prices increased along the period considered, they evidence significant differences (chart 2), as well as noticeable departures from the log-normal shape.²³ Chart 3 shows the kernel smoothing of Brent two-month returns up to 12 August.

The Brent RND functions estimated are presented in chart 4. They show a rightward movement, with some variance reduction. As we can see from charts 5 to 8, the Brent densities on 12 August evidence a striking difference regarding the first derivative in the left tail of the distribution, which, according to equations (10) or (17b), can be taken as an indicator of a risk premium increase.

they may be considered as including a futures contract and a plain-vanilla European option for the maturity date.

²² Notice that oil option contracts expiry in the previous month to the settlement.

²³ According to Sundaresan (1984), equilibrium spot prices are lognormally distributed along the optimal extraction path.

In Chart 9, Brent risk-aversion functions for several dates and two-month maturity are presented.²⁴ It can be seen that these functions have a very irregular shape and some negative values, which is in contrast with the result expected for a risk-averse representative agent. However, these values are consistently around zero. These irregularities may arise from some of the assumptions imposed, concerning namely the RND estimation technique, the kernel bandwidth or the proxy chosen for the SD, and/or from options mispricing.²⁵

The most striking result is the risk-aversion increase for lower wealth levels in July 1999. In this date, our estimations suggest that for an expected Brent price of USD 8, the risk aversion would be circa USD 3,5, while it was around zero in April 1999. In order to transform the risk-aversion in a value for the risk premium, a figure for the scaling factor in equation (5) must be used. Approaching $\text{var}(\varepsilon)$ by the product between the variance of returns and the squared strike prices, the risk premium would be near 2.4 USD for the same expected Brent price in July 1999. For higher prices, closer to the futures levels, the risk premium is virtually nil, though it decreases at a slower pace on 12 August 1999.

When comparing Brent RNDs for several maturities in the last day reported, one can conclude that risk-neutral expected values point to a Brent price decrease (chart 10). Another relevant difference regards the higher positive skewness in the longest maturities considered. Similar conclusions are obtained in what concerns to expectations of WTI prices (chart 11).

²⁴ The density derivatives were computed by arithmetic approximation. Given that a very thin grid was used, it can be taken as a reasonable proxy for the analytic derivative.

²⁵ See Jackwerth (1997) for possible explanations of his estimation results. The shape of risk-aversion functions estimated is similar to those obtained by Ait-Sahalia and Lo (2000).

Comparing the Brent risk-aversion functions, our estimates show significant risk-aversion only for the one-month maturity and for the lowest strike prices (chart 12). In fact, for an expected value around USD 14, the risk-aversion is around USD 4. Using the previously mentioned approach for the risk premium, its value for the same expected price is close to USD 3. Additionally, it can be concluded once more that the risk aversion is around zero for expected values above USD 15.

Decreasing risk-aversion functions in the terms to maturity are in accordance with the stylised fact that in most market days the oil futures curve exhibits a negative slope.²⁶ It can be also interpreted as evidencing expectations that the current price increase is perceived as a short-term phenomenon.

The figures around zero for the risk-aversion may be considered in line with what could be anticipated, as they respect to short maturities.²⁷ However, low risk-aversion levels are also obtained when WTI options with longer maturities are considered (chart 13), with slightly higher values for shorter maturities.

Our results of roughly nil risk-aversion levels seem to be corroborated by the differences between the Brent futures prices and the ex-post realised spot prices. In fact, according to chart 14, the average differences are around zero for the whole term spectrum available and there is an upward sloped term structure of standard-deviation (chart 15). Thus, assuming estimation errors with zero mean and a given standard deviation, those average differences may be taken as a proxy for the risk premium.

²⁶ According to Litzenberger and Rabinowitz (1995), between February 1984 and April 1992, the nine months West Texas Intermediate futures price was backwardated (i.e. had a negatively sloped futures price curve) most of the time (more than 75% of the time). An identical conclusion is stated in Considine and Larson (1996) for 1988-1994.

Though almost all risk-aversion functions vary around zero, the curvature of the Brent risk-aversion function estimated for the one-month maturity in the 12th August seems to be too pronounced to allow an adequate fitting by a risk-aversion function implied by a CRRA utility function. Conversely, its shape is not too far from that implied by a HARA utility function (chart 16).²⁸

7. Conclusions

Information on investor's preferences may be extracted from spot and option prices. From Brent oil data we concluded that risk aversion is typically small for levels near the futures prices, for terms to maturity up to four months. Using data on WTI oil prices, the results obtained suggest that the risk premium is still close to zero for terms up to around 24 months. Consequently, one can argue for using futures prices as a proxy for the expected value of oil price.

The results seem to be confirmed by the average differences between the Brent futures prices and the ex-post realised spot prices, which vary around zero.

The risk-aversion functions estimated seem to be compatible with HARA utility functions, though they exhibit high volatility in the tails. More robust conclusions may be obtained by estimating non-parametrically the distribution of the risk-aversion functions, e.g. using bootstrapping methods.

²⁷ Brent options for higher maturities are not sufficiently liquid to be considered.

²⁸ The implied HARA utility function results from the parameters offering the lowest sum of squared differences to the estimated risk-aversion function.

References

Adão, Bernardino, Nuno Cassola and Jorge Barros Luís (1997), “Extracting information from options premia: the case of the return of the Italian Lira to the ERM of the EMS”, Banco de Portugal, Quarterly Economic Bulletin, December.

Ait-Sahalia, Yacine and Andrew W. Lo (2000), “Nonparametric Risk Management and Implied Risk Aversion”, *Journal of Econometrics*, No. 94, pp. 9-51.

Arrow, Kenneth (1964), “The Role of Securities in the Optimal Allocation of Risk Bearing”, *Review of Economic Studies*, 31, pp. 91-96.

Bahra, Bhupinder (1996), “Implied Risk-Neutral Probability Density Functions from Option Prices: Theory and Application”, *Bank of England Economic Bulletin*, August 1996.

Bick, Avi (1990), “On Viable Diffusion Price Processes of the Market Portfolio”, *The Journal of Finance*, Vol. XLV, No.2, June.

Bick, Avi (1987), “On the Consistency of the Black-Scholes Model with a General Equilibrium Framework”, *Journal of Financial and Quantitative Analysis*, No. 22, pp. 259-275.

Breeden, Douglas e Robert Litzenberger (1978), “Prices of State-Contingent Claims Implicit in Option Prices”, *Journal of Business*, October 1978, 51(4), pp. 621-51.

Campbell, John. Y., Andrew W. Lo e A. Craig MacKinlay (1997), *The Econometrics of Financial Markets*, Princeton University Press.

Considine, Timothy J. and Donald F. Larson (1996), “Uncertainty and the Price for Crude Oil Reserves, *mimeo*.

Coutant, Sophie (1999), “Implied Risk Aversion in Option Prices”, *mimeo*.

Cox, John and Stephen Ross (1976), “The Valuation of Options for Alternative Stochastic Processes”, *Journal of Financial Economics*, 3, 145-166.

Debreu, G. (1959), *Theory of Value*, John Wiley and Sons, New York.

Derman, E. and I. Kani (1994), “Riding on the Smile”, *Risk*, 7, February, 32-39.

Dupire, B. (1994), “Pricing with a smile”, *Risk*, 7, January, 18-20.

Harrison, M. and D. Kreps (1979), “Martingales and Arbitrage in Multiperiod Securities Markets”, *Journal of Economic Theory*, 20, 381-408.

He, Hua and Hayne Leland (1993), “On Equilibrium Asset Price Processes”, *Review of Financial Studies*, No. 6, pp. 593-617.

Huang, Chi-fu and Robert H. Litzenberger (1988), Foundations for Financial Economics, North-Holland.

Hull, John (1997), *Options, Futures, and Other Derivatives* (Third Edition), Prentice-Hall.

Ingersoll, Jonathan E. (1987), Theory of Financial Decision Making, Rowman and Littlefield.

Jackwerth, Jens Carsten (1997), "Recovering Risk Aversion from Option Prices and Realized Returns", Institute of Finance and Accounting Working Paper 261.

Jackwerth, Jens C. e Mark Rubinstein (1996), "Recovering Probability Distributions from Option Prices", *Journal of Finance*, Dec.96, Vol.51, No.5, pp. 1611-1631.

Litzenberger, Robert H. and Nir Rabinowitz (1995), "Backwardation in Oil Futures Markets: Theory and Empirical Evidence", *Journal of Finance*, Vol. L, No. 5, December.

Melick, William R. and Charles P. Thomas (1997), "Recovering an Asset's Implied PDF from Option Prices: An Application to Crude Oil during the Gulf Crisis", *Journal of Financial and Quantitative Analysis*, Vol. 32, No.1, March.

Neuhaus, Hans (1995), "The information content of derivatives for monetary policy: implied volatilities and probabilities", *Deutsche Bundesbank Economic Research Group*, discussion paper No. 3/95.

Rosenberg, Joshua and Robert F. Engle (1997), "Option hedging using empirical pricing kernels", NBER WP No. 6222.

Rubinstein, Mark (1994), "Implied Binomial Trees", *Journal of Finance*, No. 49, pp. 771-818.

Shimko, David (1993), "Bounds on probability", *Risk*, Vol. 6, No. 4, pp. 33-37.

Söderlind, Paul e Lars E.O. Svensson (1997), "New Techniques to Extract Market Expectations from Financial Instruments", *Journal of Monetary Economics* 40, p.p. 383-429.

Sundaresan, Suresh (1984), "Equilibrium Valuation of Natural Resources", *Journal of Business*, Vol. 57, No. 4.

Wang, S. (1993), "The Integrability Problem of Asset Prices", *Journal of Economic Theory*, No. 59, pp. 199-213.

Appendix A - Condition for risk-aversion

If x is a random variable with density function denoted by $f(x)$ and mean represented by \bar{x} and G is a strictly concave function of x with finite derivatives of n -th order in the region between \bar{x} and x ($x = \bar{x} + h$), then using Taylor series we have:

$$(A1) \quad \begin{aligned} G(x) &= G(\bar{x}) + (x - \bar{x})G'(\bar{x}) + \frac{1}{2}(x - \bar{x})^2 G''(\bar{x}) + \dots + (x - \bar{x})^n \frac{1}{n!} G^{(n)}(\bar{x}) \\ &= G(\bar{x}) + (x - \bar{x})G'(\bar{x}) + \frac{1}{2}(x - \bar{x})^2 G''(\bar{x}^*) \end{aligned}$$

with \bar{x}^* between \bar{x} and x .

Then

$$(A2) \quad \begin{aligned} E[G(x)] &= \int_{-\infty}^{+\infty} G(x)f(x)dx \\ &= G(\bar{x}) \int_{-\infty}^{+\infty} f(x)dx + G'(\bar{x}) \int_{-\infty}^{+\infty} (x - \bar{x})f(x)dx + \frac{1}{2} \int_{-\infty}^{+\infty} G''(\bar{x}^*)(x - \bar{x})^2 f(x)dx \end{aligned}$$

Given that the first integral on the RHS is equal to one (as it is the sum of all density values) and the second one is nil (as it is the difference between the expected value of x and itself), we have:

$$(A3) \quad E[G(x)] = G(\bar{x}) + \frac{1}{2} \int_{-\infty}^{+\infty} G''(\bar{x}^*)(x - \bar{x})^2 f(x)dx$$

As G is a strictly concave function, $G'' < 0$ and $E[G(x)] < G(\bar{x})$.

Therefore, $E[U(W + \varepsilon)] < U(E[W + \varepsilon]) = U(W)$.

Appendix B - Kernel estimation²⁹

Our goal is to obtain a sufficiently smooth probability density function from the observed values of a series. Therefore, we have to get the histogram from the series and smooth the histogram.³⁰

The probability density function is the first derivative of the distribution function. Thus, for each value of the variable the density may be computed from the difference between two close values of the distribution function:

$$(B1) \quad p_v(v) = \lim_{h \rightarrow 0} \frac{\text{Pr } ob(V < v + h) - \text{Pr } ob(V < v - h)}{[(v + h) - (v - h)]} = \lim_{h \rightarrow 0} \frac{\text{Pr } ob(v - h < V < v + h)}{2h}$$

being h usually known as the bandwidth.

A common proxy for the probability of a given value is the ratio between the number of observations inside the bin centred on that value ($\#(v)$), with bandwidth h , and the total number of observations (n). By definition, the area of the histogram is equal to one, as the histogram is the discrete representation of the density function:

$$(B2) \quad \sum_{s=-\infty}^{+\infty} p_v^*(v_s) \cdot h = 1$$

being $p_v^*(v_s)$ the value of the histogram in a bin centred on v_s . Thus, one conclude that

$$(B3) \quad \sum_{s=-\infty}^{+\infty} p_v^*(v_s) = \frac{1}{h}$$

²⁹ This appendix is based on lecture notes by Yacine Ait-Sahalia.

As the density values correspond to the probabilities multiplied by a constant (and the sum of all probabilities is one), the histogram may be estimated as following:

$$(B4) \quad p(v_s) = \frac{1}{h} \cdot \frac{\#(v)}{n}$$

From (B1) or (B4), an estimator for a centred histogram is:

$$(B5) \quad p_v^*(v) = \frac{\#(v)/n}{2h} = \frac{\#(v)}{2nh}$$

This density estimator gives the same positive weight to all observations in the bin and a zero weight to the remaining observations. In fact, if we define a uniform density function as

$$(B6) \quad K(u) = \begin{cases} 1/2 & \text{if } |u| < 1 \\ 0 & \text{otherwise} \end{cases}$$

the estimator for the centred histogram in (B5) may be written as

$$(B7) \quad p_v^*(v) = \frac{1}{nh} \sum_{i=1}^n K\left(\frac{v - V_i}{h}\right)$$

From (B7) it is clear that when the distance between each V_i and the v for which the density is being computed is higher than h , i.e., when the observations are outside the bin, the weight is nil. Otherwise, the observations get a uniform

³⁰ Kernel smoothing techniques can also be implemented to filter a series, instead of its histogram

weight of $1/2$. One can also see from (B7) that an estimator for the density value on each Y_T corresponds to the average of all terms $\frac{1}{h} K\left(\frac{v - V_i}{h}\right)$, i.e., the values of the kernel density for all V_i ($i = 1, \dots, n$) divided by the bandwidth.

In order to obtain a smooth density function, it is more adequate to use a smooth kernel, being the Gaussian kernel the most popular:

$$(B8) \quad K(u) = \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{u^2}{2}\right)}$$

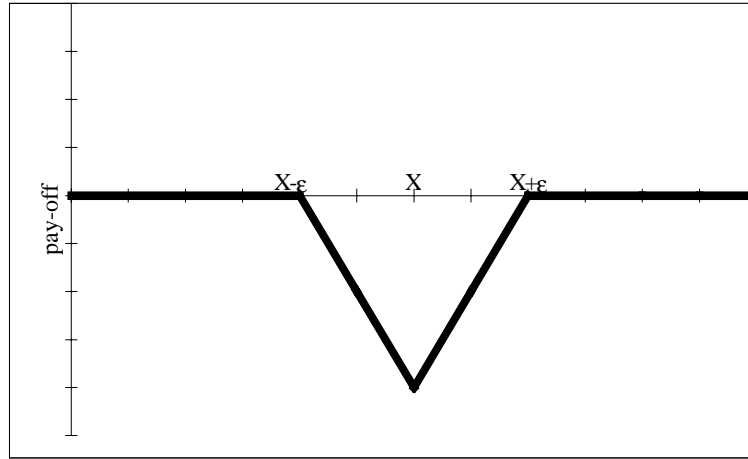
Therefore, in order to get the Gaussian kernel density value for a given v , one has just to compute the average of the normal density values for all observations, with v and h , respectively, as the mean and the variance.

(see, for instance, Campbell *et al.* (1997), chapter 12.

Appendix C – State-contingent claims and option prices

Following Breeden and Litzenberger (1978), a portfolio resulting from buying two call-options with strike price X and selling two put-options, with strike prices $X-\varepsilon$ and $X+\varepsilon$, has a pay-off function usually called butterfly spread.^{31, 32} As we can see from Figure 1, the pay-off is nil outside the interval $[X - \varepsilon, X + \varepsilon]$ and, when ε approaches zero, it becomes close to the symmetric of the Dirac function centred on X .³³

Figure 1 – Pay-off function of a butterfly spread



Note: pay-off excluding option prices.

By definition, the price of the symmetric of the butterfly spread presented is:

$$(C1) \quad D(X; \varepsilon) = \frac{[C(X + \varepsilon) - C(X)] - [C(X) - C(X - \varepsilon)]}{\varepsilon}$$

³¹ In the present case, it is a *short butterfly spread*. Conversely, the symmetric butterfly spread characterised by buying two call options with strike prices $X-\varepsilon$ and $X+\varepsilon$, and selling two call options, both with strike price X is a long butterfly spread (see, for instance, Hull (1997)). This spread has a non-negative pay-off, similar to an inverted butterfly.

³² These spreads are not traded in structured exchanges, but only in *over-the-counter* markets.

³³ Function that assumes 1 in X and 0 in the remaining strike prices.

Dividing (C1) by ε , the limit of (C1) when ε tends to zero is an approximation to the second derivative of the call-option price function, i.e., according to (32), the RND discounted by the riskless interest rate:³⁴

$$(C2) \quad \lim_{\varepsilon \rightarrow 0} \frac{D(X; \varepsilon)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{[C(X + \varepsilon) - C(X)] - [C(X) - C(X - \varepsilon)]}{\varepsilon^2} = \frac{\partial^2 C(X)}{\partial X^2}$$

In derivative exchange traded options, strike prices are spaced by small intervals, though not necessarily close to zero. Thus, Neuhaus (1995) proposes the following discrete approximation:

$$(C3) \quad \frac{\partial^2 C(X)}{\partial X^2} \approx \left[\frac{C(X_{i+1}) - C(X_i)}{X_{i+1} - X_i} - \frac{C(X_i) - C(X_{i-1})}{X_i - X_{i-1}} \right] / \left[\frac{1}{2} (X_{i+1} - X_{i-1}) \right]$$

³⁴ If the premium is paid only at redemption, the discount factor in (21) becomes one.

Chart 1
Kernel densities of 2-month returns of Brent Oil Barrel

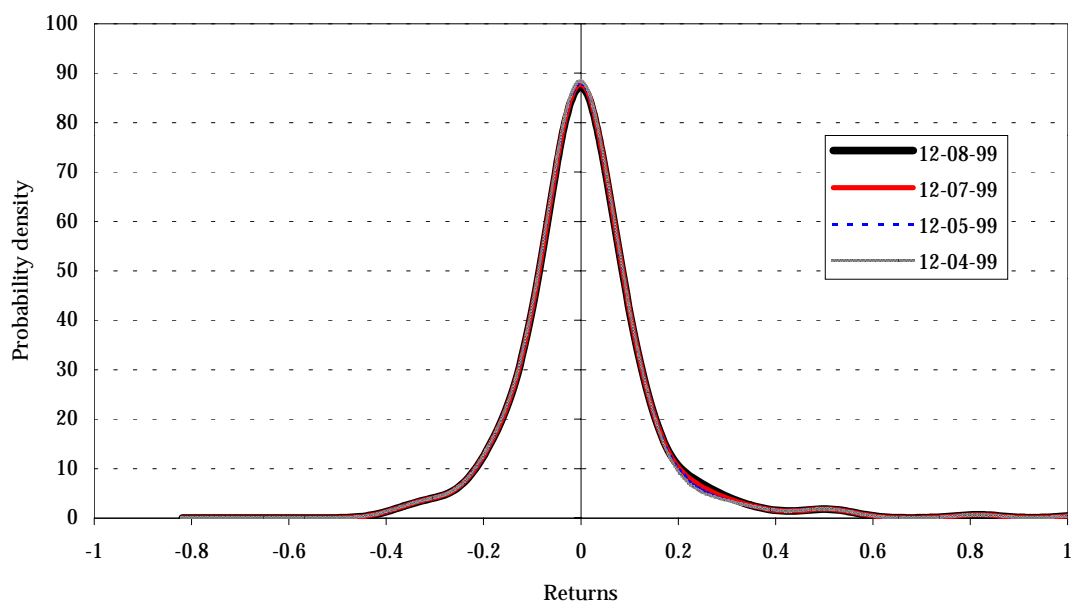


Chart 2
Kernel densities of 2-month forward prices of Brent oil barrel

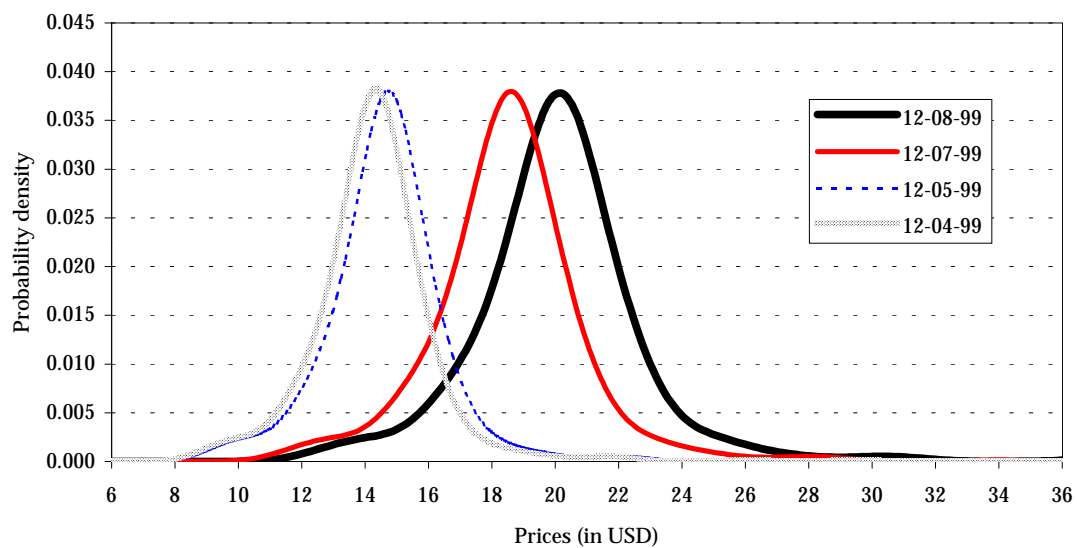


Chart 3
Kernel Density Estimator of 2-month returns of Brent oil barrel
January 1977-August 1999

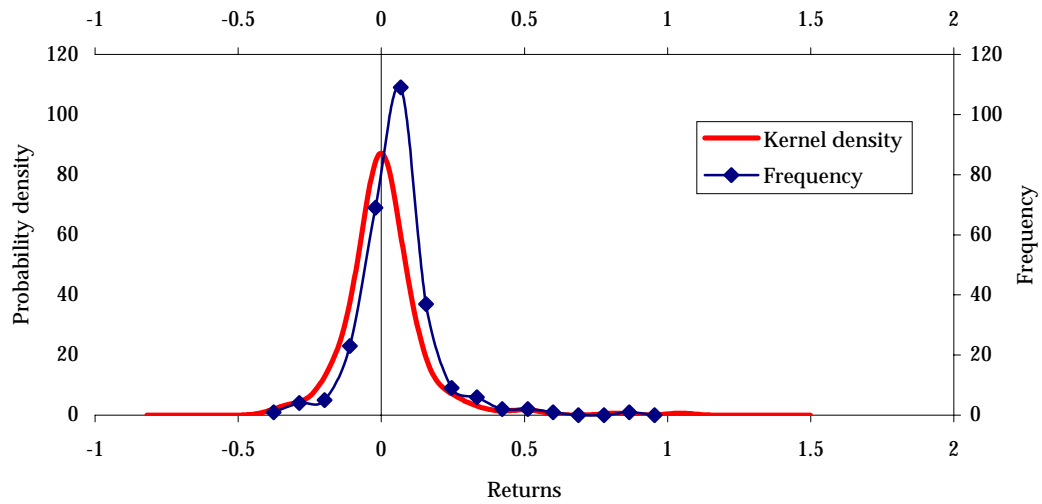


Chart 4
RND functions of 2-month forward price of Brent oil barrel

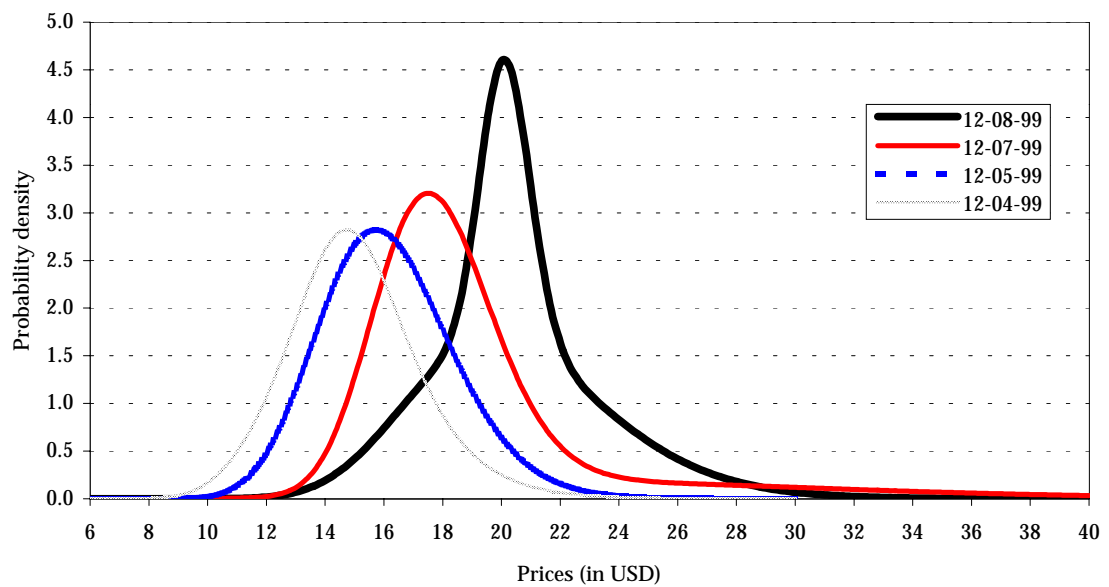


Chart 5
RND and SD of 2-month forward price of Brent oil barrel
in 12-08-99

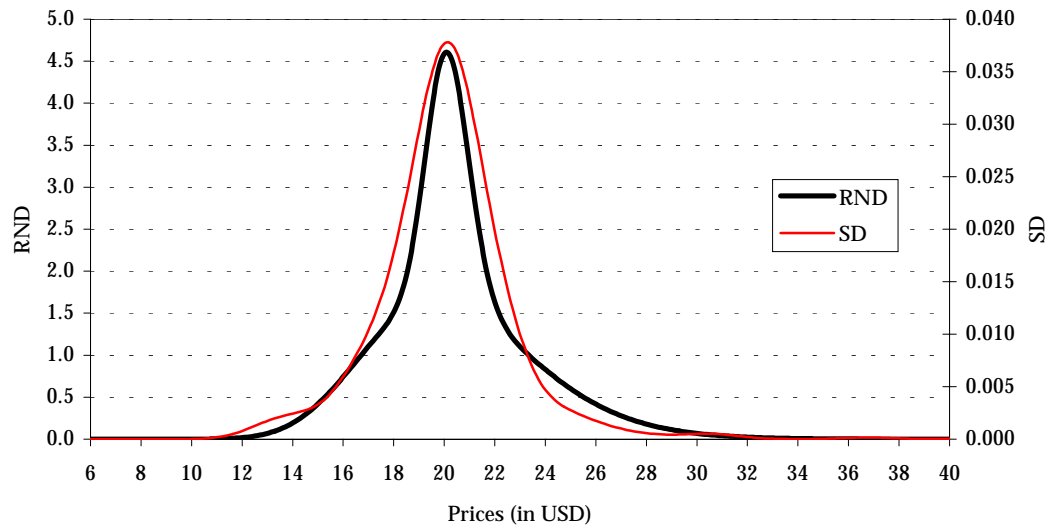


Chart 6
RND and SD of 2-month forward price of Brent oil barrel
in 12-07-99

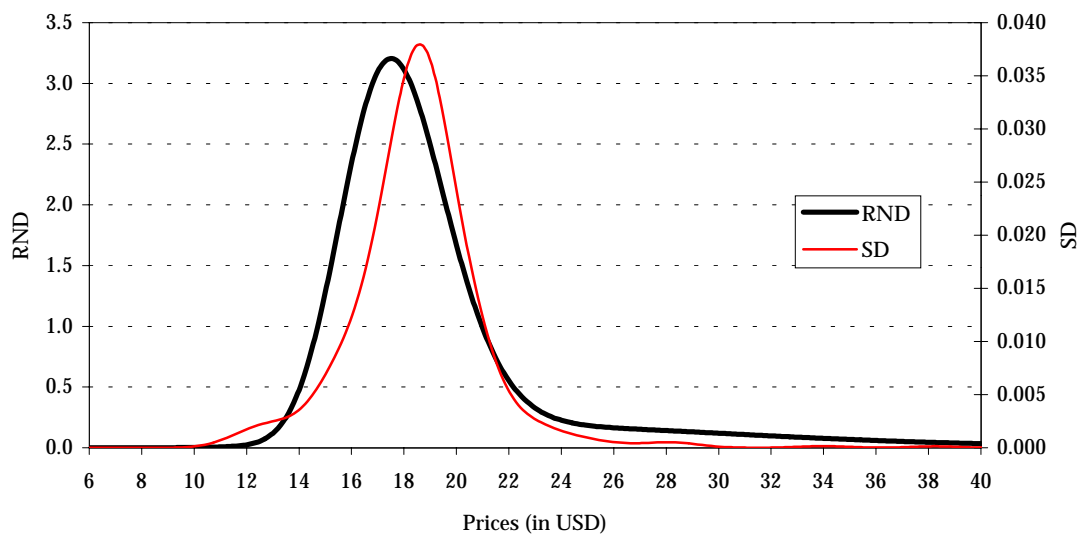


Chart 7
RND and SD of 2-month forward price of Brent oil barrel
in 12-05-99

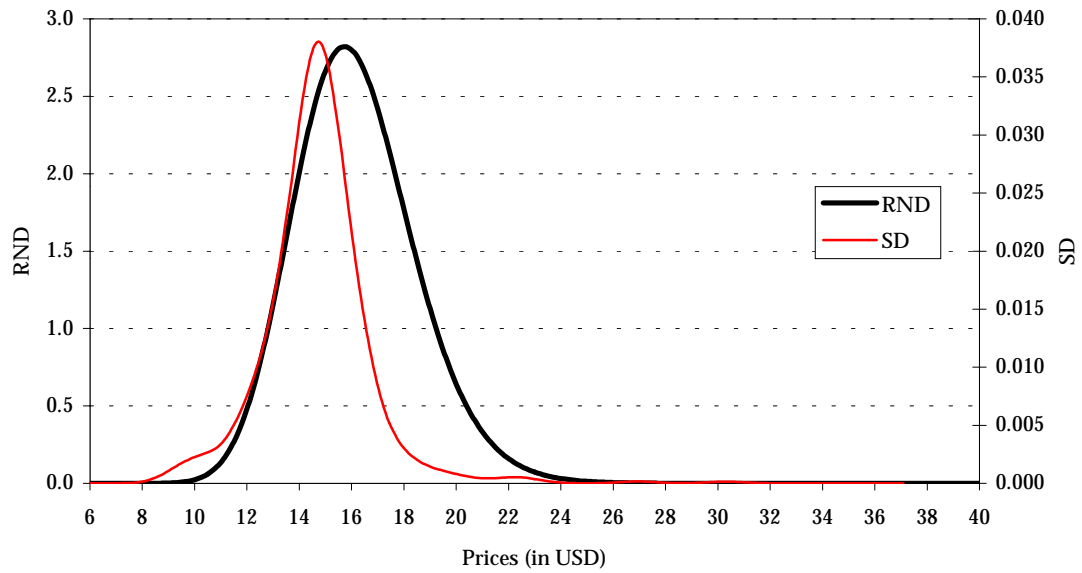


Chart 8
RND and SD of 2-month forward price of Brent oil barrel
in 12-04-99

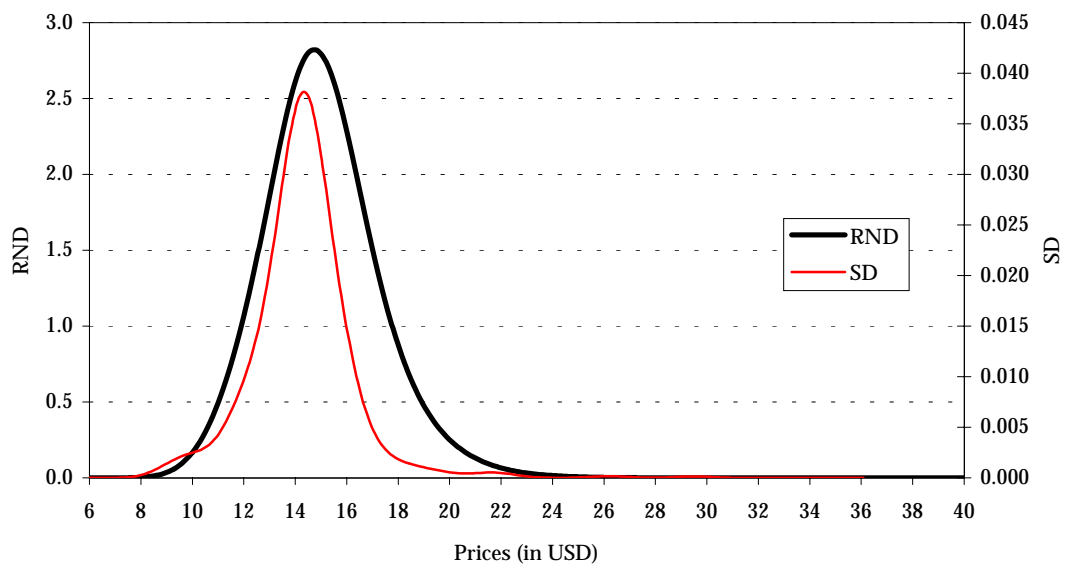


Chart 9
Risk-aversion functions of 2-month forward prices of Brent oil barrel

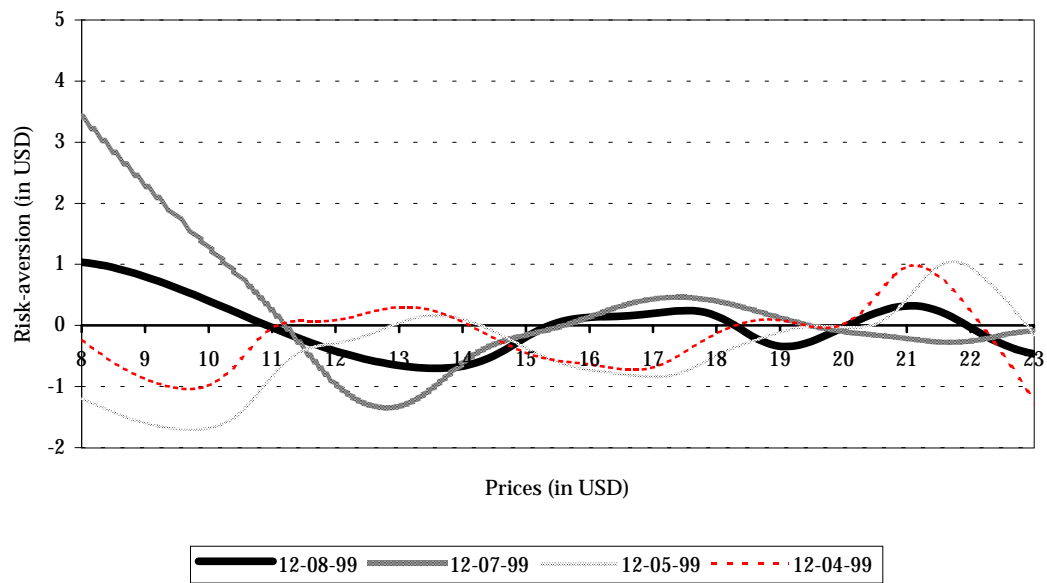


Chart 10
RND of forward prices of Brent oil barrel in 12-08-1999

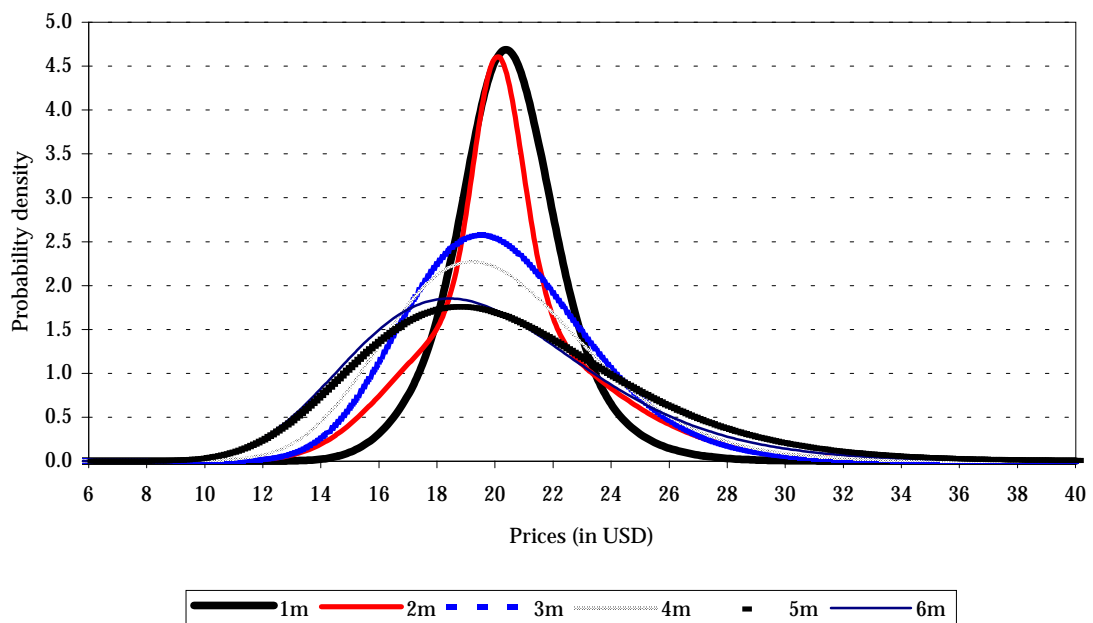


Chart 11
Observed WTI oil prices and risk-neutral expectations

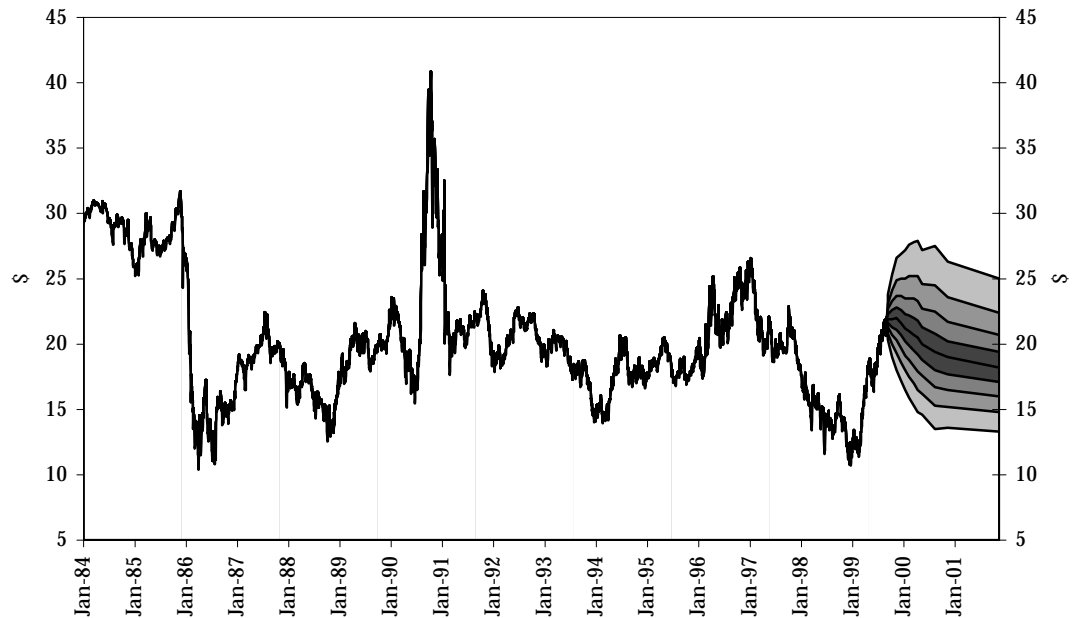
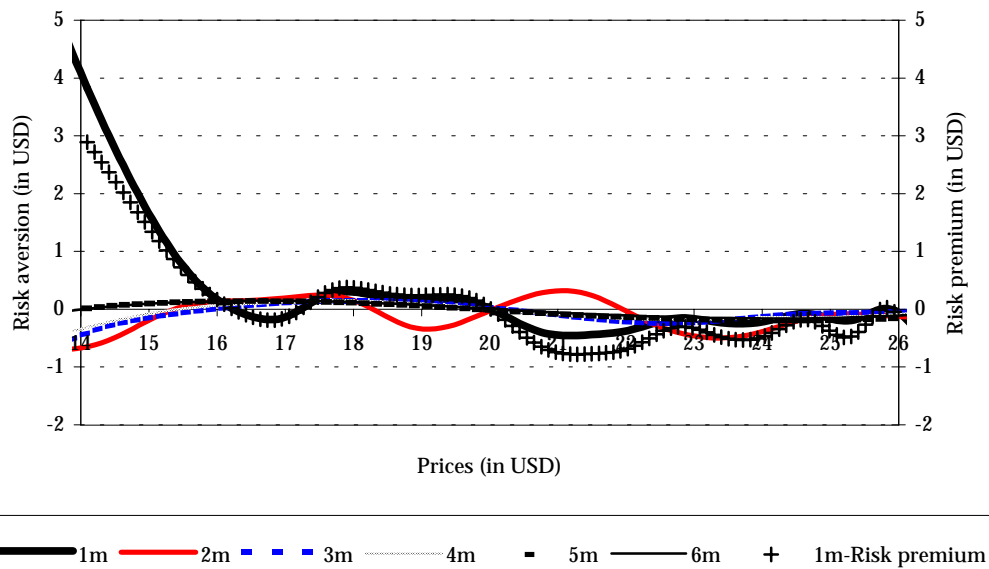


Chart 12
Risk aversion and risk premium implicit in Brent oil futures options and spot prices in 12.08.99 for several terms to maturity



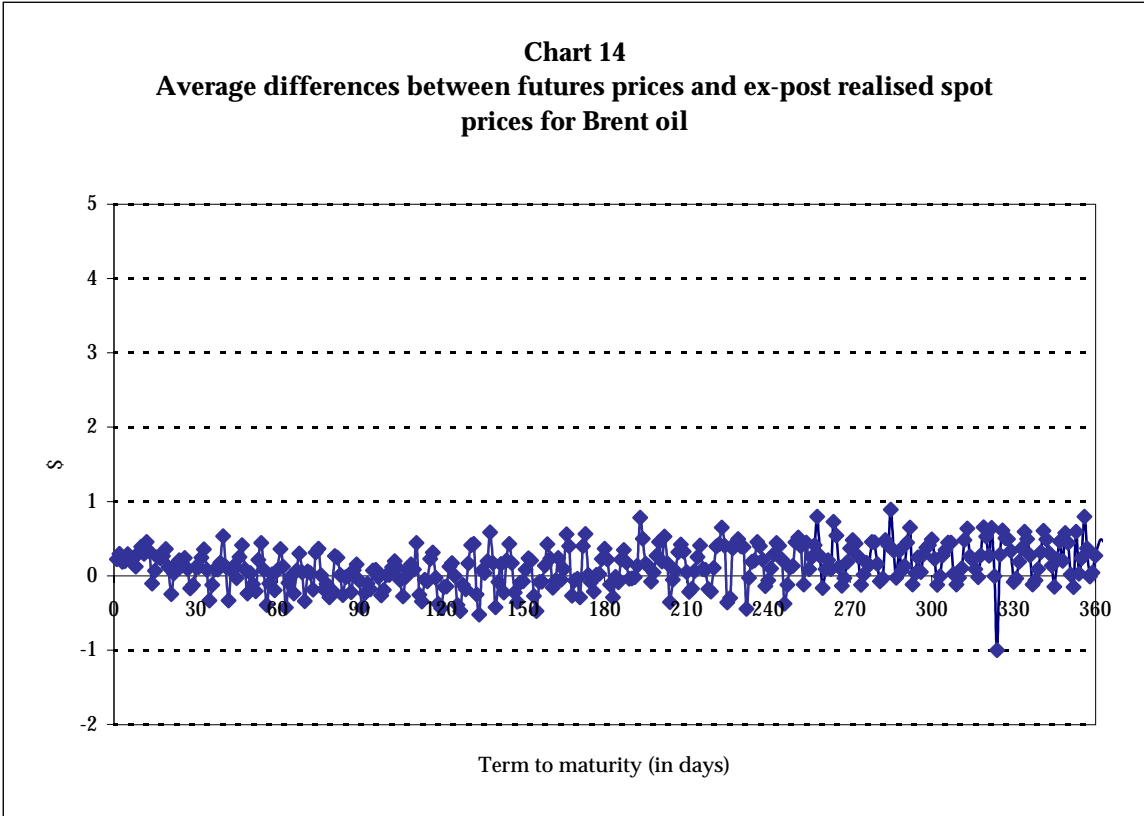
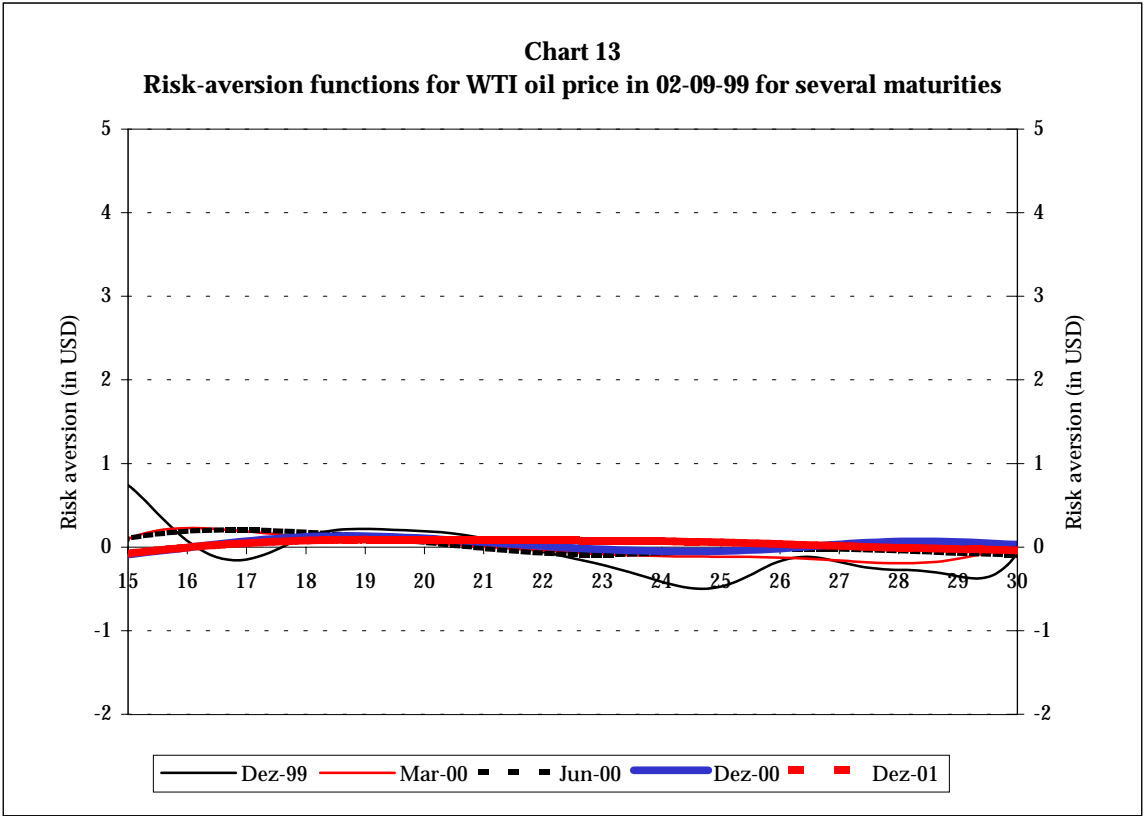


Chart 15
Standard-deviations of the differences between futures prices and ex-post realised spot prices for Brent oil

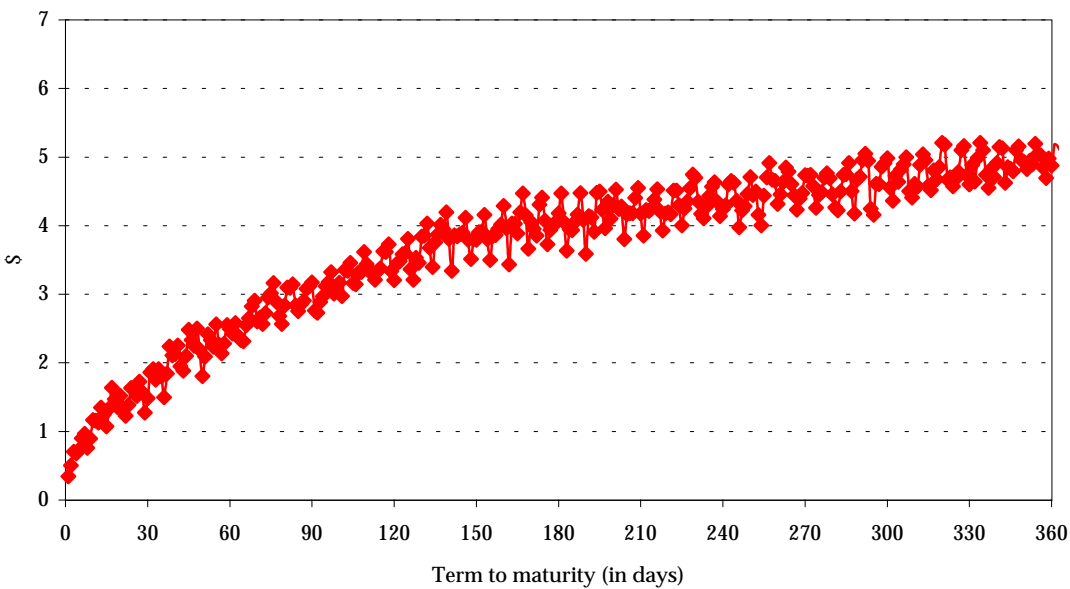


Chart 16
Risk-aversion and utility functions in 12-08-1999 for Brent oil prices (one-month maturity)

