WAVELETS IN ECONOMICS*

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ABSTRACT

The aim of this article is to highlight the usefulness of wavelet analysis in economics. Wavelet analysis is a very promising tool as it represents a refinement of Fourier analysis. In particular, it allows one to take into account both the time and frequency domains within a unified framework, that is, one can assess simultaneously how variables are related at different frequencies and how such relationship has evolved over time. Despite the potential value of wavelet analysis, it is still a relatively unexplored tool in the study of economic phenomena. The basic theoretical building blocks are reviewed and some empirical applications are provided.

1. Introduction

Time domain analysis is, far from doubt, the most widespread approach in the economic literature to study time series. Through such approach, the evolution of individual variables is modelled and multivariate relationships are assessed over time. Another strand of literature focus on the frequency domain. Frequency domain analysis is a complementary tool to time domain analysis. In particular, with spectral analysis, one can investigate the importance of different frequency components for the behaviour of a variable and the relationship between variables at the frequency level.

Wavelets analysis reconciles both approaches, in the sense that both time and frequency domains are taken into account. Hence, wavelets are a very promising tool as they represent a refinement in terms of analysis. Despite its potential usefulness, wavelets have been more popular in fields other than economics. For example, in geophysics, for the analysis of oceanic and atmospheric flow phenomena, seismic signals and climatic data; in medicine, for heart rate monitoring, breathing rate variability and blood flow and pressure; in engineering, for the assessment of machine process behaviour; just to name a few (see, for example, Adisson (2002) for a comprehensive overview). The two most well-known real-life applications of wavelets are the FBI algorithm for fingerprint data compression and the JPEG algorithm for image compression.

Although there are still relatively few papers in economics resorting to wavelet analysis, such analysis can provide fruitful insights about several economic phenomena. In fact, as mentioned by Ramsey (2002), "Wavelets are treated as a 'lens' that enables the researcher to explore relationships that previously were unobservable" while "... the ability to apply a new 'lens' to inspect the relationships in economics and finance provides great promise for the development of the discipline". For instance, the pioneer work of Ramsey and Lampart (1998a,b) draws on wavelets to study the relationship between several macroeconomic variables, namely money supply and output in the first case and consumption and income in the second. A survey concerning wavelet applications in economics is provided, for example, by Crowley (2007).

* The opinions expressed in the article are those of the author and do not necessarily coincide with those of Banco de Portugal or the Eurosystem. Any errors and omissions are his sole responsibility.

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The aim of this article is to review the basic building blocks underlying the continuous wavelet transform and discuss some empirical applications.[1] Recent work using the continuous wavelet transform includes Crowley and Mayes (2008), Rua (2010), Aguiar-Conraria and Soares (2011a), Rua and Silva Lopes (2012) who resort to wavelets for business cycle analysis, Rua and Nunes (2009) assess the international comovement of stock market returns, Aguiar-Conraria and Soares (2011b) study the relationship between oil prices and industrial production, Rua (2012) investigates the link between money growth and inflation in the euro area and Rua and Nunes (2012) propose wavelet-based measures of market risk, among others.

Despite the growing literature in the last few years, there is clearly scope to widen further the application of wavelet analysis in economics. Wavelet analysis has a huge potential as it allows one to unveil relationships between economic variables in the time-frequency space, that is, it allows one to assess simultaneously how variables are related at different frequencies and how such relationship has evolved over time. On the one hand, in a continuously changing economic environment, capturing the time dimension is obviously crucial for the assessment of time-varying behaviour. On the other hand, as argued, for instance, by Clive Granger, the 2003 Nobel Prize in economics, there is no reason to believe that economic variables should present the same relationship at all frequencies. Hence, taking into account the frequency dimension can also be extremely important for the economic analysis.

The article is organised as follows. In section 2, the basic building blocks underlying wavelet analysis are reviewed. In section 3, some empirical applications are discussed and section 4 concludes.

2. From Fourier analysis to wavelet analysis

In 1807, Jean Baptiste Joseph Fourier, a French mathematician, claimed that any periodic function can be expressed as an infinite sum of sine waves and cosine waves of various frequencies. Such idea led to the development of the well-known Fourier transform. The Fourier transform is the conventional method for studying the frequency content of a signal and it involves the projection of a series onto an orthonormal set of trigonometric components (see, for example, Priestley (1981)). In particular, the Fourier transform uses a basis of sines and cosines of different frequencies to determine how much of each frequency the signal contains. The Fourier transform of the time series $x(t)$ is given by

$$F_x(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-i\omega t} dt$$

where ω is the angular frequency and $e^{-i\omega t} = \cos(\omega t) - i\sin(\omega t)$ according to Euler's formula.

During the nineteenth century the Fourier transform solved many problems in physics and engineering. However, throughout the twentieth century, mathematicians, physicists, and engineers came to realize a drawback of the Fourier transform. The Fourier transform does not allow the frequency content of the signal to change over time and therefore it has trouble reproducing signals that have time-varying features. In other words, it can tell us how much of each frequency exists in the signal but it does not tell us when in time these frequency components exist.

To overcome such limitation it has been suggested the short-time Fourier transform. As the name suggests, the basic idea is to use the Fourier transform for short periods of time. It consists in applying a short-time window to the signal and performing the Fourier transform within this window as it slides across all the data.

However, any time-frequency analysis is limited by the Heisenberg uncertainty principle. In 1927, the physicist Werner Heisenberg stated that the position and the velocity of an object cannot both be measured

1 There are other variants of the wavelet transform such as the discrete wavelet transform (see, for example, Rua (2011)).

exactly at the same time even in theory. In signal processing terms, this means that it is impossible to know simultaneously the exact frequency and the exact time of occurrence of this frequency in a signal. In fact, there is a trade-off between time and frequency resolution. This means that for narrow windows one gets good time-resolution but poor frequency resolution whereas for wide windows one gets good frequency resolution and poor time-resolution.

The problem with the short-time Fourier transform is that it uses constant length windows. These fixed length windows give the uniform partition of the time-frequency space. When a wide range of frequencies is involved, the fixed time window tends to contain a large number of high frequency cycles and a few low frequency cycles which results in an overrepresentation of high frequency components and an underrepresentation of the low frequency components. Hence, as the signal is examined under a fixed time-frequency window with constant intervals in the time and frequency domains, the short-time Fourier transform does not allow an adequate resolution for all frequencies.

In contrast, the wavelet transform uses local base functions that can be stretched and translated with a flexible resolution in both frequency and time. In the case of the wavelet transform, the time resolution is intrinsically adjusted to the frequency with the window width narrowing when focusing on high frequencies while widening when assessing low frequencies. Allowing for windows of different size makes it possible to improve the frequency resolution of the low frequencies and the time resolution of the high frequencies. This means that, a certain high frequency component can be located better in time than a low frequency component. On the contrary, a low frequency component can be located better in frequency compared to a high frequency component. As it enables a more flexible approach in time series analysis, wavelet analysis is seen as a refinement of Fourier analysis.

The above discussion can be illustrated through Chart 1. For a time series in the time domain each point contains information about all frequencies. In contrast, in the case of the Fourier transform, every point in the frequency domain contains information from all points in the time domain. For the short-time Fourier transform, the time-frequency plane is divided using a constant length window whereas for the wavelet transform the window width is adjusted to the frequency.

The continuous wavelet transform of a time series $x(t)$ can be written as

$$W_x(\tau, s) = \int_{-\infty}^{+\infty} x(t)\psi_{t,s}^*(t)dt$$

where * denotes the complex conjugate.[2] Hence, the wavelet transform decomposes a time series $x(t)$ in terms of some basis functions (wavelets), $\psi_{\tau,s}(t)$, analogous to the use of sines and cosines in Fourier analysis. The term wavelet means a small wave. The smallness refers to the condition that this function is of finite length. The wave refers to the condition that this function is oscillatory. These basis functions are derived from the so-called mother wavelet $\psi(t)$ and are defined as

$$\psi_{\tau,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-\tau}{s}\right)$$

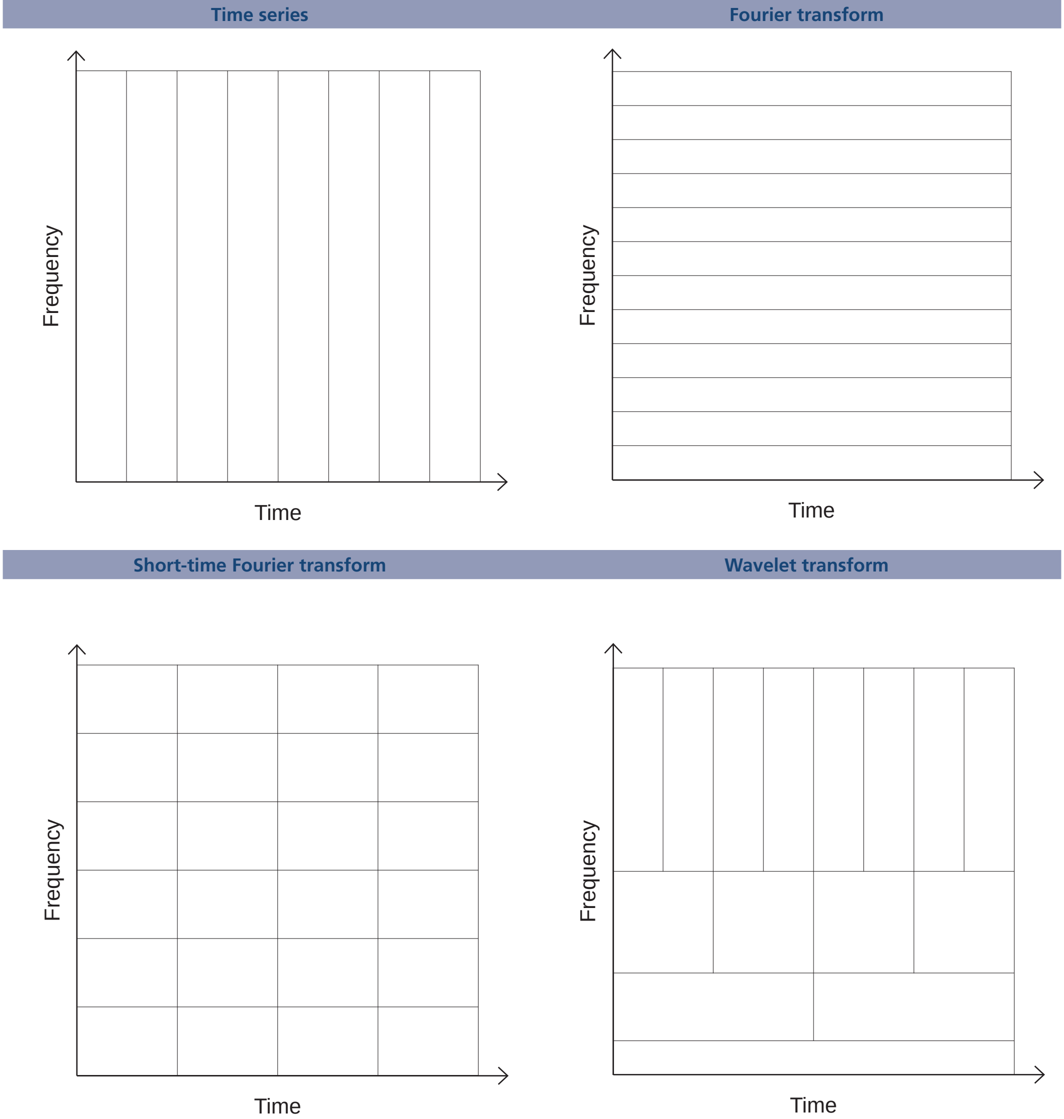
where τ determines the time position and s is the scale. In terms of frequency, low scales capture rapidly changing details, that is, high frequencies, whereas higher scales capture slowly changing features, that is, low frequencies.

To be a mother wavelet, $\psi(t)$ must fulfil certain criteria (see, for example, Percival and Walden (2000)). There are a number of functions that can be used for this purpose. The most commonly used mother wavelet for the continuous wavelet transform is the Morlet wavelet.

2 As the continuous wavelet transform at a given point in time uses information of neighbouring data points, results should be read carefully close to the beginning or the end of the time series.

Chart 1

COMPARISON OF THE TIME-FREQUENCY PROPERTIES



Geologists usually locate underground oil deposits by making loud noises. Because sound waves travel through different materials at different speeds, geologists can infer what kind of material lies under the surface by sending seismic waves into the ground. However, seismic signals contain lots of abrupt changes in the wave as it passes from one rock layer to another. As discussed earlier, the Fourier transform is unable to retain all this information. In 1981, Jean Morlet, a geophysicist working for a French oil company, developed what are now known as Morlet wavelets to solve signal processing problems for oil prospection.

In particular, the Morlet wavelet can be written as

$$\psi(t) = \pi^{-\frac{1}{4}} e^{i\omega_0 t} e^{\frac{-t^2}{2}}$$

One can see that the Morlet wavelet consists of a complex sine wave within a Gaussian envelope. One of the advantages of the Morlet wavelet is its complex nature which allows for both time-dependent amplitude and phase for different frequencies. The parameter ω_0 controls the number of oscillations within the Gaussian envelope. By increasing (decreasing) ω_0 one achieves better (poorer) frequency localization but poorer (better) time localization. In practice, setting ω_0 to 6 provides a good balance between time and frequency localization. Moreover, for ω_0= 6, the wavelet scale s is almost equal to the Fourier period which eases the interpretation of wavelet analysis. See, for example Adisson (2002) for further details on the Morlet wavelet.

Likewise in Fourier analysis, several interesting quantities can be defined in the wavelet domain. For instance, one can define the wavelet power spectrum as $\left|W_x(\tau,s)\right|^2$ which measures the contribution to the variance of the series around each time and scale. Another quantity of interest is the cross-wavelet spectrum which captures the covariance between two series in the time-frequency space. Given two time series $x(t)$ and $y(t)$, with wavelet transforms $W_x(\tau,s)$ and $W_y(\tau,s)$ one can define the cross-wavelet spectrum as $W_{xy}(\tau,s) = W_x(\tau,s)W_y^*(\tau,s)$. The wavelet squared coherency is given by

$$R^2(\tau,s) = \frac{\left|S\left(s^{-1}W_{xy}(\tau,s)\right)\right|^2}{S\left(s^{-1}\left|W_x(\tau,s)\right|^2\right)S\left(s^{-1}\left|W_y(\tau,s)\right|^2\right)}$$

where $S(.)$ denotes smoothing in both time and scale. As well as in Fourier analysis, smoothing is also required; otherwise squared coherency would be always equal to one. The idea behind the wavelet squared coherency is similar to the one of squared coherency in Fourier analysis. The wavelet squared coherency measures the strength of the relationship between the two series over time and across frequencies (while the squared coherency in Fourier analysis only allows one to assess the latter). The $R^2(\tau,s)$ is between 0 and 1 with a high (low) value indicating a strong (weak) relationship. Hence, through the plot of the wavelet squared coherency one can distinguish the regions in the time-frequency space where the link is stronger and identify both time and frequency varying features.

Additionally, one can also compute the wavelet phase, which captures the lead-lag relationship between the variables in the time-frequency space. The wavelet phase difference is defined as

$$\phi(\tau,s) = \tan^{-1}\left(\frac{\Im\left(W_{xy}(\tau,s)\right)}{\Re\left(W_{xy}(\tau,s)\right)}\right)$$

where $\Re$ and $\Im$ are the real and imaginary parts, respectively. The resemblance with the analogue measure in Fourier analysis is clear. It provides information about the lead-lag relationship between the two series. However, besides providing information about the lead-lag across frequencies as in standard Fourier analysis, the wavelet phase also allows one to assess how such lead-lag relationship has changed over time.

3. Some empirical illustrations

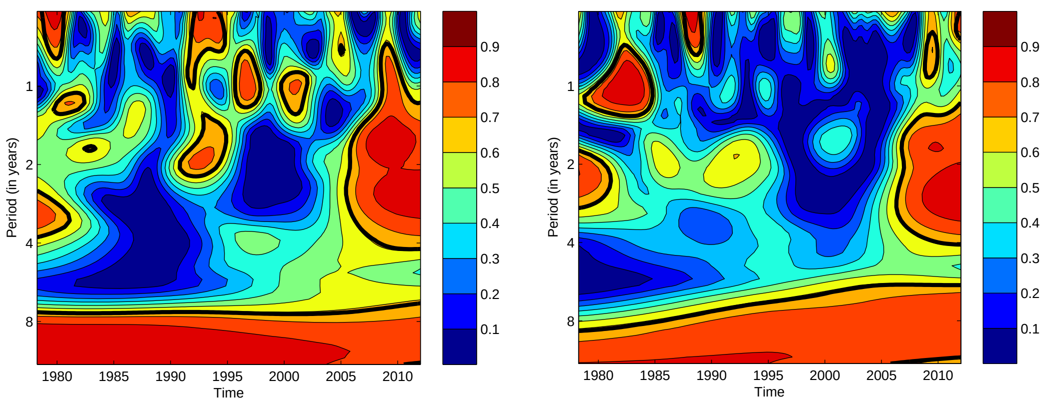
In this section, some applications of the above concepts are provided. Let us start by assessing the relationship in the time-frequency space of the Portuguese economic activity vis-à-vis the euro area as well as vis-à-vis Spain, which is the most important Portuguese trade partner. Using real GDP data from the first quarter of 1978 up to the first quarter of 2012, the wavelet squared coherency between the corresponding quarterly growth rates is presented in Chart 2. The horizontal axis refers to time while the vertical axis refers to frequency. To ease interpretation, the frequency is converted to time units (years). Hence, through the inspection of the chart one can identify both frequency bands (in the vertical axis) and time intervals (in the horizontal axis) where the series move together. The black bold line in the chart delimits the statistical significant area at the usual significance level of five per cent.

From Chart 2, one can conclude that the Portuguese economic activity has presented a high and significant link at long-term movements, namely at fluctuations that last more than 8 years, with both the

Chart 2

WAVELET SQUARED COHERENCY

Portugal vs. euro area	Portugal vs. Spain



Source: Author's calculations.

euro area and Spain over the whole sample period. At the typical business cycle frequency range, that is, for fluctuations that last more than 2 but less than 8 years, the strength of the relationship has started increasing since the beginning of 2000's and has become statistically significant since the mid-2000's reflecting an increasingly economic integration. Concerning shorter-run movements, one can identify episodes where the link has been temporarily stronger. For example, the wavelet squared coherency has been particularly high vis-à-vis the euro area during the 1992-1993 recession, vis-à-vis Spain around the 1983-1984 period and with both during the so-called Great Recession in 2009.

To assess the corresponding lead-lag relationship, the wavelet phase is plotted in Chart 3. As the wavelet phase difference can be poorly estimated when coherency is low, the statistical significant area of the wavelet squared coherency is also plotted in Chart 3. One can conclude that Portuguese economic activity lags slightly at long-term movements but at the other time-frequency regions delimited by the bold line, it oscillates between a slight lag and slight lead without presenting any noteworthy lead or lag.

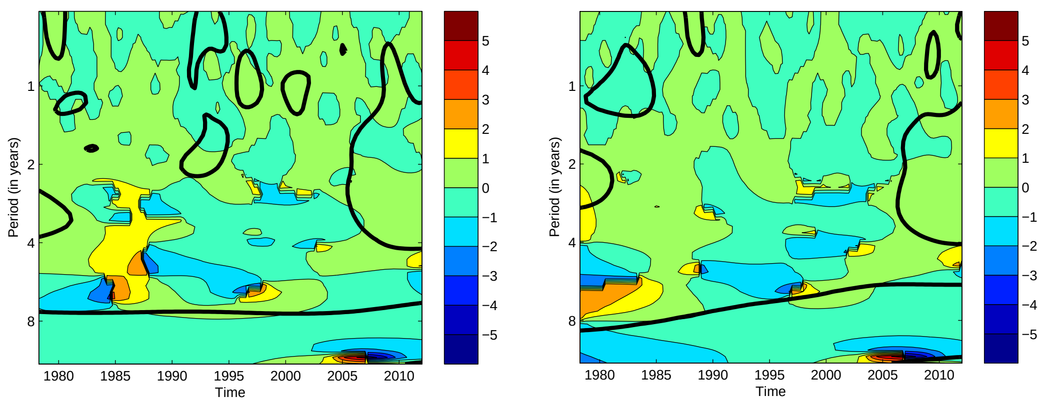
Suppose now that one is interested in measuring the contemporaneous comovement. As mentioned earlier, the wavelet squared coherency allows one to assess the strength of the relationship but it disregards how much the variables are out of phase, that is, the lead-lag. This latter information is provided by the wavelet phase difference. In other words, one can think of the first as the maximum squared correlation between the two variables which is attained when the phase difference is given by the second.[3] Within Fourier analysis, Croux, Forni and Reichlin (2001) have proposed a spectral-based measure, the dynamic correlation, which allows one to measure the comovement between two series at each individual frequency. This measure, which ranges between -1 and 1, is conceptually similar to the contemporaneous correlation between two series in the time domain. However, unlike the correlation coefficient in the time domain, one now obtains a comovement measure that can vary across frequencies. Rua (2010) proposes a wavelet-based measure which can be seen as a generalisation of the dynamic correlation measure suggested by Croux, Forni and Reichlin (2001) since it provides information about contemporaneous comovement not only at the frequency level but also over time. This feature is of striking importance for assessing, for example, the degree of synchronization of macroeconomic fluctuations across countries or regions which plays a key role on the discussion about the attractiveness of economic integration.

3 The same reasoning applies to the analogous measures in Fourier analysis (see, for example, Rua and Nunes (2005)).

Chart 3

WAVELET PHASE

Portugal vs. euro area	Portugal vs. Spain



Source: Author's calculations.
Note: A positive value denotes a lead whereas a negative one corresponds to a lag (in years).

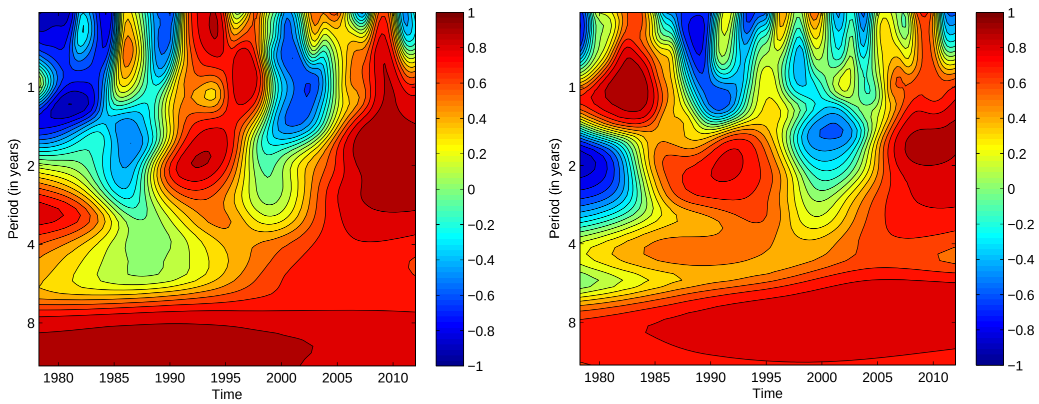
In Chart 4, the results obtained with the measure proposed by Rua (2010) are presented. Qualitatively, the findings from Chart 4 are not that different from those resulting from Chart 2, reflecting the fact that there is no substantial lead-lag relationship. From Chart 4, it becomes clear that synchronization has always been high at long-term fluctuations. At the typical business cycle frequency range, synchronization has become gradually higher since the establishment of the monetary union in 1999. This higher synchronization was also extended to short-run fluctuations during the Great Recession but one should note that afterwards there is evidence of decoupling.

In order to take on board more than two series when assessing comovement, Croux, Forni and Reichlin (2001) have extended the dynamic correlation to the multivariate case and named this generalised measure as cohesion. Cohesion is essentially a weighted average of the dynamic correlations between

Chart 4

CONTEMPORANEOUS COMOVEMENT IN THE TIME-FREQUENCY SPACE

Portugal vs. euro area	Portugal vs. Spain



Source: Author's calculations.

all possible pairs of series within a group of variables. For instance, this measure can provide a useful summary statistic on the degree of synchronization across countries or regions while avoiding the problem of choosing a base country or region. In a similar fashion to Croux, Forni and Reichlin (2001), Rua and Silva Lopes (2012) have extended the bivariate measure proposed by Rua (2010) to the more general case in order to obtain a measure of cohesion in the time-frequency space. The wavelet-based cohesion also varies between -1 and 1 and it allows one to quantify the extent of cohesion among several series at different frequencies and investigate if such relationship has changed over time.

Let us consider the long time series for annual GDP growth provided by Angus Maddison (available at www.ggdc.net/maddison) updated with the latest IMF World Economic Outlook data. In particular, it is considered the sample period from 1871 up to 2011 for several countries (namely Austria, Belgium, Denmark, Finland, France, Germany, Italy, Netherlands, Norway, Sweden, Switzerland, UK, Portugal, Spain, Australia, New Zealand, Canada, USA, Brazil, Chile, Uruguay, Japan and Sri Lanka) accounting for almost 60 per cent of the world GDP in 1990. Using GDP weights, the resulting wavelet-based cohesion is displayed in Chart 5. A key finding emerges. The business cycle synchronization has never been as high as the observed during the latest economic and financial crisis, when one considers the last 140 years. This evidence unveils the widespread nature of such event and the current degree of the world economic integration.

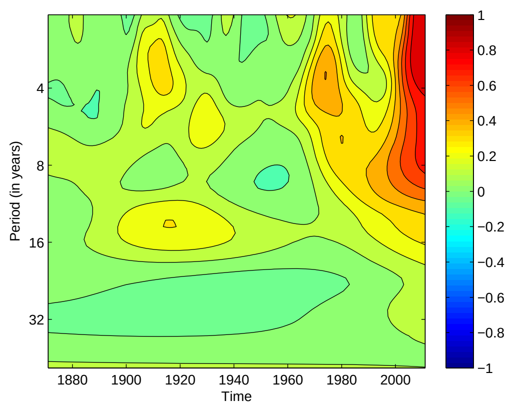
4. Conclusions

The aim of this article is to motivate the reader to the usefulness of wavelet analysis in economics. However, the above discussion does not intend to be an exhaustive description of wavelet analysis. Instead, the goal of the article is to provide an intuitive and brief overview of the main tools related with the continuous wavelet transform. Firstly, the basic concepts underlying wavelet analysis are addressed as well as its relationship with the standard Fourier analysis. Afterwards, some empirical applications are provided so as to illustrate the use of the described tools.

Despite the growing literature in the last few years, there is clearly scope to widen further the application of wavelet analysis. In fact, wavelet analysis allows one to unveil relationships between economic variables in the time-frequency space, that is, it allows one to assess simultaneously how variables are related at different frequencies and how such relationship has evolved over time. This can be of striking importance for the study of economic behaviour in a continuously changing world.

Chart 5

WORLDWIDE COHESION



Source: Author's calculations.

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