

ON THE CYCLICAL SENSITIVITY OF REAL WAGES*

*Pedro Portugal***

*Anabela Carneiro****

*Paulo Guimarães*****

“O que está na pessoa se deve estimar: tudo o mais é da fortuna”

“Quem muito estima cousas pequenas, nunca faz nenhuma grande”

D. Francisco de Portugal, 1º Conde de Vimioso

Real wage flexibility in Portugal

The Portuguese labour market has, for many decades, been conspicuous on account of its low, strongly counter-cyclical, unemployment rates. Based on widespread evidence of the very low mobility level of labour, researchers have naturally pointed to real wage flexibility as being the main reason for this result. Over the course of this period, there has been an accumulation of convincing research, suggesting a strong cyclical sensitivity of aggregate wages to the unemployment rate.¹ Reference should be made to the following cautionary note, set out in the box “*Real wage flexibility in Portugal*”, in the 1998 report of the Banco de Portugal.

“It should be borne in mind that the configuration influencing the institutional framework governing the Portuguese labour market over the course of the last decade, was contemporaneous with relatively high price growth. The recent change in the monetary system, associated with a regime of low inflation may be the cause of significant and unpredictable changes in the parameters defining the determination of wages in Portugal, i.e. the conditions governing real wage flexibility may not be necessarily guaranteed in the future, under the Portuguese economy’s new framework.”

In a similar vein, the following concern was expressed in the box entitled “*Nominal and Real Wage Rigidity: A Microeconomic Approach*” in the 2004 report of the Banco de Portugal:

“Strong nominal wage rigidity may, in a low inflation regime, seriously constrain corporate behaviour and lead firms, when confronted with the need to react to negative shocks (which could, for example, be triggered by an intensification of market product competition, to prefer to adjust employment

* Over the course of its various “reincarnations” this work has benefited from the most helpful comments and suggestions of Marta Abreu, Bernardino Adão, António Antunes, Isabel Horta Correia, Cláudia Duarte, Ricardo Félix, Vítor Gaspar, José António Machado, Carlos Robalo Marques, Pedro Martins, Maximiano Pinheiro and Pedro Raposo. The authors are also grateful to Lucena Vieira for her competent, diligent and efficient computational assistance. Finally, a special thank goes to Sónia Torres for providing the statistical series. The opinions expressed in the article are those of the authors and do not necessarily coincide with those of Banco de Portugal or the Eurosystem. Any errors and omissions are the sole responsibility of the authors.

** Banco de Portugal, Economics and Research Department.

*** Universidade do Porto, CEF.UP.

**** Universidade do Porto, University of South Carolina, CEF.UP.

(1) See Luz and Pinheiro (1993), Gaspar and Luz (1997), Dias, Esteves and Félix (2004), Marques (2009) and Martins, Marques and Portugal (2009).

rather than wages, thus increasing unemployment.”

With historically high unemployment levels, a fresh look should be taken at the relationship between wages and the unemployment rate. An important dimension of wage flexibility is real wage cyclical-ity and how real wages react to changes in economic activity. Employment adjustments over the aggregate labour demand curve trigger a counter-cyclical reaction of real wages. Alternatively, the inter-temporal substitution of labour by leisure over the dynamic labour supply curve will generate real wage cyclical-ity in tandem with the economic cycle.

Heterogeneity of workers, firms and jobs

A characterisation of the behaviour of real wages may, however, be partly obfuscated by changes in the composition of the workforce over the course the cycle. The use of aggregate wage indicators irremediably confounds the effect of changes in wage dispersion, distribution of hours worked and composition of the labour force. An additional implication of the use of aggregate data is the implicit assumption that the relationship between real wages and the economic cycle is the same for all workers, establishments, industries or professions.

The bias introduced by a greater tendency towards retaining skilled workers in recessionary peri-ods (and the hiring of unskilled workers in periods of expansion) has been well documented in the literature. In the absence of any control for worker heterogeneity, this generates an illusion of wage counter-cyclical-ity. The change in the sectoral composition of employment over the economic cycle, comprising the creation and destruction of firms may also spuriously influence the wage cyclical-ity measurements. In turn, the emergence of well paid jobs during an expansion and the change to low paid jobs in the downward stage of the economic cycle will tend to artificially amplify the indication of wage cyclical-ity *vis-à-vis* the economic cycle (Gertler and Trigari, 2009).

This article aims to reassess the real wage cyclical-ity in Portugal, taking into consideration the het-erogeneity of workers, jobs and pay policies.² The study requires access to longitudinal databases with an unusually high level of information content and the use of estimation techniques particularly suited to the presence of various types of observed and unobserved heterogeneity. To this end, we made use of the individual records of the Quadros de Pessoal from 1986 to 2007.

The treatment of heterogeneity firstly involved a distinction between newly hired workers and exist-ing workers within a firm, with a view to the consideration of the difference, emphasised by econom-ic theory, in the wages of newly hired employees and those of other workers. Secondly, the analysis was organised in such a way as to permit the conditional inference of the observation of individual characteristics (namely, gender, age and workers' academic qualifications). Lastly, the estimation technique used enabled to control for worker characteristics, which are unknown, but constant over the estimation period. This included motivation, discipline, creativeness or leadership skills for work-ers, corporate competence, organisational structure, market dominance or product specialisation

(2) The idea of considering simultaneously firm and job fixed effects in the estimation derived from various insightful discussions with Mark Gertler and Antonella Trigari.

for firms and task complexity, complementarity with sophisticated equipment, etc. for jobs. In sum, a set of fixed effects per worker, firm and job, i.e. a collection of dummy variables for each of the effects was included in the regression model. An algorithm guaranteeing the exact solution to the ordinary least squares estimation involving a linear regression model with various high-dimensional fixed effects was specially developed by Carneiro, Guimarães and Portugal (2010). To the best of the authors' knowledge, an estimation with three types of high-dimensional fixed effects has never before been presented.³

An econometric model with three high-dimensional fixed effects

The basic specification of the regression equation is set out below:

$$\ln w_{ijt} = \lambda_i + \gamma_f + \theta_j + x_{it}\beta + \alpha t + \delta t^2 + \varphi \text{ingresso}_{ijt} + \xi_s \text{ciclo}_t + \xi_h \text{ciclo}_t * \text{ingresso}_{ijt} + u_{ijt}$$

in which w_{ijt} is the monthly inflation-adjusted wage of an individual i , in a job j of the firm f , in the period t . λ_i represents a worker fixed effect, γ_f a firm fixed effect and θ_j a job fixed effect.⁴ x is a vector of time-varying individual observable characteristics, such as age (and its square) and education. t and t^2 define a quadratic time trend and the variable ciclo corresponds to the economic cycle indicator. As we are particularly interested in comparing the real wage cyclicalities of workers who stay with a firm for consecutive years and newly hired workers, a dummy variable indicating the presence of a newly hired worker (tenure less than 12 months) and an interaction term between the latter and the cycle indicator was also included in the model. The parameters of interest are ξ_s and ξ_h . When the unemployment rate is used as the economic cycle indicator, ξ_s measures the semi-elasticity of real wages in relation to the unemployment rate for workers who stay with a firm for two consecutive years, and ξ_h the increase in the same semi-elasticity associated with newly hired workers.

It is easy, albeit time-consuming, to estimate a regression model with high-dimensional fixed effects

As our database consists of a panel of annual observations with multiple observations per worker and firm, it is possible to isolate the effect of these observed and unobserved characteristics of workers, firms and jobs which remain constant over time. This can be achieved through the introduction of fixed or random effects, although the fixed effects option is preferable as it does not impose any restriction on the correlation between observed and unobserved effects.

The introduction of a fixed effect in a linear regression model is relatively simple. For example, the introduction of a fixed effect per firm consists on the introduction of a firm-specific dummy variable. If there is a very large number of firms, as in the case under consideration, the application of the least squares estimator formula requires a large scale matrix inversion which makes it impractical. Fortunately, there is a simple solution to the least squares problem that does not require the introduction

(3) Readers who are less than "enchanted" by the "magic" of the estimation of this type of model may, at no great risk, abstain from reading the next two sections. All such readers need to do is conceive that it is possible to estimate a model with 6,171,261 dummies identifying each worker, 520,147 dummies identifying each firm and 108,035 dummies identifying each professional category.

(4) Jobs are identified on the basis of a worker's professional category as defined in the collective wage bargaining agreement.

of dummy variables in the regression. This is known as the “within-groups” estimator and derives from the direct application of the Frish-Waugh-Lovell regression theorem. In practical terms, the within transformation consists in subtracting the average of each group (in this case firms) to the original variables of the model.

The issue becomes more complicated when it involves two or more high dimensional fixed effects. This is the situation we are dealing with in which the aim is to control simultaneously for fixed effects in 520,147 firms, 6,171,261 workers and 108,035 jobs. In this case, even if the estimation of one of the fixed effects using the within-groups estimator is avoided, it would still be impractical to estimate a model including dummy variables for the other fixed effects. In a prominent work, Abowd *et al* (1999) dealt with the problem of the estimation of a linear regression model which includes two high dimensional fixed effects and suggested several approximated solutions for the least square estimates. This was followed up by Abowd *et al* (2002) who suggested an iterative algorithm which provided the exact solution to this problem.

In practice, the implementation of this latter method, when applied to very large databases, has been problematical. There is also no obvious way of extending the method to more than two high dimensional fixed effects. In light of such problems, a cyclical algorithm leading to the exact solution of a linear regression model with two or more high-dimensional fixed effects was developed. The algorithm, although simple and slow, has the advantage of requiring relatively little in terms of computational resources. In the Appendix we present a detailed description of the algorithm.

Wage bargaining in Portugal

Private sector workers’ wages, in Portugal, are subject to two levels of restriction. The first, simply defines the guaranteed minimum wage which establishes a wage floor for most workers. The second level is defined on the basis of collective bargaining between employers’ associations and the unions which stipulates the minimum wage for each professional category.

The wage agreement comprising the central, albeit not unique, element of the negotiations, may derive from sectoral (the mostly common) firm or multi-firm agreements. Notwithstanding the fact that the agreement is only legally binding upon the parties involved in the negotiations, i.e. workers who are union members and firms which are affiliated with employers’ associations, the Ministry of Labour and Social Solidarity (MTSS) systematically uses extension mechanisms to broaden the coverage of the collective wage bargaining agreement to all firms and workers in the sector.

Firms often have advantage in paying their workers more than the amounts agreed in the collective bargaining. The main motivation of firms in paying wages higher than those set out in the agreements is to retain workers who, having been involved in selection and training processes, demonstrate that they are well suited for their respective jobs.

Cyclical sensitivity of real wages

The cyclical sensitivity of real wages may be condensed in the reaction of wages to the unemployment rate. The first line of Table 1 shows the semi-elasticity of real wages to the unemployment rate, when only exercising control over the observed heterogeneity across individuals.⁵ Accordingly, a one percentage point increase in the unemployment rate corresponds to a 1.685 per cent decrease in the real wages of workers who stay with a firm for consecutive years and a reduction of 2.319 (1.685+0.634) per cent in the real wages of newly hired workers.⁶

This evolution could evidently derive both from a change in the composition of employment and a modification in the reaction of wages, when the workforce over the period in question is homogeneous. The consideration of the effects associated with the unobserved characteristics of workers shows that a significant part of the loss of cyclical sensitivity of real wages is effectively associated with changes in the composition of employment. The fact that a comparison between the first and second lines of Table 1 shows that the recomposition of the workforce over the course of the economic cycle generates a counter-cyclical bias is consistent with the well documented fact that in times of recession firms tend to retain their most highly skilled workers.

In contrast, the sectoral recomposition of employment deriving from the creation and destruction of firms over the economic cycle, tends to generate a pro-cyclical bias (line three of Table 1).

Lastly, also controlling for the heterogeneity of jobs, considerably increases wage cyclicality *vis-à-vis*

Table 1

| REAL WAGE CYCLICALITY (N =30 906 573) | | | |
|--|-------------------|--|-------------------|
| Cyclical variable | OLS | | |
| | Stayers | | New-hires |
| Unemployment rate | -1.685 (0.336) | | -0.634 (0.317) |
| Worker Fixed Effect | | | |
| Unemployment rate | -1.883 (0.384) | | -0.814 (0.222) |
| Worker and Firm Fixed Effect | | | |
| Unemployment rate | -1.683 (0.320) | | -0.503 (0.162) |
| Worker, Firm and Job Fixed Effect | | | |
| Unemployment rate | -2.094 (0.384) | | -0.594 (0.147) |

Source: *Quadros do Pessoal* (1986-2007).

Note: Robust standard errors in parenthesis.

(5) The dependent variable is defined as the monthly inflation-adjusted wage, excluding overtime. The regressions include, as independent variables, in addition to cyclical variables, a quadratic trend, the indication of the presence of a newly hired worker (with less than 12 months in the job), as well as gender, age (and its square) and a worker's academic qualifications.

(6) The conventional standard error estimates associated with the estimator of the coefficient of the variable which measures the economic cycle will tend to be dramatically undervalued, as this variable contains only temporal variation. The problem is expeditiously resolved through the calculation of a clustered-robust standard error.

vis the unemployment rate. A one percentage point increase in the unemployment rate induces a 2.094 per cent reduction in the real wages of a worker who stays with a firm for two consecutive years and a 2.688 per cent reduction in the case of a newly hired worker. This result appears to suggest that the unobserved component of the recomposition of jobs follows a cyclical trajectory similar to the unobserved component of the recomposition of the workforce, reinforcing the idea of a more pronounced trend towards the maintenance of better paid jobs in recessionary periods with the most poorly paid jobs being filled in periods of expansion.

In short, the estimates set out in Table 1 are generally indicative of wage flexibility over the period 1986-2007, particularly in the case of newly hired workers.

Wage agreements and wage cushions

In Cardoso and Portugal (2005), the wage cushion is defined as being the log difference between the observed real basic monthly wage and the real monthly wage agreed in the wage bargaining negotiations for the respective professional category. Curiously, it has been noted that the greater real wage cyclical of newly hired workers has been decisively affected by the cyclical of the wage cushion. The wages agreed in wage bargaining negotiations are highly sensitive to the evolution of the unemployment rate both in the case of workers who stay with a firm or newly hired workers, although, in the latter case firm-specific contractual dispositions also play an important role in explaining the behaviour of real wages over the cycle (Table 2).

A revealing breakdown of the unemployment rate⁷

A particularly illustrative way of characterising the evolution of the unemployment rate is to decompose this indicator into the probability of finding and the probability of losing a job (Charts 1 and 2).⁸ The behaviour of these indicators is, once again, indicative of the well known sclerosis of the Portuguese labour market, translating into a low intensity of flows between unemployment and employment and between employment and unemployment (Blanchard and Portugal, 2001; Varejão and Portugal, 2007). The average probability of the value of finding a job (19.5 per cent) is less than half of the estimate of the US economy's 46 per cent (Shimer, 2005). In turn, the average value of the probability of losing a job, at 1.4 per cent, is less than half of the estimate of 3.5 per cent for the US economy.

In this context, it is of interest to note that in the Portuguese labour market it is the duration of unemployment (inverse to the probability of finding a job) which is the most decisive factor in the evolution of the unemployment rate (Torres, 2009).

(7) We are grateful for Olivier Blanchard's suggestion on how to decompose the unemployment rate.

(8) The calculation of these probabilities complies with Shimer's formulation (2005) which has minimum information requirements (Torres, 2009). In the case of the probability of losing a job, the expression is the ratio between the number of short term unemployed (less than three month), u_{t+1}^s , and the volume of employment, e_t : $\frac{u_{t+1}^s}{e_t}$. In the case of the probability of finding a job the equations is as follows: $\frac{u_t - u_{t+1} - u_{t+1}^s}{u_t}$, in which u_t is the unemployment pool for quarter t . The unemployment series was harmonized to permit the use of the definition of unemployment in a restricted sense over the whole of the period.

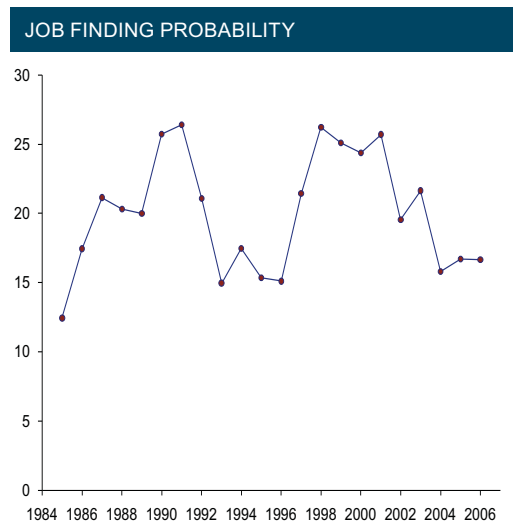
Table 2

| REAL WAGE CYCLICALITY BY WAGE DEFINITION (N =30 906 573) | | |
|--|-----------------------------------|-------------------|
| Cyclical variable | Worker, firm and job fixed effect | |
| | Bargained Wage | |
| | Stayers | New-hire |
| Unemployment rate | -1.981 (0.410) | -0.055 (0.054) |
| | Wage Cushion | |
| | Stayers | New-hire |
| Unemployment rate | -0.113 (0.184) | -0.539 (0.122) |
| | Monthly Wage | |
| | Stayers | New-hire |
| Unemployment rate | -2.094 (0.384) | -0.594 (0.147) |
| | Hourly Wage | |
| | Stayers | New-hire |
| Unemployment rate | -2.197 (0.597) | -0.482 (0.157) |

Source: *Quadros do Pessoal* (1986-2007).

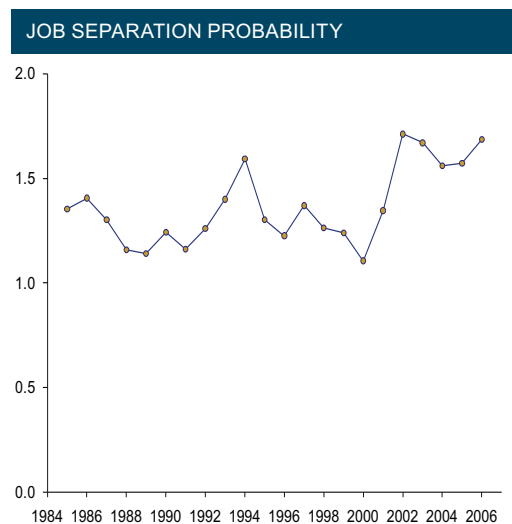
Note: Robust standard errors in parenthesis.

Chart 1



Sources: *INE* and Sónia Torres calculations.

Chart 2



Sources: *INE* and Sónia Torres calculations.

Real wages, in general, react both to the change in the probability of finding a job and to the change in the probability of losing a job. Accordingly, a one percentage point increase in the probability of finding a job will correspond to a 0.51 per cent increase in the real wages of workers who start a new job (line two of Table 3). As opposed to the above, a one percentage point increase in the probability of losing a job will correspond to a 9.5 per cent decrease in the real wages of workers who start a new job (line three of Table 3). However, when the bands between the changes of the two probabilities are considered, the magnitude of the effect of the aggregate indicators on wages is similar.⁹

Table 3

| Cyclical variable | REAL WAGE CYCLICALITY BY CYCLICAL VARIABLE (N =30 906 573) | |
|----------------------------|--|-------------------|
| | Worker, firm and job fixed effect | |
| | Stayers | New-hires |
| Unemployment Rate | -2.094 (0.384) | -0.594 (0.147) |
| Job Separation Probability | 0.418 (0.075) | 0.088 (0.026) |
| Job Finding Probability | -10.3 (4.2) | 0.8 (1.2) |

Source: *Quadros do Pessoal* (1986-2007).

Note: Robust standard errors in parenthesis.

The recent fall in wage cyclicality

Lastly a re-examination of this issue of the real wage cyclicality, in light of the new institutional framework imposed by euro membership in 1999, is of interest.

To analyse the impact of the change in the monetary system in Portugal on the relationship between real wages and the unemployment rate, the base model was estimated to include a binary variable identifying the period 1999-2007.

The estimates presented in the first panel of Table 4 show evidence of strong wage flexibility for the period 1986-1998, particularly for newly hired workers. There is, however, a clear indication of a significant decrease in the real wage cyclicality since 1999. Semi-elasticity, in the post-membership period was 1.808 percentage points lower for stayers and 2.468 (1.808+0.660) percentage points lower for newly hired workers.

However, admitting that there is a certain time lag in the transmission of the effect of the change in the monetary system to the labour market, and, in particular to wage bargaining, then the indication of the drop in wage cyclicality is even more expressive. For workers who stay with a firm for two consecutive years the effect associated with the unemployment rate for the period 2002-2007 is virtually nil, whereas for newly hired workers the semi-elasticity of wages to the unemployment rate

(9) These results are consistent with those achieved by Addison, Machado and Portugal (2010) who show that reservation wages diminish with the length of unemployment and with Carneiro and Portugal (2008) who establish a negative relationship between wages and the probability of separation through dismissal.

Table 4

| THE FALL IN THE REAL WAGE CYCLICALITY (N = 30 906 573) | | | | |
|--|-----------------------------------|-----------------------|-------------------|-----------------------|
| Cyclical variable | Worker, firm and job fixed effect | | | |
| | Stayers | | New-hires | |
| | Coefficient | Change in coefficient | Coefficient | Change in coefficient |
| | 1986-1998 | 1999-2007 | 1986-1998 | 1999-2007 |
| Unemployment Rate | -2.612 (0.709) | 1 808 (1.213) | -0.990 (0.175) | 0.66 (0.215) |
| | 1986-2000 | 2001-2007 | 1986-2000 | 2001-2007 |
| Unemployment Rate | -2.460 (0.560) | 2.462 (1.213) | -0.955 (0.117) | 0.612 (0.165) |

Source: *Quadros do Pessoal* (1986-2007).

Note: Robust standard errors in parenthesis.

is minus 0.343 per cent. It is, to a certain extent, hardly reassuring to know that these results do not derive from changes in the composition of workers, firms or professional categories, as these results are obtained by controlling for the composition effects.

With due proviso for the care to be taken over the interpretation of these results, owing to the uncertainty of the timeline associated with the measurement of the economic cycle, claim may naturally be made to the loss of real wage sensitivity as one of the mechanisms responsible, inter alia, for the Portuguese economy's current macroeconomic imbalances (Blanchard, 2007).

On the inadequate architecture of the wage formation system in Portugal

In the debate over the Portuguese economy the essentiality of the link between unemployment and wage behaviour is mysteriously ignored. In the interpretation of these authors, the continued increase in the duration of unemployment, in conjunction with the drop in the sensitivity of real wages to conditions in the Portuguese labour market, reflects mirrors an evolution of bargained wages that are out of sync with feasible wages.¹⁰

The indication of the decrease in real wage cyclicity may be associated with the inadequacy of mechanisms for determining wages in low inflation regimes. The use of across-the-board procedures to ensure that agreements are extended to cover the whole of the sector tend to exacerbate the already strong nominal wage rigidity.

In turn, the growing generosity of the unemployment benefit system, particularly as regards the potential duration has undoubtedly facilitated the tolerability of unemployment status, in favour of persistently higher and less sensitive reservation wages and therefore higher unemployment.

In the Portuguese economy's current institutional framework, the successive increases in labour costs in the form of increases in minimum wages, impacting particularly on the marginal adjustment fringes of the Portuguese labour market, may lead to an undesirable fall in the arrival rate of job offers and, accordingly, a higher rate of unemployment.

(10) See Blanchard and Portugal (2001).

The current architecture of the wage formation system no longer appears capable of guaranteeing necessary real wage flexibility. In this context, and as in the case of other European countries, consideration of the introduction of decentralised wage bargaining mechanisms allowing firms to opt in or opt out of sectoral wage agreements appears to be an effective route to promote wage flexibility.

REFERENCES

- Abowd, J., R. Creecy and F. Kramarz, "Computing Person and Firm Effects Using Linked Longitudinal Employer-Employee Data", *mimeo*, 2002.
- Abowd, J., F. Kramarz and D. Margolis, "High Wage Workers and High Wage Firms", *Econometrica*, 67, 1999.
- Addison, J. T., Machado, J. A. and Pedro Portugal, "The Reservation Wage Unemployment Nexus," Banco de Portugal, Working Paper 26, 2010.
- Blanchard, O., "Adjustment within the euro. The difficult case of Portugal," *Portuguese Economic Journal*, 6, 2007.
- Blanchard, O. and P. Portugal, "What Hides behind an Unemployment Rate: Comparing Portuguese and U. S. Labor Markets", *American Economic Review*, 91, 2001.
- Cardoso, A. and P. Portugal, "Contractual Wages and the Wage Cushion under Different Bargaining Settings", *Journal of Labor Economics*, 23, 2005.
- Carneiro A. and P. Portugal, "Wages and the Risk of Displacement," *Research in Labor Economics*, 18, 2008.
- Carneiro, A., Guimarães, P. and P. Portugal, "Real Wages and the Business Cycle: Accounting for Worker, Firm, and Job Title Heterogeneity", *mimeo*, 2010.
- Dias, F., Esteves, P. and R. Félix, "Uma Nova Avaliação das Estimativas da NAIRU para a Economia Portuguesa", Banco de Portugal, *Boletim Económico - Primavera 2004*.
- Gaspar, V. and S. Luz, "Desemprego e Salários em Portugal", Banco de Portugal, *Boletim Económico - Dezembro 1997*.
- Gertler, M. and A. Trigari (2009), "Unemployment Fluctuations with Staggered Nash Bargaining", *Journal of Political Economy*, 117.
- Guimarães, P. and P. Portugal, "A Simple Feasible Alternative Procedure to Estimate Models with High-dimensional Fixed Effects", *Stata Journal*, 4, 2010.
- Luz, S. and M. Pinheiro, "Desemprego, Vagas e Crescimento Salarial", Banco de Portugal, *Boletim Trimestral - Junho 1993*.
- Marques, C. , "Wage and Price Dynamics in Portugal", *ECB working paper*, 2009.
- Shimer, R., *Reassessing the Ins and Outs of Unemployment*, NBER, 2005.
- Torres, S. (2009), *Unemployment, Worker Flows and Wages Formation in the Portuguese Labour Market, Doctoral Dissertation in Economics*, Universidade do Porto, Faculdade de Economia.
- Varejão J. and P. Portugal, "Employment Dynamics and the Structure of Labor Adjustment Costs", *Journal of Labor Economics*, 25, 2007.

APPENDIX

To understand how the algorithm operates we begin by defining a linear regression model in matrix form including a fixed dimensional effect n ,

$$Y = X\beta + D1\alpha + \varepsilon$$

Here X is a matrix with a dimension $M \times k$ containing the observed explanatory variables, $D1$ is a matrix with a dimension $M \times n$ containing the n dummy variables relative to the fixed effect and α and β are the vectors with the model's regression coefficients. Knowledge of the least squares solution for α would enable us to calculate the vector $D1\alpha$ (with the dimension $M \times 1$), adding it as an explanatory variable to a linear regression between Y and X . This strategy would enable the least square estimates of β to be obtained. On the other hand, knowledge of the least squares solution for β , would easily enable us to obtain the least squares estimates for α .

In this case, the least square estimates for α would be the group averages of the elements of the vector $u = Y - X\beta$. This means that the model may be estimated using an algorithm which alternates between the estimation of α and β . A possible strategy for implementing the algorithm would be the following sequence of iterations:

- 1) Obtain the initial values for β by regressing Y on X ;
- 2) Calculate the residuals u using the last estimate of β ;
- 3) Estimate α calculating the averages per group of elements of u ;
- 4) Estimate β by regressing Y on X and including an additional variable, $D1\alpha$, calculated as the last estimate of α ;
- 5) Return to step 2 and iterate until obtaining convergence;

It should be noted that this algorithm only requires the estimation of regressions with $k+1$ explanatory variables and the calculation of averages per group of regression residuals.

Although it works, this strategy is inefficient because, as we have already discussed, estimates β could be calculated by running of the transformed Y on the transformed and X variables. Let us now consider a situation in which there are two fixed high dimensional effects. The situation can be represented, as set out below,

$$Y = X\beta + D1\alpha + D2\gamma + \varepsilon$$

in which $D2$ is a matrix with a dimension $M \times p$ containing p columns indicating membership to the second group and γ is a vector of parameters.

In this case, the above proposed algorithm may be easily modified to accommodate this new situation. Now, we only need to alternate between the solution of β , α and γ . Accordingly, to estimate β we implement the regression from Y on X , including two additional variables containing the values of α and γ for each observation. At each step, we obtain estimates for α calculating the group averages of the residuals $u = Y - X\beta - D2\gamma$ with the estimates of γ being similarly obtained. This means that we

can obtain the exact least squares solution without the need to invert a high dimensional matrix. The implementation of the algorithm requires the calculation of various regressions with $k+2$ explanatory variables and averages per group of estimation residuals.

If we wish to include a third fixed effect in the regression, we can implement the above described regressions using the within-groups estimator to avoid the direct estimation of third effect coefficients, i.e. the only requirement is that we subtract the average calculated for the groups comprising the third fixed effect from all of the variables entering into the regressions of the above described algorithm.

A noted disadvantage of this method is the slow convergence rate of the cyclical algorithms. However, it is possible to accelerate the algorithm by retaining the estimates of α (or γ) produced in the last iterations and using them to adjust the convergence trajectory of the estimates of the fixed effect coefficients.

The standard error estimates associated with the estimation of β may also be obtained avoiding the inversion of a high dimensional matrix. The standard error estimates can be calculated through the application of the Frisch-Waugh-Lovell regression theorem. The strategy consists of firstly expurgating the three fixed effects (using the above described algorithm) of the Y and each of the X variables and then running regression between the transformed Y and X variables. This regression, in addition to producing the correct estimates for β , also produces the correct estimates of the standard errors (whether or not robust) provided that the degrees of freedom associated with the estimate of the variance of the perturbation term are correct. For more details see Guimarães and Portugal (2010).