A Macroeconomic Model with a Financial Sector.[†]

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PRELIMINARY AND INCOMPLETE

ABSTRACT. This paper studies a macroeconomic model in which financial experts borrow from less productive agents. We pursue four sets of results: (i) The economy is prone to instability and occasionally enters volatile episodes. As volatility spikes agents precautionary motive increases depressing prices even further. Log-linear approximations fail to capture these non-linear effects that can cause economies to be significantly depressed for long periods of time. (ii) Endogenous risk during volatile episodes increases asset price correlations. (iii) Financial experts impose a negative externality on each other and on the labor sector by not maintaining adequate capital cushion, and funding structure. (iv) While risk sharing within the financial sector (through securitization and derivatives contracts) reduces many inefficiencies, it can also amlify systemic risks.

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1. Introduction

Many standard macroeconomic models are based on *identical* households that invest directly without financial intermediaries. This *representative agent* approach can only yield realistic macroeconomic predictions if, in reality, there are no frictions in the financial sector. Yet, following the Great Depression, economists such as Fisher (1933), Keynes (1936) and Minsky (1986) have attributed the economic downturn to the failure of financial markets. The current financial crisis has underscored once again the importance of the financial sector for the business cycles.

Central ideas to modeling financial frictions include *heterogeneous* agents and *leverage*. One class of agents - let us call them *experts* - have superior ability or greater willingness to manage and invest in productive assets. Because experts have limited net worth, they end up borrowing from the second class of agents - let us call them *households* - who are less skilled at managing or less willing to hold productive assets.

Existing literature uncovers two important properties of these models, *persistence* and *amplification*. Persistence is related to the wealth distribution between the two types of agents: low net worth of experts in a given period results in depressed economic activity, and low net worth of experts in the next period. The causes of amplification are leverage and the feedback effect of prices. Through leverage, expert net worth absorbs a magnified effect of each shock, such as new information about the potential future earning power of current investments. When the shock is aggregate, affecting many experts at once, it results in decreased demand for assets and a drop in asset prices, further lowering the net worth of experts, further feeding back into prices, and so on. Thus, each shock passes through this infinite amplification loop, and asset price volatility created through this mechanism is sometimes referred to as *endogenous risk*. Bernanke, Gertler and Gilchrist (1999) and Kiyotaki and Moore (1997) build a macro model with these effects, and study linearized system dynamics around the steady state.

In this paper, we emphasize the feedback between volatility dynamics and precautionary hoarding motive. As volatility increases experts are increasingly concerned about hitting the funding constraint in the future, leading to depressed prices. The precautionary effects add to the prevalent loss spiral: an initial shock erodes net worth of leveraged expert investors leading to lower prices and even further losses. We build a model to study full equilibrium dynamics, not just near the steady state, and argue that steady-state analysis misses important effects. Specifically, while the system is characterized by relative stability, low volatility and reasonable growth for the most part, occasional large losses can plunge the system into a regime with high volatility. These crises episodes are highly nonlinear, and strong amplifying feedback loops during these incidents may take the system way below the steady state, resulting in significant inefficiencies, disinvestment, and slow recovery. Interestingly, the stationary distribution is double-humped shaped suggesting that (without government intervention) the dynamical system spends a significant amount of time in the crisis state once thrown there.

The amplification of shocks through prices is much milder near the steady state than below the steady state in our model because experts choose their capital cushions endogenously. In the normal regime, experts choose their capital ratios to be able to withstand reasonable losses. Excess profits are paid out (as bonuses, dividends, etc) and mild losses are absorbed by reduced payouts to raise capital cushions to a desired level. Thus, normally experts are fairly unconstrained and are able to absorb moderate shocks to net worth easily, without a significant effect on their demand for assets and market prices. However, in response to more significant losses, experts choose to reduce their positions, affecting asset prices and triggering amplification loops. The stronger asset prices react to shocks to the net worth of experts, the stronger the feedback effect that causes further drops in net worth, due to depressed prices. Thus, it follows that below the steady state, when experts feel more constrained, the system becomes less stable as the volatility shoots up.

While original shocks affect the values of individual assets held by experts, feedback effects affect the prices of all assets held by experts. As a result, endogenous risk and excess volatility created through the amplification loop makes asset prices significantly more correlated cross-sectionally in crises than in normal times.

There are externalities - generally experts lever up too much funded with short-term debt by taking on too much risk and by paying out funds too early. Experts impose an externality on the labor sector since when choosing their leverage they do not take fully into account the costs of adverse economic conditions that result in crises. Also, there are ‘firesale’ externalities within the financial sector when households can provide a limited liquidity cushion by absorbing some of the assets in times of crises. When levering up, experts do not take into account that they hurt other experts’ ability to sell to households in times of crises. On top of it, low fire-sale prices also lower the fraction of outside equity financial experts can raise from households in times of crisis. Put together, this can also lead to overcapacity.

Finally, we study the effects of securitization and financial innovation. Securitization of home loans into mortgage-backed securities allows institutions that originate loans to unload some of the risks to other institutions. More generally, institutions can share risks through contracts like credit-default swaps, through integration of commercial banks and investment banks, and through more complex intermediation chains (e.g. see Shin (2010)). To study the effects of these risk-sharing mechanisms on equilibrium, we add idiosyncratic shocks to our model. We find that when expert can hedge idiosyncratic shocks among each other, they become less financially constrained and take on more leverage, making the system less stable. Thus, while securitization is in principle a good thing - it reduces the costs of idiosyncratic shocks and thus interest rate spreads - it ends up amplifying systemic risks in equilibrium.

Literature review. Financial crises are common in history - having occurred at roughly 10-year intervals in Western Europe over the past four centuries, according Kindleberger (1993). Crises have become less frequent with the introduction of central banks and regulations that include deposit insurance and capital requirements (see Allen and Gale (2009) and Cooper (2008)). Yet, the stability of the financial system has been brought into the spotlight again by the events of the current crises, see Brunnermeier (2009).

The existence of the financial system is premised on the heterogeneity of agents in the economy – lenders and borrowers. In Bernanke and Gertler (1989), entrepreneurs have special skill and borrow to produce. In Kiyotaki (1998), more productive agents lever up by borrowing from the less productive ones, in Geanakoplos (2003) more optimistic and in Garleanu and Pedersen (2009) less risk-averse investors lever up. Intermediaries can facilitate lending – for example Diamond (1984) shows how intermediaries reduce the cost of borrowing. Holmström and Tirole (1997, 1998) also propose a model where both where both intermediaries and firms are financially constrained. Philippon (2008) looks at the financial system plays in helping young firms with low cash flows get funds to invest. In these models, financial intermediaries are also levered.

Leverage leads to amplification of shocks, and prices can play an important role in this process. Negative shocks erode borrowers' wealth, and impair their ability to perform their functions of production or intermediation. Literature presents different manifestations of how this happens. Shleifer and Vishny (1992) argue that when physical collateral is liquidated, its price is depressed since natural buyers, who are typically in the same industry, are likely to be also constrained. Brunnermeier and Pedersen (2009) study liquidity spirals, where shocks to institutions net worth lead to binding margin constraints and fire sales. The resulting increase in volatility brings about a spike in margins and haircuts forcing financial intermediaries to delever further. Maturity mismatch between the assets that borrowers hold and liabilities can lead to runs, such as the bank runs in Diamond and Dybvig (1983), or more general runs on non-financial firms in He and Xiong (2009). Allen and Gale (2000) and Zawadowski (2009) look at network effects and contagion. In Shleifer and Vishny (2009) banks are unstable since they operate in a market influenced by investor sentiment.

These phenomena are important in a macroeconomic context – and many papers have studied the amplification of shocks through the financial sector near the steady state, using log-linearization. Prominent examples include Bernanke, Gertler and Gilchrist (1999), Carlstrom and Fuerst (1997) and Kiyotaki and Moore (1997) and (2007). More recently, Christiano, Eichenbaum and Evans (2005), Christiano, Motto and Rostagno (2005, 2007), Cordia and Woodford (2009), Gertler and Karadi (2009) and Gertler and Kiyotaki (2009) have studied related questions, including the impact of monetary policy on financial frictions.

We argue that the financial system exhibits the types of instabilities that cannot be adequately studied by steady-state analysis, and use the recursive approach to solve for full equilibrium dynamics. Our solution builds upon recursive macroeconomics, see Stokey and Lucas (1989) and Ljungqvist and Sargent (2004). We adapt this approach to study the financial system, and enhance tractability by using continuous-time methods, see Sannikov (2008) and DeMarzo and Sannikov (2006).

A few other papers that do not log-linearize include He and Krishnamurthy (2008 and 2009) and Mendoza (2010). Perhaps most closely related to our model, He and Krishnamurthy (2008) also model experts, but assume that only experts can hold risky assets. They derive many interesting asset pricing implications and link them to risk aversion. In contrast to He and Krishnamurthy (2008) we focus on the risk-neutral case and look at not only individual asset prices, but also in cross-section. We also study system dynamics through its stationary distribution, and analyze externalities and the effects of securitization.

Our result that pecuniary externalities lead to socially inefficient excessive borrowing, leverage and volatility can be related to Bhattacharya and Gale (1987) in which externalities arise in the interbank market and to Caballero and Krishnamurthy (2004) which study externalities an international open economy framework. On a more abstract level these effects can be traced back to inefficiency results within an incomplete markets general equilibrium setting, see e.g. Stiglitz (1982) and Geanakoplos and Polemarchakis (1986). In Lorenzoni (2007) and Jeanne and Korinek (2009) funding constraints depend on prices that each individual investor takes as given. Adrian and Brunnermeier (2008) provide a systemic risk measure and argue that financial regulation should focus on these externalities.

We build our analysis around a basic model, which we present in Section 2. The basic model has only two types of agents - borrowers and lenders - and it is purposefully designed to have no externalities. We solve the basic model and illustrate how full equilibrium dynamics differs from steady-state dynamics. In Subsection 2.2 we microfound the capital structure. Subsection 2.3 takes a detour to show how the basic model fits within a broader framework, which includes the chain of intermediation. Subsection 2.4 is devoted to asset pricing implications. We study externalities in Section 3, and the effects of securitization in Section 4.

2. The Model

We follow a modular design principle. We start with a fairly simple framework and add new modeling elements and endogenize assumptions as we go along.

2.1 The Baseline Model

Model setup. We consider an economy populated by households and financial experts (who in the later part of the paper pass their funds on to more productive households). Since, experts are better at managing capital, they find it profitable to invest in projects, such as productive firms, entrepreneurial ventures, home loans, etc. This investment may be in form of an equity or risky debt stake, or in form of a derivative contract that allows the firm to manage risk more efficiently.

We assume that experts and households are risk-neutral. Households discount rate is r, while experts own discount rate is $\rho > r$. We are imagining a story in which households hold money to ensure themselves against future shocks (large purchases, accidents, etc). Because of the option value of holding money, households are willing to lend it to experts (banks) at rate r, which is lower than their discount rate.[1]

Physical capital k_t produces output at rate

$$y_t = a\, k_t.$$

where a is a parameter.

Experts can create new capital through internal investment i_t. When held by an expert, capital stock k_t evolves according to

$$dk_t = (\Phi(i_t/k_t) - \delta)\, k_t\, dt + \sigma\, k_t\, dZ_t$$

where the function $\Phi(i_t/k_t)$ reflects (dis)investment costs. A higher internal investment rate, i_t, increases the capital stock. We assume that the function $\Phi(.)$ is concave reflecting the fact that the marginal impact of internal investment on capital is decreasing. Similarly, disinvestment lowers the capital stock. Due to "technological illiquidity" large scale disinvestments are less effective. We assume that $\Phi(0) = 0$, so in the absence of new investment capital depreciates at rate δ when managed by experts. Households are less productive and do not have an internal investment technology. Also, when managed by households, capital depreciates at a faster rate $\underline{\delta} > \delta$. The law of motion of k_t when managed by households is

$$dk_t = -\, \underline{\delta}\, k_t\, dt + \sigma\, k_t\, dZ_t.$$

Capital is also subject to exogenous aggregate Brownian shocks Z_t, which reflect the fact that one learns over time how "effective" the capital stock is.[2] Note that k_t reflects the "efficiency units" of capital, measured in output rather than in simple units of physical capital (number of machines). Hence, dZ_t also captures changes in expectations about the future productivity of capital. In this sense our model is also linked to the literature on connects news to business cycles.

There is a market for physical capital, in which experts can buy and sell capital among each other, and sell it to households. Denote the market price of capital, which is determined *endogenously* in our model, by p_t, and its law of motion by

$$dp_t = \mu_t^p\, p_t\, dt + \sigma_t^p\, p_t\, dZ_t.$$

Note that p_t follows a diffusion process without loss of generality. Since the option to sell

[1] Of course, in a model with money rate r will depend on the banks' demand for deposits and the point in the economic cycle. We ignore these effects in our model.
[2] Alternatively, one can also assume that the economy experiences aggregate TFP shocks a_t. However, in order to preserve the tractable scale invariance property one has to assume that a_t-shocks are persistent and modify $\Phi(.)$ to $\Phi(i_t/y_t)$.

capital to households is always there, the Gordon growth formula tells us that in equilibrium $p_t \geq \underline{p} = a/(r+\underline{\delta})$, the households' valuation of capital. Initially we assume that if households buy capital from experts, they cannot speculate and resell back the capital to the more productive experts.

Experts' balance sheets. An essential ingredient of our model is that any expert who manages capital k_t must absorb at least a fraction of risk that affects the value of the capital. The total risk can be divided into *exogenous* risk from Brownian shocks that affect k_t directly and *endogenous* risk that affects p_t, the market valuation of k_t.

Under the simplest framework that delivers all the main results, experts hold capital on the asset side of their balance sheet and issue short-term debt, which is risk-free for one instant, and outside equity, as shown in Figure 1. Experts can only offload a fraction (1-α) of the total risk. Note that cash flows to outside investors can be split arbitrarily between debt and equity-holders, by Modigliani and Miller (1958). We choose a particular capital structure that makes debt risk-free, because it simplifies exposition.

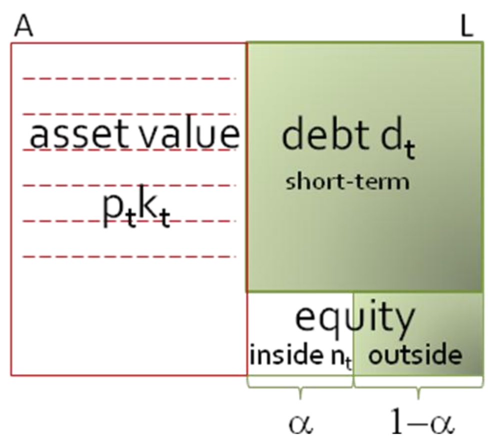

Figure 1. Expert balance sheet with inside and outside equity.

In Section 3 we justify balance sheets as an outcome of contracting, subject to informational problems. In addition, we fully model the intermediary sector that monitors and lends to more productive households.

The dynamic evolution of balance sheets. The experts' decisions how much to lever up depend not just on the current price level and individual expert's net worth, but also on the whole future law of motion of prices. That is, experts have to choose *dynamic* trading strategies to maximize their payoffs. There is a trade-off that greater leverage leads to both higher profit and greater risk. Greater risk means that experts will suffer greater losses exactly in the events when they value funds the most - after negative shocks when prices become depressed and profitable opportunities arise. The subsequent analysis shows how this trade-off leads to an equilibrium choice of leverage.

Note that experts do not fully exploit their debt capacity since they are concerned whether they can rollover their debt in the future and ultimately have to fire-sale their assets. The experts' demand for capital and the aggregate amount of capital available in the economy together determine the spot price of capital p_t, through the market-clearing condition. The experts' willingness to hold capital depends on their net worth. Thus, exogenous shocks Z_t feed into prices through their effect on the experts' net worth.

The rate of profit and risk from holding capital can be quantified from the laws of motion of k_t and p_t. Using Ito's lemma, without any sales or purchases of new capital the value of the assets on the balance sheet evolves according to

$$d(k_t p_t) = (\Phi(i_t/k_t) - \delta + \mu_t^p + \sigma\sigma_t^p)\,(k_t p_t)\,dt + (\sigma + \sigma_t^p)\,(k_t p_t)\,dZ_t.$$

The asset side of experts balance sheet increases with investment i_t by $\Phi(i_t/k_t)$ minus depreciation δ and average price increase reflected by μ_t^p. The term, $\sigma\sigma_t^p$, is due to Ito's lemma and reflects the positive covariance between the Z_t-shock to capital and price volatility.[3] The equation also has two risk terms. *Exogenous* risk $(k_t p_t)\,\sigma\,dZ_t$ comes from shocks dZ_t that directly affect k_t. In contrast, *endogenous* risk stems from the market valuation of capital p_t, which depends on the experts' willingness to hold assets and their net worth's. We will see how the level of endogenous risk in equilibrium depends on feedback effects within the financial sector and the experts' constraints. In turn a high level of endogenous risk can lead to greater *precautionary* motive, as experts hoard more cash in volatile time waiting to pick up the assets at low prices at the bottom.

In addition output ak_t net of investment i_t can be used to pay off debt. Before payouts to equity holders, debt evolves according to

$$dd_t = (r\,d_t + i_t - a\,k_t)\,dt,$$

where cash outflows like interest payment $r\,d_t$ and internal investment costs increase debt level, while $a\,k_t$ is output, i_t reduce debt level. As a result, the value of equity $e_t = p_t k_t - d_t$ changes according to

$$de_t = r\,e_t\,dt + a\,k_t\,dt - i_t\,dt + (k_t p_t)\,[(\Phi(i_t/k_t) - \delta + \mu_t^p + \sigma\sigma_t^p - r)\,dt + (\sigma + \sigma_t^p)\,dZ_t].$$

While the risk is shared proportionately between inside and outside equity holders, the expected return is not the same. Outside equity holders require an expected return of *r* on their investment of $e_t^o = (1 - \alpha)\,e_t$, so the value of outside equity evolves as

$$de_t^o = r\,(1 - \alpha)\,e_t\,dt + (k_t p_t)\,(1 - \alpha)\,(\sigma + \sigma_t^p)\,dZ_t.$$

The expert receives everything that is left after debt holders and outside equity holders are paid off. The expert's net worth $n_t = p_t k_t - d_t - e_t^o$ changes according to

$$dn_t = r\,n_t\,dt + a\,k_t\,dt - i_t\,dt + (k_t p_t)\,[(\Phi(i_t/k_t) - \delta + \mu_t^p + \sigma\sigma_t^p - r)\,dt + \alpha(\sigma + \sigma_t^p)\,dZ_t].$$

[3] The version of Ito's lemma we use is the product rule $d(X_t Y_t) = dX_t\,Y_t + X_t\,dY_t + \sigma_X\,\sigma_Y\,dt$.

In addition, experts may consume their net worth (e.g. by paying out bonuses). When this happens, the expert's net worth decreases by the amount of payout dc_t.

Equilibrium. Our strategy for solving for the equilibrium is to combine the experts' dynamic optimization problems (expressed via Bellman equations) with the market clearing conditions. Among the choices experts make, the amount of internal investment i_t is a static choice: it is optimal to maximize

$$k_t\, p_t\, \Phi(i_t/k_t) - i_t.$$

The first-order condition is $p_t\, \Phi'(i_t/k_t) = 1$ (marginal Tobin's q) implies that the optimal level of investment and the resulting growth rate of capital are functions of the price, i.e.

$$i_t/k_t = \iota(p_t) \qquad \text{and} \qquad \Phi(i_t/k_t) - \delta = g(p_t).$$

Investment $i_t = \iota(p_t)\, k_t$ maximizes the drift of n_t and has no effect on the volatility of n_t for any amount of capital k_t.

In contrast, expert choices of the amount of capital to hold k_t and the amount to consume dc_t are dynamic. A condition for the optimality of these choices can be expressed in terms of the experts' value functions, which summarize how the experts' continuation values depend on their wealth. The following lemma shows that expert value functions are proportionate to their wealth, because of the assumption that experts are atomistic and act competitively.

Lemma 1. There exists a process f_t such that the value function of any expert with net worth n_t is of the form $f_t\, n_t$.

Proof. Consider two experts 1 and 2 with net worth's n_t^1 and n_t^2. Denote by u_t^1 and u_t^2 the maximal expected utilities that these experts can get in equilibrium from time t onwards. We need to show that $u_t^1/n_t^1 = u_t^2/n_t^2$. Suppose not, e.g. $u_t^1/n_t^1 > u_t^2/n_t^2$. Denote by $\{k_s, dc_s, s \geq t\}$ the optimal dynamic trading and consumption strategy of expert 1, which attains utility u_t^1, i.e.

$$u_t^1 = E_t\left[\int_t^\infty e^{-\rho(s-t)} dc_{t+s}\right].$$

Because the strategy is feasible, the process

$$dn_s^1 = r\, n_s^1\, dt + a\, k_s\, dt - \iota(p_s)\, k_s\, dt + (k_s\, p_s)\, [(g(p_s) + \mu_s^p + \sigma\sigma_s^p - r)\, dt + (\sigma + \sigma_s^p)\, dZ_s] - dc_s$$

stays nonnegative. Let $\theta = n_t^2/n_t^1$, and consider the strategy $\{\theta k_s, \theta dc_s, s \geq t\}$ of expert 2. This strategy is also feasible, because it leads to the non-negative wealth process $n_t^2 = \theta n_t^1$, and it delivers the expected utility of θu_t^1 to player 2. Thus, $u_t^2 \geq \theta u_t^1$, leading to a contradiction.

Therefore, for all experts their expected utility under the optimal trading strategy is proportional to wealth. It follows that $f_t = u_t^1/n_t^1 = u_t^2/n_t^2$. QED

In equilibrium f_t depends on the market conditions: current asset prices and price dynamics. Denote the law of motion of f_t by

$$df_t = \mu_t^f \, dt + \sigma_t^f \, dZ_t.$$

When taking positions, experts take into account expected profit and losses, as well as the values of f_t in states where profit and losses are realized. They are willing to pay price x_t an asset that pays x_{t+s} at time t+s, such that

$$f_t \, x_t = E_t \, [e^{-\rho s} \, f_{t+s} \, x_{t+s}],$$

since the value of a dollar of net worth at time t is f_t and at time t+s, f_{t+s}. Thus, $e^{-\rho t} f_{t+s}$ is the stochastic discount factor with which experts evaluate their investment opportunities at time t. It should price any asset on the experts' balance sheets, and determine the optimal amount of investment in case of diminishing returns to scale from holding an asset (as it is the case in Section 4). Also, experts should consume, converting a dollar of net worth into a dollar of utility, only when $f_t = 1$. The following lemma formalizes this logic, and characterizes the optimal strategy of any expert.

Lemma 2. *Consider the process*

$$F_t = \int_0^t e^{-\rho s} dc_s + e^{-\rho t} n_t f_t .$$

Under the optimal strategy $\{k_t, c_t\}$ of an expert with net worth n_t, F_t is a martingale. Under any arbitrary strategy, F_t is a supermartingale.

Proof. The maximal payoff that an expert can obtain at time t is

$$n_t f_t \geq E_t\left[\int_t^{t+s} e^{-\rho(s'-t)} dc_{s'} + e^{-\rho s} n_{t+s} f_{t+s}\right],$$

with equality if the agent follows an optimal strategy between time t and $t + s$, since $n_{t+s} f_{t+s}$ is the maximal payoff that the agent can attain from time $t + s$ onwards. Therefore,

$$F_t = \int_0^t e^{-\rho s} dc_s + e^{-\rho t} n_t f_t \geq E_t\left[\int_0^{t+s} e^{-\rho s'} dc_{s'} + e^{-\rho(t+s)} n_{t+s} f_{t+s}\right] = E_t\left[F_{t+s}\right],$$

with equality if the agent follows the optimal strategy. QED

To draw a useful corollary from Lemma 2, let us differentiate F_t with respect to time t, and study the drift of F_t:

$$dF_t = e^{-\rho t}(dc_t - \rho n_t f_t + dn_t f_t + n_t df_t + \alpha(k_t p_t)(\sigma + \sigma_t^p)\sigma_t^f dt) \Rightarrow$$

$$\frac{dF_t}{e^{-\rho t}} = dc_t(1-f_t) - (\rho - r)n_t f_t dt + k_t(a - \iota(p_t))dt +$$
$$(k_t p_t)[(g(p_t) + \mu_t^p + \sigma\sigma_t^p - r)dt + \alpha(\sigma + \sigma_t^p)dZ_t] f_t + n_t df_t + \sigma_t^f \alpha(k_t p_t)(\sigma + \sigma_t^p)dt.$$

The optimal strategy $\{dc_t, k_t\}$ maximizes the drift of F_t, and the maximal drift equals zero by Lemma 2.

Proposition 1. In equilibrium

(a) $f_t \geq 1$ at all times, and experts consume only when $f_t = 1$. If ever f_t were less than 1, the drift of F_t could be made arbitrarily large by choosing large dc_t

(b) the first-order condition with respect to k_t must hold for the *market-clearing* value of k_t, which satisfies $\eta_t = n_t/k_t$. Differentiating the drift of f_t with respect to k_t, we obtain[4]

$$\frac{a - \iota(p_t)}{p_t} + g(p_t) + \mu_t^p + \sigma\sigma_t^p - r + \frac{\sigma_t^f}{f_t}\alpha(\sigma + \sigma_t^p) = 0 \qquad (*)$$

(c) By setting the drift of F_t to zero and using the first-order condition with respect to k_t, we find that the drift of f_t satisfies

$$\mu_t^f = (\rho - r) f_t \qquad (**)$$

Proof. This proposition is a direct corollary of Lemma 2.

Can we characterize equilibrium prices p_t and value functions f_t from equations (*) and (**)? In our economy, the key state variables are the aggregate expert net worth N_t across all expert of unit mass and the aggregate amount of capital K_t in the economy. Because everything is proportionate with respect to K_t, we get scale invariance and the key state variable is the ratio

$$\eta_t = N_t/K_t.$$

[4] Note that in our baseline model, if the first-order condition holds at the market-clearing value of k_t, then it holds for all k_t by linearity. This is not the case in a more general version of the model with idiosyncratic shocks ,which we study in Section 6.

Thus, in a Markov equilibrium[5] in our economy p_t and f_t are functions of η_t, so

$$p_t = p(\eta_t) \quad \text{and} \quad f_t = f(\eta_t).$$

From this point onwards, our strategy for characterizing the equilibrium is straightforward: we plug functions $p(\eta_t)$ and $f(\eta_t)$ into equations (*) and (**), and through multiple mechanical applications of Ito's lemma derive differential equations that functions p and f must satisfy. Lemma 3 derives the law of motion of $\eta_t = N_t/K_t$ from the equations for dN_t and dK_t.

Lemma 3. The equilibrium law of motion of η_t is

$$d\eta_t = (r - g(p_t) + \sigma^2)(\eta_t - p_t)\, dt + (a - \iota(p_t) + \mu_t^p\, p_t)\, dt + (\alpha\,(\sigma + \sigma_t^p)\, p_t - \sigma\eta_t)\, dZ_t - d\zeta_t,$$

where $d\zeta_t = dC_t/K_t$ and dC_t is aggregate payout to experts.

Proof. Aggregating over all experts, the law of motion of N_t is

$$dN_t = r\, N_t\, dt + K_t\, [\, (a - \iota(p_t) + (g + \mu_t^p + \sigma\sigma_t^p - r)\, p_t)\, dt + \alpha\,(\sigma + \sigma_t^p)\, p_t\, dZ_t] - dC_t,$$

where C_t is are aggregate payouts, and the law of motion of K_t is

$$dK_t = g(p_t)\, K_t\, dt + \sigma\, K_t\, dZ_t.$$

Combining the two equations, and using Ito's lemma, we get a desired expression for η_t. QED

Proposition 2 uses Ito's lemma to derive μ_t^p, μ_t^f, σ_t^p, and σ_t^f, and plugs them into equations (*) and (**) to back out the differential equations for $p(\eta)$ and $f(\eta)$.

Proposition 2. The equilibrium domain of functions $p(\eta)$ and $f(\eta)$ is an interval $[0, \eta^*]$. For $\eta \in [0, \eta^*]$, these functions can be computed from the differential equations

$$p''(\eta) = \frac{2[p_t\mu_t^p - ((r - g(p_t) + \sigma^2)(\eta - p_t) + a - \iota(p_t) + p_t\mu_t^p)\, p'(\eta)]}{(\sigma_t^\eta)^2}$$

$$f''(\eta) = \frac{2[(\rho - r) f_t - ((r - g(p_t) + \sigma^2)(\eta - p_t) + a - \iota(p_t) + p_t\mu_t^p)\, f'(\eta)]}{(\sigma_t^\eta)^2}$$

where $p_t = p(\eta_t)$, $f_t = f(\eta_t)$

[5] We also prove that the equilibrium in our baseline model is unique and Markov without imposing Markov structure a priori - see Corollary to Proposition 5.

$$\mu_t^p = -\left(\frac{a-\iota(p_t)}{p_t} + g(p_t) + \sigma\sigma_t^p - r + \frac{\sigma_t^f}{f(\eta)}\alpha(\sigma+\sigma_t^p)\right),$$

$$\sigma_t^\eta = \frac{\sigma(\alpha p_t - \eta)}{1-\alpha p'(\eta)},\ \sigma_t^p = \frac{p'(\eta)\sigma(\alpha p_t - \eta)}{p_t(1-\alpha p'(\eta))},\quad \text{and} \qquad \sigma_t^f = \frac{f'(\eta)\sigma(\alpha p_t - \eta)}{1-\alpha p'(\eta)}.$$

Function $p(\eta)$ is increasing, $f(\eta)$ is decreasing, and the boundary conditions are

$$p(0) = \underline{p},\ \ f(\eta^*) = 1,\ \ p'(\eta^*) = 0,\ \ f'(\eta^*) = 0 \qquad \text{and} \quad \lim_{\eta\to 0} f(\eta) = \infty.$$

Proof. First, we derive expressions for the volatilities of η_t, p_t and f_t. Using the law of motion of η_t from Lemma 3 and Ito's lemma, the volatility of p_t is given by

$$p_t\, \sigma_t^p = p'(\eta)\,(\alpha\,(\sigma + \sigma_t^p)\, p_t - \sigma\eta_t) \Rightarrow\ \ \sigma_t^p p_t = \frac{p'(\eta)\sigma(\alpha p_t - \eta)}{(1-\alpha p'(\eta))}.$$

The expressions for σ_t^η and σ_t^f follow immediately from Ito's lemma.

The expression for μ_t^p follows directly from the first order condition with respect to k_t in Proposition 1. The differential equation for $p''(\eta)$ follows from the law of motion of η_t again and Ito's lemma: the drift of p_t is given by

$$\mu_t^p\, p_t = p'(\eta_t)\, [(r - g(p_t) + \sigma^2)\,(\eta_t - p_t) + (a - \iota(p_t) + \mu_t^p\, p_t)] + \tfrac{1}{2}\,(\sigma_t^\eta)^2\, p''(\eta_t).$$

Also, $\mu_t^f = (\rho - r)\, f_t$ and similarly Ito's lemma implies that

$$f'(\eta_t)\, [(r - g(p_t) + \sigma^2)\,(\eta_t - p_t) + (a - \iota(p_t) + \mu_t^p\, p_t)] + \tfrac{1}{2}\,(\sigma_t^\eta)^2\, f''(\eta_t) = (\rho - r)\, f(\eta_t).$$

Finally, let us justify the five boundary conditions. First, because in the event that η_t drops to 0 experts are pushed to the solvency constraint and must liquidate any capital holdings to households, we have $p(0) = \underline{p}$. Second, because η^* is defined as the point where experts consume, expert optimization implies that $f(\eta^*) = 1$ (see Proposition 1). Third and fourth, $p'(\eta^*) = 0$ and $f'(\eta^*) = 0$ are the standard boundary condition at a reflecting boundary. If one of these conditions were violated, e.g. if $p'(\eta^*) < 0$, then any expert holding capital when $\eta_t = \eta^*$ would suffer losses at an infinite expected rate.[6] Likewise, if $f'(\eta^*) < 0$, then the drift of $f(\eta_t)$ would be infinite at the moment when $\eta_t = \eta^*$, contradicting Proposition 1. Fifth, if η_t ever reaches 0, it becomes absorbed there. If

[6] To see intuition behind this result, if $\eta_t = \eta^*$ then $\eta_{t+\varepsilon}$ is approximately distributed as $\eta^* - \varpi$, where ϖ is the absolute value of a normal random variable with mean 0 and variance $(\sigma_t^\eta)^2\, \varepsilon$. As a result, $\eta_{t+\varepsilon} \sim \eta^* - \sigma_t^\eta\, \mathrm{sqrt}(\varepsilon)$, so $p(\eta_{t+\varepsilon}) = p(\eta^*) - p'(\eta^*)\, \sigma_t^\eta\, \mathrm{sqrt}(\varepsilon)$. Thus, the loss per unit of time ε is $p'(\eta^*)\, \sigma_t^\eta\, \mathrm{sqrt}(\varepsilon)$, and the average rate of loss is $p'(\eta^*)\, \sigma_t^\eta / \mathrm{sqrt}(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

any expert had an infinitesimal amount of capital at that point, he would face a permanent price of capital of $\underline{p}$. At this price, he is able to generate the return on capital of

$$\frac{a-\iota(\underline{p})}{\underline{p}}+g(\underline{p})>r$$

without leverage, and arbitrarily high return with leverage. In particular, with high enough leverage this expert can generate a return that exceeds his rate of time preference ρ, and since he is risk-neutral, he can attain infinite utility. It follows that $f(0) = \infty$.

Finally, note that we have five boundary conditions required to solve a system of two second-order ordinary differential equations with an unknown boundary η^*. QED

For completeness, we show that the equilibrium characterized in Lemma 3 is unique not only among equilibria that are Markov in η_t but among all competitive rational expectations equilibria.

Proposition 2. Our economy has a unique equilibrium, which is described by Proposition 1.

We defer the proof until Section 3 - this proposition is a corollary of Proposition 5.

Figure 2 shows an example, in which we computed functions $f(\eta)$ and $p(\eta)$ numerically. We set $r = 5\%$, $\rho = 6\%$, $\delta = 2\%$, $\underline{\delta} = 5\%$, $\underline{p} = 10$, $a = 1$, and $\sigma = 0.2$ and assume an investment function $\Phi(.)$ such that the cost of generating growth g is

$$\underline{p}\,(g + \delta) - 0.1(r - g)^{1/2} + 0.1(r + \delta)^{1/2}.$$

Note the investment cost is 0 when the capital depreciates at rate δ (i.e. $g = -\delta$), and it is possible to recover at least $\underline{p}$ units of output per unit of capital as capital is liquidated at the infinite rate (i.e. $g = -\infty$).

As expected, asset prices $p(\eta_t)$ increase when experts have more net worth. At the same time, experts get more value per dollar of net worth when prices are depressed and they can buy assets cheaply, so function $f(\eta_t)$ is decreasing.

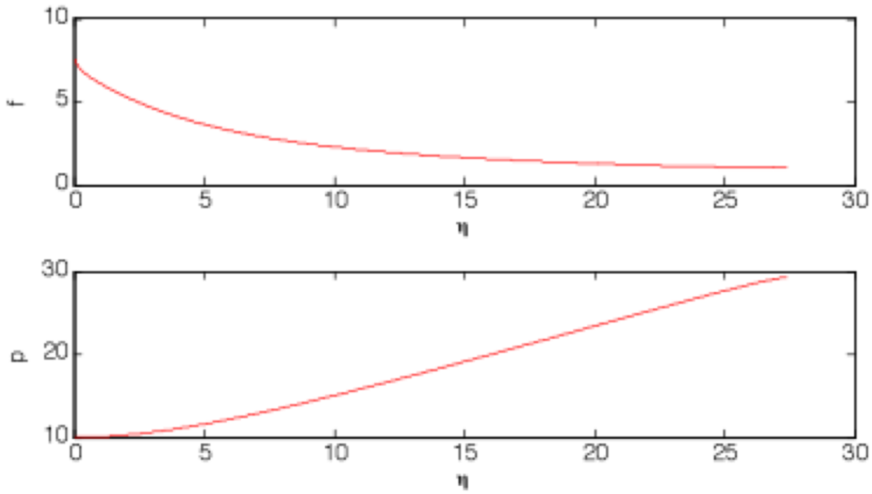

Figure 2. The marginal component of experts' value function and the price of capital as functions of η.

Equilibrium Dynamics. Since $f(\eta)$ is a decreasing function with $f(\eta^*) = 1$, experts are consuming only when $\eta_t = \eta^*$. Thus the equilibrium law of motion of η_t is given by

$$d\eta_t = (r - g(p_t) + \sigma^2)\, \eta_t\, dt + (a - \iota(p_t) - (r - g(p_t) + \sigma^2)p_t + \mu_t^p)\, dt + (\alpha\, (\sigma + \sigma_t^p)\, p_t - \sigma\eta_t)\, dZ_t$$

on $[0, \eta^*)$, and it is characterized by a reflecting boundary at η^*, which is caused by the aggregate consumption/payouts.

To get a better sense of equilibrium dynamics, Figure 3 shows the *drift* and *volatility* of η_t for our computed example. We see that the drift is positive for all $\eta_t < \eta^*$, as experts earn interest on their funds and make profit in expectation from their risky investments. The expected rate of profit per unit of net worth is particularly high for low η_t. Since η^* is a reflecting boundary, it is the point of attraction of the system since in expectation the system gravitates towards η^*. Point η^* is analogous to the *steady state* in traditional macro models, such as BGG and KM. Of course, while in expectation the system always moves towards η^* due to drift, it may be shocked away from η^* due to volatility.

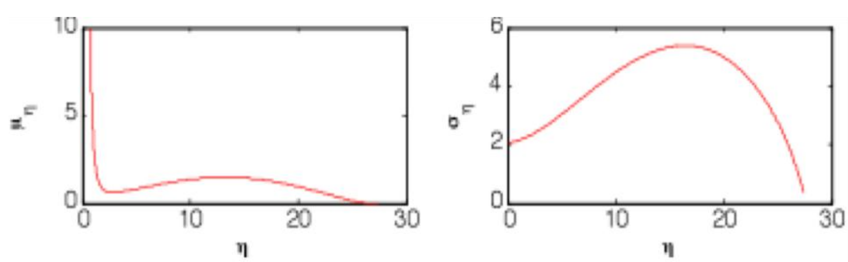

Figure 3. The drift and volatility of η in equilibrium.

While the drift dynamics of the system is stabilizing, *volatility* dynamics exhibits salient instabilities. From Figure 4 we see that volatility is $\cap$-shaped. In particular, near η^* volatility is quite low, but below η^* volatility becomes much higher. We need to discuss (1) what determines the volatility, (2) what are the implications of the shape of the volatility function on equilibrium dynamics and (3) how equilibrium dynamics predicted by our model are different from the dynamics under log-linearized solutions of BGG and KM.

Volatility is determined by fundamental shocks (i.e. *exogenous* risk), and the degree to which they are amplified within the system (i.e. *endogenous* risk). Endogenous risk is measured by the volatility if the valuation process p_t. From Lemma 3, the volatilities of η_t and p_t are given by

$$\sigma_t^\eta = \frac{\sigma(\alpha p_t - \eta_t)}{p_t(1-\alpha p'(\eta_t))} \quad \text{and} \quad \sigma_t^p = \frac{p'(\eta_t)\sigma(\alpha p_t - \eta_t)}{p_t(1-\alpha p'(\eta_t))} \qquad (***).$$

These expressions can be understood through the cycle of amplification, shown in Figure 4. An exogenous shock of dZ_t changes K_t by $dK_t = \sigma K_t dZ_t$, and has an immediate effect on the net worth of experts of the size $dN_t = \alpha p_t \sigma K_t dZ_t$. The immediate effect is that the ratio η_t of net worth to total capital changes by $\sigma (\alpha p_t - \eta_t) dZ_t$, since[7]

$$d(N_t/K_t) = (dN_t K_t - N_t dK_t)/K_t^2 = \sigma (\alpha p_t - \eta_t) dZ_t.$$

Note that $\alpha p_t/\eta_t$ is the leverage ratio (total assets to total equity), and when p_t is larger compared to η_t, shocks get magnified through *leverage*. However, there is another effect - the feedback effect through prices. When η_t drops by $\sigma (\alpha p_t - \eta_t) dZ_t$, price p_t drops by $p'(\eta_t) \sigma (\alpha p_t - \eta_t) dZ_t$, leading to further deterioration of the net worth of experts, which feeds back into prices, and so on. Figure 5 illustrates this self-reinforcing feedback loop.

[7] In this thought experiment, we consider how a shock to capital translates into η_t at a single *instant* of time, and therefore we ignore the effects of the drift.

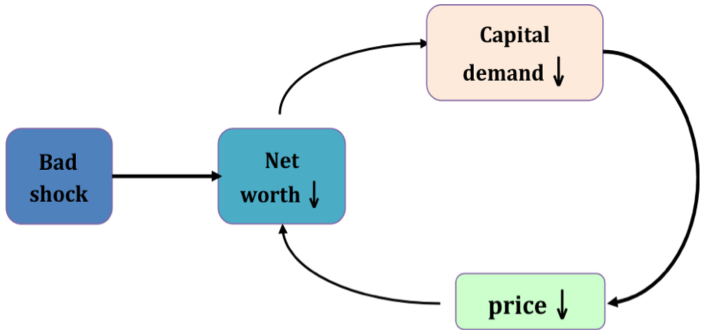

Figure 4: The cycle of amplification.

The strength of the feedback effect is measured by the reaction of prices to the net worth of experts, $p'(\eta)$. When $p'(\eta)$ is higher, then each exogenous shock to the system becomes more amplified as the feedback effects converge. The amplification effect is captured by $1 - \alpha p'(\eta)$ in the denominator of (***) (and if $p'(\eta)$ were ever greater than $1/\alpha$, then the feedback effect would be completely unstable, leading to infinite volatility). To summarize, while exogenous risk is constant in our model, endogenous risk depends on the strength of the feedback loops.

It turns out that in our equilibrium there is no amplification at η^* and a lot of amplification below η^*, leading to a $\cap$-shaped form of volatility. A crucial feature of our model that drives this result is that payouts are chosen endogenously. As a result, payouts happen at point η^* where experts are relatively unconstrained. At that point shocks to experts net worth's become absorbed through adjustments to payouts, and so they have no effect on the experts' demand for capital or prices. Therefore, $p'(\eta^*) = 0$, and there is no amplification at η^*. In contrast, below η^* experts become constrained, and so shocks to their net worth's immediately feed into their demand for assets.

The $\cap$-shaped form of volatility implies that the system is relatively stable near its "steady state" of η^*, but becomes unstable below the steady state as the volatility shoots up. Figure 5 shows the stationary distribution of η_t. Starting from any point $\eta_0 \in (0, \eta^*]$ in the state space, the density of the state variable η_t converges to the stationary distribution in the long run as $t \to \infty$. Stationary density also measures the average amount of time that the variable η_t spends in the long run near each point. We see that the stationary density is high near η^*, which is the attracting point of the system, but very thin in the middle region below η^* where the volatility is high. The system moves fast through regions of high volatility, and so the time spent there is very short. As we can see from a sample path of η_t on the right panel of Figure 5, these excursions below the steady state are characterized by high uncertainty, and occasionally may take the system very far below the steady state. At the extreme low end of the state space, assets are essentially valued by unproductive households, with $p_t \sim \underline{p}$, and so the volatility is low. The stationary distribution has a large positive mass way below the steady state, so the system spends significant amounts of time there.

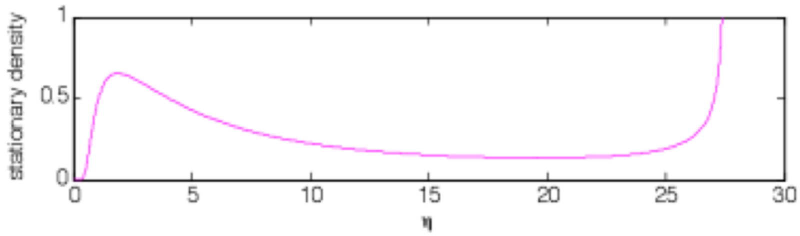

Figure 5. The stationary density of η_t and sample paths of η_t.

Papers such as BGG and KM do not capture the distinction between relatively stable dynamics near the steady state, and much stronger amplification loops below the steady state - but why? An amplification cycle like that presented in Figure 4 is a feature of both BGG and KM, but the solution method of log-linearizing near the steady state implicitly assumes that the strength of amplification effects is even throughout the state space. However, log-linearization is a valid approximation only if the system does not exhibit instabilities like those presented in Figure 5. Log-linearized solutions can capture amplification effects of various magnitudes as the steady state is placed in a particular part of the state space by a choice of an *exogenous* parameter (such as exogenous drainage of expert net worth in BGG). However, such an exogenous parameter forces the system to behave in a completely different way in order to zoom the magnifying glass of log-linearization to a particular region. With *endogenous* payouts, the steady state naturally falls in the relatively unconstrained region where amplification is low, and amplification below the steady state is high.

Proposition A1 in the appendix provides equations that characterize this stationary distribution.

2.2 Endogenizing the capital structure

In our baseline model we made several simplifying assumptions, which we try to relax or justify in the following two subsections. First, rather than simply assuming that entrepreneurs have to hold a fixed fraction α of the equity, we microfound this conclusion using a moral hazard argument. Second, we explicitly model the financial sector by introducing intermediaries that have the capability to reduce financial frictions between productive and unproductive households. In Subsection 2.4 we add idiosyncratic shocks to study various asset pricing implications.

So far, we simply assumed that experts have to retain "skin in the game" and hence can only offload a fraction 1-α of risk. We now endogenously derive this restriction using informational frictions. For convenience, we model asymmetric information frictions as moral hazard, and assume that productive households can invest in a negative NPV pet projects from which he derives a private benefit of $b < 1$ per unit of value destroyed. The financial expert will forgo his pet project if he is liable for a fraction α of this loss such that

$$\alpha \geq b.$$

This constraint is the one-shot deviation condition. Appendix A justifies this constraint formally using the theory of optimal dynamic contracts, in which the contracting variable is the market value of assets $k_t\, p_t$. By assuming that contracts depend on the market value of capital $k_t\, p_t$ instead of k_t directly, we allow for an amplification channel in which market prices affect the expert's net worth. This assumption is consistent with what we see in the real world, as well as with the models of Kiyotaki and Moore (1997) and Bernanke, Gertler and Gilchrist (1999). We assume that contracting directly on k_t is difficult because we view k_t not as something objective and static like the number of machines, but rather something much more forward looking, like the expected NPV of assets under a particular management strategy. Moreover, even though in our model there is a one-to-one correspondence between k_t and output, in a more general model this relationship could be different for different types of projects, and could depend on the private information of the expert. Furthermore, output can be manipulated, e.g. by underinvestment. In extensions of our model, we relax the contracting assumption by allowing the expert to hedge some of the risks of $k_t p_t$ (e.g. see the Section 4 on securitization).

The contracting problem determines fraction of risk α that has to be borne by the expert which, together with the requirement that outside investors must receive a required return of *r*, pins down the cash flows that go to inside equity n_t. The incentive constraint also implies a solvency constraint, since it is possible to reward and punish the expert only as long as $n_t > 0$.

Note that we assumed for simplicity that private benefits are proportional to the value that has been destroyed, and does not depend on the market valuation of capital. Alternatively, one could assume that experts get the benefit of b units of *output* per unit of capital destroyed, leading to the incentive constraint of $\alpha_t \geq b/p_t$. In this case an additional amplification mechanism would emerge, as a price decline would tighten the moral hazard constraint further. That is, the incentive constraint requires a higher α_t in downturns, when equilibrium prices p_t are depressed. This observation is consistent with higher informational asymmetry and lower liquidity in downturns.[8] This property of α_t also creates an additional reason why experts find it harder to hold assets in downturns -

[8] See Leland and Pyle (1977) where managers must retain a greater fraction of equity when the informational asymmetry is greater, or DeMarzo and Duffie (1999) where informational sensitivity leads to lower liquidity.

because they must retain a greater fraction of risk.[9]

2.3 Modeling the financial sector explicitly

In our baseline setting we modeled the financial intermediary sector is only implicitly. In this subsection we justify our baseline setting by arguing that all the insights carry over to a richer model with an explicit financial intermediary sector. Funds are channeled from the less productive households to more productive (experts) households through the financial sector. As before direct lending is subject to informational frictions. However, the financial sector has the ability to mitigate these frictions. Instead of the networth of the expert, now the combined networth of expert households and the financial sector will form the basis of our state variable. Indeed, we provide conditions under which the two networths are perfect substitutes.

Figure 6 depicts a more general financing structure, in which more productive experts hold capital, lever up and receive funds from intermediaries. Financial intermediaries issue debt claims as well as outside equity towards less productive households.

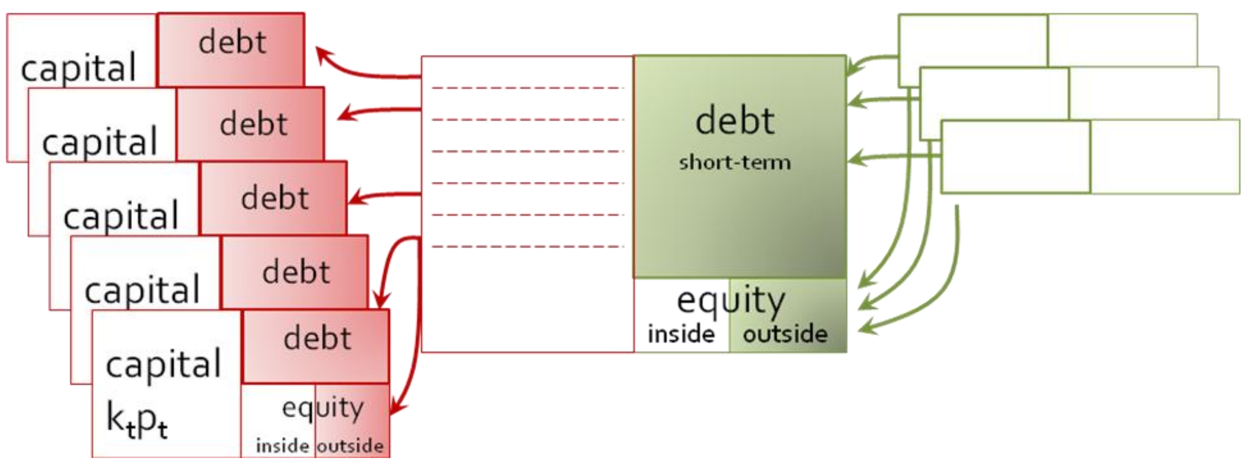

Figure 6. Balance sheets structures of experts and financial intermediaries

Such a funding structure arises endogenously if one has to overcome two layers of moral hazard problems, as e.g. in Holmström and Tirole (1997). As before, productive households have to hold inside equity of at least

$$\alpha_t^E \geq b(m_t),$$

where the productive households' private benefits from shirking $b(m_t) < 1$ are now decreasing in the monitoring effort, m_t, of the financial intermediary. Put differently, by

[9] In a version of our model where $\alpha_t = b/p_t$ and when households can provide liquidity support by buying assets temporarily in downturns (see Section 3), the equilibrium exhibits procyclical leverage in the region where households hold some of the assets. The reason is that α_t increases when p_t falls, making it harder for the financial sector to hold assets. Procyclical leverage is consistent with what we observe in investment banks in practice.

increasing the monitoring intensity, m_t, one can lower the productive households' inside equity share necessary to incentivize the productive households. Note that the above constraint always binds in equilibrium, i.e. $\alpha_t^E = b(m_t)$, since otherwise the productive household would issue more outside equity and scale up its production.

Assume that the monitoring intensity is not directly observable to outside investors. By not monitoring, each financial intermediary can get a private benefit of $c(m) < 1$ per unit of value destroyed through faster depreciation of capital (as the productive household is also shirking due to the lack of monitoring). Hence, financial intermediaries also have to be incentivized and they have to be exposed to a fraction

$$\alpha_t^I \geq c(m_t),$$

of total risk. Higher monitoring effort requires the financial intermediary to get more involved in running the project, and so $c(m_t)$ is increasing in m_t. Thus, more monitoring requires that intermediaries hold a larger fraction of the overall risk. Note that intermediaries' incentive constraint is also always binding since it otherwise would always be profitable to scale up the projects.

Overall, productive households and financial intermediaries together hold a fraction of $\alpha_t := \alpha_t^E + \alpha_t^I$. The remaining fraction $(1-\alpha_t)$ of the risk is held by the unproductive households in form of outside equity. In our setting it is irrelevant to what extent the outside equity issued by productive households is directly held by unproductive households or indirectly through outside equity issued by financial intermediaries. The same holds for debt issuance. We assume that for all m, the total benefit that productive households and intermediaries can derive per unit of capital destroyed is less than 1 ($b(m) + c(m) < 1$) and that the damage they can cause by shirking is significant, so that it is always suboptimal to allow them to consume benefits.

One can easily see that the net worths of productive households and of the financial intermediaries are substitutes. Proposition 3 states if both groups of investors share the same preference ordering and the sum of $b(m) + c(m)$ is a constant for all m, the two net worths are perfect substitutes. Hence, in this case we can without loss of generality collapse productive households and financial intermediaries to a single economic entity called "experts", as we did in our baseline setting.

Proposition 3. If the sum of $b(m) + c(m)$ is constant for all m, productive households and financial intermediaries can be merged to single entities, "experts", as their net worths are perfect substitutes.

Proof: Since in equilibrium both incentive constraints $\alpha_t^E \geq b(m_t)$ and $\alpha_t^I \geq c(m_t)$ hold with equality, $\alpha_t = b(m_t) + c(m_t)$. Hence, the total share of the risk held together by productive households and financial intermediaries is invariant to changes in m_t. QED

Note that productive households need not have any net worth at all if maximum monitoring makes monitoring perfect such that private benefits b are pushed to zero. That is, in this case total net worth N_t can be equated with financial intermediaries' net worth and our baseline model holds literally.

2.4 Idiosyncratic shocks and asset-pricing implications

Our equilibrium analysis implies interesting results for asset pricing - predictability, excess volatility, and an increase in correlation across various assets in the cross section at times of crises. To derive these important results we extend our model to allow expert-specific idiosyncratic shocks. It is important to distinguish between the price of physical capital, p_t, and the price of outside equity held by households.

Physical capital. The price of physical capital is determined by experts' Bellman equation and the first-order condition with respect to k. They imply that the value n_t of any portfolio of risky capital and cash satisfies

$$\rho f(\eta_t) n_t \, dt = E[d(f(\eta_t) n_t)]$$

when internal investment is done optimally, according to $i_t/k_t = \iota(p_t)$. It follows that any portfolio held by an expert can be priced using the stochastic discount factor

$$e^{-\rho t} f(\eta_t)/f(\eta_0).$$

Our model predicts *excess volatility*. The volatility of $p_t k_t$ is $\sigma + \sigma_t^p$, where σ is the volatility of earnings (per dollar invested). Our model also implies that asset returns are *predictable*. From the first-order condition (*), the excess expected return from investing a dollar into the risky asset is driven by the time-varying risk premium of

$$\sigma_t^\eta \, f'(\eta_t)/f(\eta_t) \, \alpha \, (\sigma + \sigma_t^p),$$

where we use $\sigma_t^f = f'(\eta_t) \, \sigma_t^\eta$. The risk premium is zero at η^*, since $f'(\eta^*) = 0$. Below η^*, the risk premium is positive.

To look at asset prices in cross section, we reinterpret the model to allow for multiple assets. Suppose that there are many types of capital, and each is hit by aggregate and type-specific shocks. Specifically, capital of type *j* evolves according to

$$dk_t^j = g \, k_t^j \, dt + \sigma \, k_t^j \, dZ_t + \sigma' \, k_t^j \, dZ_t^j,$$

where dZ_t^j is type-specific Brownian shock uncorrelated with the aggregate shock dZ_t.

In aggregate, idiosyncratic shocks cancel out and the total amount of capital in the economy still evolves according to

$$dK_t = g \, K_t \, dt + \sigma \, K_t \, dZ_t.$$

Then, in equilibrium financial intermediaries hold fully diversified portfolios and experience only aggregate shocks. The equilibrium looks identical to one in the single-asset model, with price of capital of any kind given by p_t per unit of capital.

Then

$$d\,(p_t k_t^j) = \text{drift} + (p_t k_t^j)\,(\sigma + \sigma_t^p)\, dZ_t + (p_t k_t^j)\,\sigma'\, dZ_t^j.$$

The correlation between assets i and j is

$$\mathrm{Cov}(p_t k_t^i,\, p_t k_t^j)/(\mathrm{Var}(p_t k_t^i)\mathrm{Var}(p_t k_t^j))^{1/2} = (\sigma + \sigma_t^p)^2/((\sigma + \sigma_t^p)^2 + (\sigma')^2).$$

Near the steady state $\eta_t = \eta^*$, there is only as much correlation between the prices of assets i and j as there is correlation between shocks. Specifically, $\sigma_t^p = 0$ near the steady state, and so the correlation is

$$\sigma^2/(\sigma^2 + (\sigma')^2).$$

Away from η^*, correlation increases as σ_t^p increases. Asset prices become most correlated in prices when σ_t^p is the largest, and as $\sigma_t^p \to \infty$, the correlation coefficient tends to 1.

Outside equity. Experts' outside equity can be directly held by risk-neutral households and hence the price is determined by their discount factor e^{-rt}. This implies that the discounted price processes follow a martingale. Nevertheless, the returns are negatively skewed as a negative fundamental macro shock is amplified in times of crisis. In the cross section, equity prices become more correlated at times of crises.[10] This phenomenon is important in practice as many risk models have failed to take this correlation effects into account in the recent crisis.[11]

Derivatives. Since data for crisis periods are limited, it is worthwhile to look at option prices that reflect market participants' implicit probability weights of extreme events. Our result that price volatility is higher for lower η_t-values also has strong implications for option prices.

First, it provides an explanation for "volatility smirks" of options. Since the values of options monotonically increase with the volatility of the underlying stock, option prices can be used to compute the "implied volatility" from the Black-Scholes option pricing formula. One example of a "volatility smirk" is that empirically put options have a higher implied volatility when they are further out of the money. That is, the larger the price drop has to be for an option to ultimately pay off, the higher is the implied volatility

[10] For an empirical documentation see for example Erb, Harvey and Viskanta (1994).
[11] See "Efficiency and Beyond" in *The Economist*, July 16, 2009.

or, put differently, far out of the money options are overpriced relative to at the money options. Our model naturally delivers this result as volatility in times of crises is higher.

Second, so called "dispersion trades" try to exploit the empirical pattern that the smirk effect is more pronounced for index options than for options written on individual stocks (Driessen et al. 2010). Note that index options are primarily driven by macro shocks, while individual stock options are also affected by idiosyncratic shocks. The observed option price patterns arise quite naturally in our setting as the correlation across stock prices increases in crisis times. Note that in our setting options are redundant assets as their payoffs can be replicated by the underlying asset and the bond, since the volatility is a smooth function in p_t. This is in contrast to stochastic volatility models in which volatility is independently drawn.

3. Externalities

So far, we set up our baseline model intentionally in a way that has no externalities. That is, a social planner who faces the same constraints would opt for the same unstable dynamics as the competitive outcome shown in Section 2. After establishing this equivalence we enrich our setting in two ways. First, we allow households to speculate. That is, they provide liquidity by buying capital with the intention to resell back to experts as soon as the economy recovers. While liquidity provision can act as a stabilizing force, it also introduces a *firesale externality*, which is an inefficient pecuniary externality in an incomplete markets setting. The firesale externality appears when in the event of crises (i) experts are able to sell assets to another sector (e.g. vulture investors or the government) and (ii) the new asset buyers provide a downward-sloping demand function. In this case, when levering up in good times financial institutions do not take into account that in the event of crises, its own fire sales will depress prices that other institutions are able to sell at. This effect leads to excess leverage due to competition among the financial institution, i.e. a monopolist expert would lever up less.

To illustrate externalities from the financial sector to the real economy, we add a labor market, in which households' labor income depends on the amount of capital in the economy. When levering up and choosing bonus payouts, experts do not internalize the damages that crises bring onto the labor market.

3.1 Social planner's problem in baseline model

We show that the competitive rational expectations equilibrium in our baseline model coincides with a policy that a social planner, who that faces the same constraints, would chose. Since less productive households are compensated for deferring their consumption with the discount rate r, the social planner's outcome equals the one of a monopolist expert with a discount rate ρ. Debt and the total amount of capital in the economy evolve according to

$$dD_t = (r\, D_t - a\, K_t + \iota(g)\, K_t)\, dt - dC_t \quad \text{and} \quad dK_t = gK_t\, dt + \sigma K_t\, dZ_t,$$

where dC_t is the monopolist's consumption. It is convenient to express the monopolist's value function as $h(\omega_t)K_t$, where $\omega_t = -D_t/K_t$.[12] The value function is homogenous in D_t and K_t of degree 1 because of scale invariance. From the liquidation value of assets, the monopolist's debt capacity is $D_t \leq aK_t/(r+\underline{\delta})$, and so $\omega_t \geq -a/(r+\underline{\delta})$.

Using Ito's lemma

$$d\omega_t = ((r - g + \sigma^2)\, \omega_t + a - \iota(g))\, dt - \sigma\omega_t\, dZ_t - dC_t/K_t.$$

The following proposition summarizes the Bellman equation and the optimal policy of the monopolist.

Proposition 4. The monopolist's value function solves equation

$$(\rho - g)\, h(\omega) = h'(\omega)\, [(r - g)\, \omega + a - \iota(g)] + \tfrac{1}{2}\, h''(\omega)(\sigma\omega)^2$$

with boundary conditions $h(-a/(r+\underline{\delta})) = 0$, $h'(\omega^*) = 1$ and $h''(\omega^*) = 0$. The optimal policy has investment with $\iota(g)$ with $(\omega + \iota'(g))\, h'(\omega) = h(\omega)$. Payouts occur exactly when ω_t reaches ω^*, and prevent ω_t from exceeding ω^*. Thus, technically, ω^* is the reflecting boundary for the process ω_t.[13]

Proof. The value function must satisfy the Bellman equation

$$\rho h(\omega)K\, dt = \max_{g,dC}\ dC + E[d(h(\omega)K)] =$$

$$dC + h'(\omega)\, ((r - g + \sigma^2)\, \omega + a - \iota(g) - dC)\, K + \tfrac{1}{2}\, h''(\omega)(\sigma\omega)^2\, K + h(\omega)\, gK - h'(\omega)\, \sigma^2\omega\, K.$$

When $h'(\omega) > 1$, then $dC = 0$ is optimal and the equation reduces to (*). The optimal choice of g is determined by $(\omega + \iota'(g))\, h'(\omega) = h(\omega)$.

To justify the boundary conditions, we extend function $h(\omega)$ that satisfies them beyond ω^* according to $h(\omega) = h(\omega^*) + \omega - \omega^*$, and show that the Bellman equation holds on the entire domain $[-a/(r+\underline{\delta}), \infty)$. For $\omega < \omega^*$, it holds because $h'(\omega) > 1$ and so $dC = 0$ is the optimal choice. The value function for $\omega \geq \omega^*$ can be attained by making a one-time payment of $dC/K = \omega - \omega^*$, and moreover, $dC = 0$ is suboptimal since

$$h(\omega^*) = 1,\ h''(\omega^*) = 0 \Rightarrow (\rho - g)\, h(\omega^*) = (r - g)\, \omega^* + a - \iota(g)$$

[12] It convenient to analyze the monopolist's behavior using ω_t, instead of the more economically meaningful variable $\eta_t = N_t/K_t$, because η_t depends on market prices, which are endogenous in equilibrium. Proposition 5 provides a one-to-one map between variables ω_t and η_t in equilibrium.

[13] Our analysis here can be related to Bolton, Chen and Wang (2009). They study optimal investment and payouts of a single firm, which faces output shocks (rather than capital shocks, as in our setting).

$$\Rightarrow (\rho - g)\, h(\omega) < (r - g)\, \omega + a - \iota(g) \quad \text{for all } \omega > \omega^*,$$

since $\rho > r$. QED

Figure 7 illustrates the monopolist's value function.

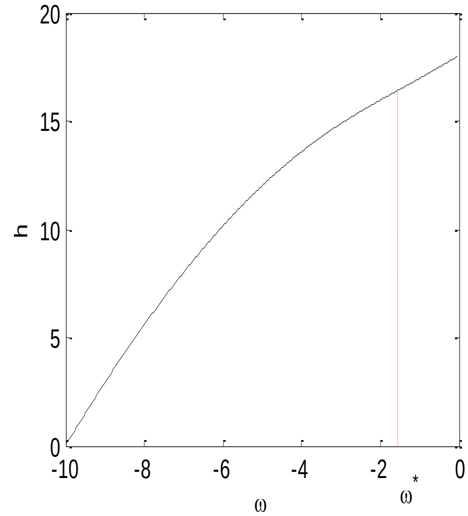

Figure 7. The value function of a monopolist expert.

For a monopolist expert, the optimal payout point ω^* is determined by the trade-off between the benefits of being able to borrow at rate r, which is less than his discount rate, to consume, and the liquidation costs that are incurred when ω_t gets close to $-a/(r+\underline{\delta})$. It is optimal to pay out when there is a sufficient amount of financial slack ω^*, which determined by Proposition 4.

Proposition 5 shows that in our baseline model, the outcome with a monopolist investor is identical to that under competition. The intuition is that even though in a competitive equilibrium experts do affect prices in the aggregate by their choices of compensation and investment, they are isolated from market prices because they do not trade in equilibrium (due to symmetry).[14]

Proposition 5. The competitive equilibrium in our baseline economy is equivalent to the outcome with a monopolist. The following equations summarize the map between the two:

$$\eta_t = h(\omega_t)/h'(\omega_t), \qquad p_t = h(\omega_t)/h'(\omega_t) - \omega_t, \qquad \text{and} \qquad f_t = h'(\omega_t).$$

Proof. First, since the monopolist chooses g and dC_t to maximize his payoff, the sum of all experts' utilities in the competitive equilibrium cannot be greater than that of a monopolist. On the other hand, each expert can guarantee his fraction of the

[14] The argument of Proposition 5 can be easily generalized to show that in the baseline model, the equilibrium is the same under oligopolistic competition as well.

monopolist's utility (weighted by his net worth) by trading to a fraction of the aggregate portfolio at time 0, and by copying the monopolist's policy in isolation thereafter. Thus, the sum of all experts' utilities in the competitive equilibrium must equal the monopolist's payoff.

It follows that the aggregate behavior in the competitive equilibrium is equivalent to the monopolist's optimal policy. In particular, since growth chosen by the monopolist satisfies $(\omega + \iota'(g))\, h'(\omega) = h(\omega)$, the competitive equilibrium has prices

$$p_t = h(\omega_t)/h'(\omega_t) - \omega_t.$$

Under these prices, $\eta_t = N_t/K_t = (p_t K_t - D_t)/K_t = p_t - \omega_t = h(\omega_t)/h'(\omega_t)$. Finally, the sum of the experts' utilities is $f_t\, N_t = h(\omega_t)\, K_t \Rightarrow f_t = h(\omega_t)/\eta_t = h'(\omega_t)$. QED

As a corollary of Proposition 5, we conclude that the competitive equilibrium in our baseline model is unique.

Corollary. In our baseline model, equilibrium prices, expert value function $f_t\, n_t$, and the law of motion of η_t are uniquely determined.

Proof. Note that the proof of Proposition 5 does not assume any properties of the competitive equilibrium (such as that it is Markov in η_t). Uniqueness follows from the uniqueness of the monopolist's optimal policy. QED

Proposition 5 provides an alternative convenient way to compute equilibria in our baseline setting, by solving a singe equation for $h(\omega)$ instead of a system of equations for $p(\eta)$ and $f(\eta)$.

3.2 Speculative households and externalities within the financial sector

Liquidity provision by speculative households. So far, we assumed that the sale of assets from experts to households in the event of crises is irreversible. However, in practice the economy has resources to pick up some of the functions of the traditional financial sector in times of crises. Investors like Warren Buffet have helped institutions like Goldman Sachs and Wells Fargo with capital infusions. More generally, governments have played a huge role in providing capital to financial institutions in various ways and induced large shifts in asset holdings (see He, Khang and Krishnamurthy (2009)). For example, in the spring of 2009, the Fed introduced the Term Asset-Backed Securities Loan Facility in order to entice hedge funds to buy some of the asset-backed securities.

Formally, denote the fraction of capital that is held by experts by ψ_t. Hence, the total capital in the economy evolves according to

$$dK_t = (\psi_t\, g(p_t) - (1 - \psi_t)\, \underline{\delta})\, K_t\, dt + \sigma\, K_t\, dZ_t.$$

Aggregate experts' networth follows

$$dN_t = rN_t\, dt + (a-\iota(p_s))\, \psi_t K_t\, dt + \psi_t(K_t p_t)\, [(g(p_s) + \mu_t^p + \sigma\sigma_t^p - r)\, dt + \alpha\, (\sigma+\sigma_t^p)\, dZ_t] - dC_t.$$

Note that experts only hold a fraction α of the total exogenous and endogenous risk as they issue outside equity claims. Mechanical application of Ito's Lemma allows us to derive

$$d\eta_t = (r - \psi_t\, g(p_t) + (1-\psi_t)\, \underline{\delta} + \sigma^2)\, \eta_t + \psi_t\, (a - \iota(p_s) + p_t(g(p_t) + \mu_t^p + (1-\alpha)\sigma\sigma_t^p - \alpha\sigma^2 - r))\, dt$$

$$+ (\psi_t\, p_t\, \alpha\, (\sigma + \sigma_t^p) - \sigma\, \eta_t)\, dZ_t.$$

Again, using Ito's lemma, $\sigma_t^p\, p_t = p'(\eta)\, \sigma_t^\eta \Rightarrow \sigma_t^\eta = \dfrac{\sigma(\psi_t \alpha p_t - \eta)}{1-\psi_t \alpha p'(\eta)}$, and also $\sigma_t^f = f'(\eta)\sigma_t^\eta$.

Following the same steps as in Section 2, expert value functions must satisfy the first-order condition

$$\frac{a-\iota(p_t)}{p_t} + g(p_t) + \mu_t^p + \sigma\sigma_t^p - r + \frac{\sigma_t^f}{f_t}\alpha(\sigma+\sigma_t^p) = 0 \qquad (*4*)$$

while households' expected excess return from investing in risky capital directly must be

$$\frac{a}{p_t} - \underline{\delta} + \mu_t^p + \sigma\sigma_t^p - r \le 0,$$

with equality when households invest a positive amount, i.e. $\psi_t < 1$. Expert value functions also need to satisfy $\mu_t^f = (\rho - r) f_t$ (see Proposition 1).

Figure 8 illustrates the functions f_t, p_t and asset holdings ψ_t by the financial sector for the parameter values a–i=1, ρ =.06, r =.05, g =.04, $\underline{\delta}$ =.05 for three different exogenous risk values σ =.025 (blue), .5 (red), and .1 (black).

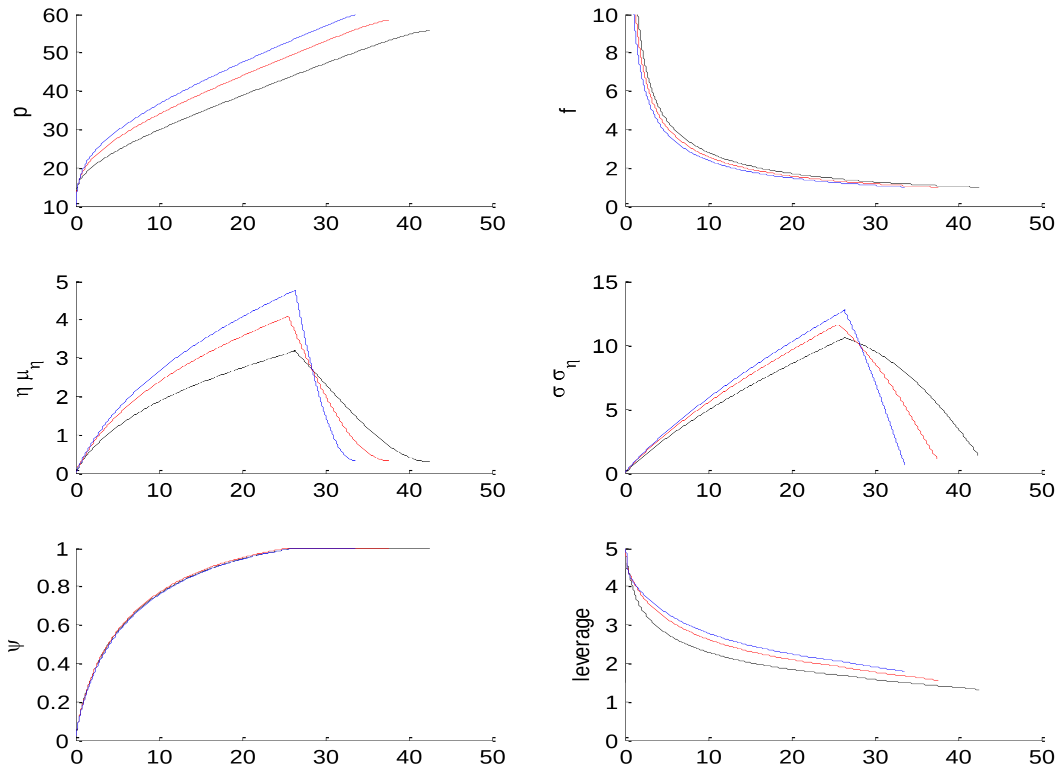

Figure 8: Equilibrium when households provide liquidity support for three different σ = .025 (blue), .05 (red), .1 (black).

We see that as η_t becomes small, $\psi_t < 1$ experts sell assets to households at market prices. As a consequence less productive households hold part of the capital. Also, market prices directly enter the experts' welfare. Several forms of externalities within the financial sector can emerge.

Pecuniary fire sale externality. Externalities within the financial sector are pecuniary externalities in an incomplete market setting.[15] They arise whenever experts' welfare depends directly on market prices, which are affected by the actions of other experts. In our baseline model of Section 2 there are no pecuniary externalities because in equilibrium experts do not trade with each other at market prices, and prices do not enter the experts' payoffs or action sets through contracts.

One type of an externality seems very prominent - the fire sale externality. This externality arises when households offer a downward-sloping demand function for assets from the financial sector during crises. Fire sale externalities stem from the fact that the economy has a bounded capacity to absorb assets when the financial sector fails. When

[15] Bhattacharya and Gale (1987) were among the first to highlight the inefficiency of a pecuniary externality. A recent application of this inefficiency within a finance context, see Lorenzoni (2007).

investing and levering up in good times, experts do not take into account that in the event of a crisis their fire-sales will depress the price at which other institutions are able to sell assets.

More formally, suppose that there are two groups of households. Sophisticated households are able to buy and resell assets from experts, while unsophisticated households cannot speculate. Sophisticated households' funds are limited, while unsophisticated households have deep pockets. Both groups of households have discount rate r, capital held in their hands depreciates at a higher rate $\underline{\delta} > \delta$ and, for simplicity, cannot invest internally.

If the value function of a sophisticated household with net worth $\underline{n}_t$ is given by $\underline{f}_t\underline{n}_t$, with

$$d\underline{f}_t = \underline{\mu}_t^f\, dt + \underline{\sigma}_t^f\, dZ_t,$$

then the first-order condition for the optimal investment strategy of sophisticated households is

$$a/p_t - \underline{\delta} + \mu_t^p + \sigma\sigma_t^p + \underline{\sigma}_t^f/\underline{f}_t\,(\sigma + \sigma_t^p) = 0. \qquad (*5*)$$

Denote by $\underline{N}_t$ the total amount of sophisticated household capital in the economy. Then equilibrium dynamics is characterized by two state variables, $\eta_t = N_t/K_t$ and $\underline{\eta}_t = \underline{N}_t/K_t$. The following proposition extends our equilibrium characterization to a model with sophisticated households.

Proposition 6. If sophisticated households can provide liquidity support, then aggregate capital in the economy follows dK_t and the state variable η_t follows the same evolution as described before. In addition, the second state variable $\underline{\eta}_t$ follows

$$d\underline{\eta}_t = (r - \psi_t\, g(p_t) + (1 - \psi_t)\, \underline{\delta} + \sigma^2)\, \eta_t\, dt + (1 - \psi_t)\, (a + (\mu_t^p - \underline{\delta} - \sigma^2 - r)p_t)\, dt +$$

$$((1 - \psi_t)\, p_t\, (\sigma + \sigma_t^p) - \sigma_t\, \underline{\eta}_t)\, dZ_t.$$

The equilibrium is characterized by four functions of $(\eta_t, \underline{\eta}_t)$, p_t, $(f_t, \underline{f}_t)$ and ψ_t, which are determined by the equations (*4*), (*5*), $(\rho - r)f_t = \mu_t^f$, and $\mu_t^{\underline{f}} = 0$.

Proof. To be completed.

Other externalities within the financial sector. If households are not financially constrained, the effect of these externalities is unclear: when the economy and the financial sector expand, the households' willingness to pick up assets also expands. When K_t grows, in equilibrium, households start absorbing assets at a larger value of N_t, and their capacity expands proportionately to the size of the economy. However, there are many natural extensions that give rise to externalities. There are externalities when experts trade, e.g. if they invest not internally but by buying capital from capital producers. Externalities may exist even without trade when the experts' contracts depend on prices, such as in the following examples:

- when experts can unload a fraction $1-\alpha_t$ of risk to outside investors, there are externalities when α_t depends on prices, for example when $\alpha_t = b/p_t$
- the terms of borrowing - the spread between the interest rate experts need to pay and the risk-free rate - may depend on prices. For example, there are externalities in the setting of Section 4, where experts face idiosyncratic jump risk.
- experts may be bound by margin requirements, which may depend both on price level and price volatility
- in asset management, the willingness of investors to keep money in the fund depends on short-term returns, and thus market prices

Overall, it may be hard to quantify the effects of many of these externalities directly, because each action has rippling effects through future histories, and there can be a mix of good and bad effects. To see how this can happen, let us explore how increased internal investment by one expert affects future values of α_t for everybody. Since volatility increases with higher leverage, investment leads to higher values of p_t and lower values of α_t in good states and vice versa in bad states. Given the mix of effects, it is best to study the overall significance of various externalities, as well as the welfare effects of possible regulatory policies, numerically on a calibrated model.

3.4 Externalities between the financial sector and real economy

To illustrate externalities of the financial sector to the real economy we model the labor market in a way that does not directly interfere with the equilibrium dynamics among financial intermediaries.

As Bernanke and Gertler (1989), suppose that households in the economy supply a fixed and inelastic amount of labor L. The production function is Cobb-Douglas in labor and capital, and it depends on the aggregate amount of capital in the economy,

$$y_t = (A\, K_t^{\alpha})\, l_t^{\alpha}\, k_t^{1-\alpha}.$$

The total amount of capital K_t in the production function reflects the idea from endogenous growth literature that technological progress increases productivity of everyone in the economy (e.g. see Romer (1986)). Recall that we do not measure capital k_t as the number of machines, but rather k_t is the cash-flow generating potential of capital under appropriate management. That is why it is difficult to quantify k_t and contract on it directly - the quantification of k_t involves something intangible. Therefore, a part of K_t is the level of knowledge and technological progress of the economy as a whole, and that part enters the production function of everyone.

In equilibrium, capital and labor is used for production proportionately, with $l_t = k_t\,(L/K_t)$. Wages per unit of labor and in the aggregate are given by

$$w_t = \alpha\, A/L^{1-\alpha}\, K_t \qquad \text{and} \qquad W_t = \alpha\, A\, L^{\alpha}\, K_t.$$

Capital owners receive output net of wages, which is

$$a\, k_t = (1 - \alpha)\, A\, L^{\alpha}\, k_t.$$

We see immediately that there are externalities between households, who supply labor, and the financial sector. Financial experts receive only a fraction 1-α of total output. Therefore, when they take actions that increase the likelihood of a downturn, such as taking on too much risk for the sake of short-term profits or paying out bonuses, they do not take into account the full extent of the damage of these downturns to the labor market.

To illustrate this point most clearly, we assume a constant marginal cost of capital production of $\varphi'(g) = a/(r + \underline{\delta})$ for $g \le g^*$, take $\varphi(g) = \infty$ for $g > g^*$, and normalize $\varphi(g^*) = 0$. That is, without investment capital grows according to

$$dk_t = g^*\, k_t\, dt + \sigma\, k_t\, dZ_t,$$

it cannot be made to grow any faster, but it can be liquidated in any amount at a constant price of $a/(r + \underline{\delta})$ per unit of capital. Under these assumptions, the experts' investment decisions are totally passive - and capital grows at rate g^* - whenever $p_t > a/(r + \underline{\delta})$. The only active decision involves bonus payouts. We call it the *passive investment economy*. The following proposition characterizes the equilibrium, which is the same with competitive investors and with a monopolist.

Proposition 7. In the passive investment economy, the equilibrium law of motion of $\omega_t = -D_t/K_t$ is given by

$$d\omega_t = ((r - g^* + \sigma^2)\, \omega_t + a)\, dt - \sigma\omega_t\, dZ_t - dC_t/K_t$$

on the interval $[-a/(r+\underline{\delta}), \omega^*]$, with a reflecting boundary at ω^* at which bonuses are paid out. The aggregate expert payoff function $h(\omega_t)K_t$ and point ω^* can be found from the equation

$$(\rho - g)\, h(\omega) = ((r - g^*)\, \omega + a)\, h'(\omega) + \tfrac{1}{2}\, (\sigma\omega)^2\, h''(\omega)$$

with boundary conditions $h(-a/(r+\underline{\delta})) = 0$, $h'(\omega^*) = 1$ and $h''(\omega^*) = 0$.

Proof. The desired conclusions follow directly from Proposition 4, which characterizes the optimal policy of a monopolist, and Proposition 5, which shows that the monopolist solution coincides with the competitive equilibrium. QED

We would like to argue that a regulator can improve social welfare by a policy that limits bonus payouts within the financial sector. Specifically, suppose that experts are not allowed to pay themselves as long as financial experts are not sufficiently capitalized (formally, until ω_t reaches some level $\omega^{**} > \omega^*$). This type of a regulation keeps capital within the financial system longer, and makes it more stable. The following proposition characterizes the equilibrium with such a regulatory policy, as well as the value functions of the experts and the households.

Proposition 8. If experts are not allowed to pay out bonuses until ω_t reaches $\omega^{**} = \omega^*$, they will pay at ω^{**}. The process ω_t follows the same equation, but with a reflecting boundary at ω^{**}. Expert value function is given by

$$\tilde{h}(\omega) = h(\omega)/h'(\omega^{**}),$$

where $h(\omega)$ is as in Proposition 5. The household value function is $H(\omega_t)K_t$, where $H(\omega)$ solves equation

$$(r - g)\, H(\omega) = \alpha A L^{\alpha} + ((r - g)\,\omega + a)\, H'(\omega) + \tfrac{1}{2}\, (\sigma\omega)^2\, H''(\omega), \qquad (**)$$

with boundary conditions $H(-a/(r+\underline{\delta})) = AL^{\alpha}/(r+\underline{\delta})$ and $H'(\omega^{**}) = -1$.

Proof. Then the household value function $H(\omega_t)K$ satisfies

$$r\, H(\omega_t)\, K_t = (a + b)\, K + ((r - g + \sigma^2)\, \omega_t + a)\, H'(\omega)\, K + \tfrac{1}{2}\, (\sigma\omega_t)^2\, H''(\omega_t)\, K + H(\omega)\, g\, K - H'(\omega)\, \sigma^2\, \omega \Rightarrow$$

To be completed.

How does such a regulatory policy affect welfare? For experts, note that $h'(\omega^{**}) > 1$ for $\omega^{**} > \omega^*$. Therefore, for a fixed level of ω_t, a restriction on compensation practices reduces expert welfare. However, since $h''(\omega^*) = 0$, $h'(\omega^{**})$ increases very little with ω^{**} near ω^*, the effect on expert welfare is second-order.

For households, for welfare analysis it is convenient to write $H(\omega)$ as a linear combination of the solutions of the homogeneous equation

$$(r - g)\, h_i(\omega_t) = ((r - g)\, \omega + a)\, h_i'(\omega) + \tfrac{1}{2}\, (\sigma\omega)^2\, h_i''(\omega).$$

Denote by h_1 and h_2 the functions that solve it with boundary conditions

$$h_1(-a/(r+\underline{\delta})) = 0,\;\; h_1'(-a/(r+\underline{\delta})) = 1,\;\; h_2(-L) = AL^{\alpha}/(r + \underline{\delta}) - \alpha AL^{\alpha}/(r - g) \;\text{ and }\; h_2(0) = 0.$$

Functions h_1 and h_2 are illustrated in Figure 9.

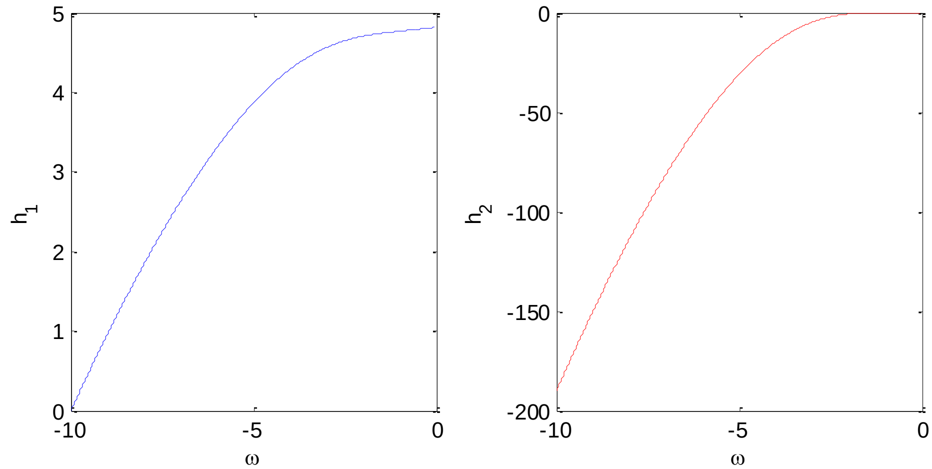

Figure 9: Solutions to the homogenous version of the household Bellman equation.

Lemma 4. Household welfare function under the policy that limits compensation for $\omega_t < \omega^{**}$ is given by

$$H(\omega) = \alpha AL^{\alpha}/(r - g) + q\, h_1(\omega) + h_2(\omega),$$

with $q = -(h_2'(\omega^{**}) + 1)/h_1'(\omega^{**})$. As ω^{**} increases, q increases.

Proof. It is easy to see that any function of the form $\alpha AL^{\alpha}/(r - g) + q_1\, h_1(\omega) + q_2\, h_2(\omega)$ satisfies the non-homogenous equation (**). Coefficient $q_2 = 1$ follows from the boundary condition $H(-a/(r+\underline{\delta})) = AL^{\alpha}/(r+\underline{\delta})$, since $h_1(-a/(r+\underline{\delta})) = 0$. Coefficient q_1 can be found from the boundary condition $H'(\omega^{**}) = -1$.

Since h_2 and h_1 are concave functions and $h_2'(\omega) < 1$ for $\omega > -a/(r+\underline{\delta})$, … *to be completed.*
QED

Because q is increasing in ω^{**}, the effect of ω^{**} on household welfare is first-order. Figure 10 shows the experts' and households' value functions for various choices of ω^{**} by the social planner.

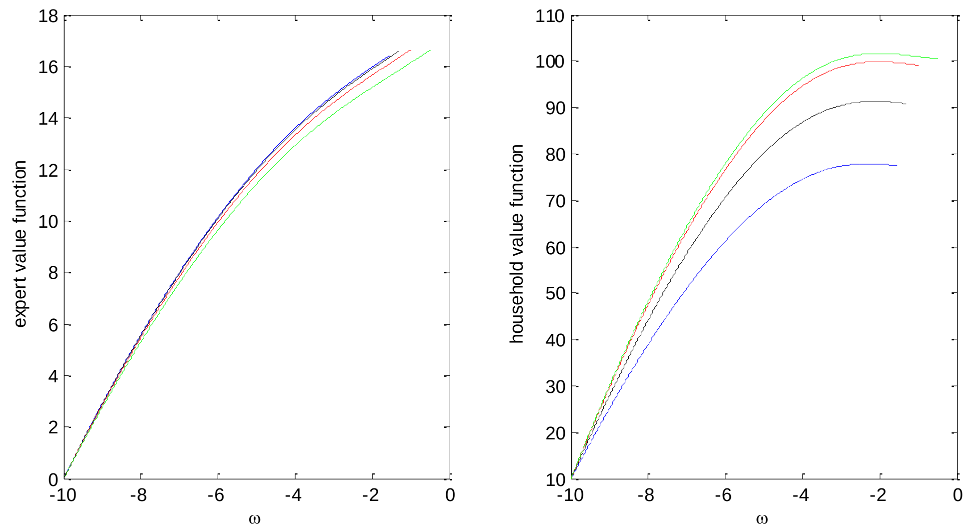

Figure 10: Value functions of experts and households for different regulatory policies; the blue functions corresponds to $\omega^{**} = \omega^*$.

We see that the central planner can improve efficiency by setting $\omega^{**} > \omega^*$. When ω^{**} is close to ω^*, the effect of policy on expert welfare is second-order, but the effect on households is first-order. Relative to the equilibrium without regulation, a social planner can implement a Pareto improvement by a policy that combines a transfer from households to the financial sector together with a regulation that limits bonus payouts. When ω_t is small, such a transfer can be interpreted as a bailout.

Without an accompanying transfer, regulation always hurts the financial experts in our baseline model. However, next we modify our baseline model to highlight possible externalities within the financial sector. In such a context, regulation can be welfare-improving even without accompanying transfers.

4. Idiosyncratic Shocks and Securitization

By securitization/hedging, we refer to various mechanisms by which financial institutions can share risks among each other. These mechanisms include pooling and tranching, by which the issuer can diversify and slice risks. Credit default swaps, and various options and futures contracts allow financial institutions to hedge specific risks. Furthermore, more efficient risk-sharing can be attained by longer intermediation chains between households and borrowers (e.g. see Shin (2010)).

A model with idiosyncratic shocks. A natural simple way to capture these phenomena, is to augment our baseline model to allow idiosyncratic shocks, which may be hedged within the financial sector, in the same spirit as BGG. Specifically assume that capital k_t managed by expert *i* evolves according to

$$dk_t = g\, k_t\, dt + \sigma\, k_t\, dZ_t + k_t\, dJ_t^i,$$

where dJ_t^i is an idiosyncratic Poisson loss process. As BGG we make the simplifying assumption that when experts get bigger, their idiosyncratic shocks are amplified proportionately, that is, there is no diversification of idiosyncratic shocks within any expert.

Losses after an idiosyncratic jump are characterized by the distribution function $F : [0, 1] \to [0, 1]$, which describes the percentage of capital that is recovered in the event of a loss. We can capture additional volatility effects by allowing the intensity of losses $\lambda(\sigma_t^p)$ to depend on the volatility asset prices p_t. This assumption is consistent with the general idea that interest rate spreads and margins are set by debt holders who worry about potential losses (which depend on volatility). It can be justified through an informal story that idiosyncratic shocks have to do with liquidity (such as the difficulty to find an acceptable buyer and having to sell assets at fire-sale prices). Note that this assumption would be vacuous under steady-state analysis, since price volatility is constant near the steady state.

To extend our agency model to idiosyncratic losses, we assume that an expert may generate losses for benefit extraction, getting b units of private financial benefit from a single unit of lost physical capital. While the expert's stake α_t in the assets prevents losses of size n_t/b or less, we assume that costly state verification is possible to prevent larger losses.[16] We assume that if verification is immediate when an expert simulates a loss in order to steal money, the fraud is revealed and the expert cannot get any private benefit. As in BGG, we assume that the verification cost is a fraction $c \in (0,1)$ of the amount of capital recovered.[17]

Default and costly state verification occur when the value of the assets $v = p_t k_t$ falls below the value of debt $d = v - n_t/\alpha_t$ i.e. k_t falls by more than $n_t/(\alpha_t p_t) = n_t/\alpha_t$. Note that the expected loss in the event of default is

$$v\, H(d/v) = v \int_0^{d/v} (d/v - x)\, dF(x)$$

where x is the fraction of assets left after loss. Default occurs in the event that $x < d/v$. The expected verification cost is

[16] It is optimal to trigger verification if and only if k_t^i drops below n_t/b because, as we will see later, in equilibrium the expert is risk-neutral towards idiosyncratic risk that does not lead to default.

[17] The basic costly state verification framework, developed by Townsend (1979) and adopted by Bernanke-Gertler and Gilchrist (1999) is a two-period contracting framework. At date 0, the agent requires investment I from the principal, and at date 1 he receives random output y distributed on the interval $[0, \bar{y}]$. The agent privately observes output y, but the principal can verify it at a cost. The optimal contract under commitment is a standard debt contract. If the agent receives $y \geq D$, the face value of debt, then he pays the principal D and there is no verification. If $y < D$, the agent cannot pay D and costly state verification (bankruptcy) is triggered, and debtholders receive all of output.

$$v\ C(d/v) = v \int_0^{d} c\ x d\ F(x)$$

To break even, households who lend money to the expert must get not only the interest rate ρ, but also compensation for the possible losses and verification costs at a total rate of $\lambda(\sigma_t^p)\ V\ (L(d/v) + C(d/v))$. This quantity defines the spread that the expert needs to pay to borrow from households.

As in the remainder of the paper, we now narrow down analysis to the case when $\alpha_t = 1$, which occurs when the agent is able to fully extract the value of lost capital from the losses he generates.

As before, the equilibrium is characterized by the state variable η_t, and prices $p_t = p(\eta_t)$ and the expert's value function $f_t = f(\eta_t)$ are functions of η_t. The net worth of an individual expert evolves according to

$$dn_t = r\ n_t\ dt +$$

$$k_t\ [(a - \iota(g) - (r + \lambda(\sigma_t^p)\ (L(\xi_t) + C(\xi_t)) - g + dJ_t^i)\ p_t + \mu_t^p + \sigma\sigma_t^p)\ dt + (\sigma p_t + \sigma_t^p)\ dZ_t] - dc_t,$$

where $\xi_t = 1 - n_t/(p_t k_t)$ is the expert's leverage ratio.

In the aggregate, idiosyncratic losses cancel out and total expert capital evolves according to

$$dN_t = rN_t\ dt + K_t\ [(a - \iota(g) - (r + \lambda(\sigma_t^p)C(\xi_t) - g)p_t + \mu_t^p + \sigma\sigma_t^p)\ dt + (\sigma p_t + \sigma_t^p)\ dZ_t] - dC_t,$$

where the term $L(\xi_t)$ disappears because of limited liability. The modified law of motion of $\eta_t = N_t/K_t$ is

$$d\eta_t = (r - g + \sigma^2)\ \eta_t\ dt + (a - \iota(g) - (r + \lambda(\sigma_t^p)C(\xi_t) - g + \sigma^2)p_t + \mu_t^p)\ dt$$

$$+ (\sigma p_t + \sigma_t^p - \sigma_t\ \eta_t)\ dZ_t - d\zeta_t.$$

The Bellman equation and the first-order condition with respect to k_t are now

$$(\rho - r)\ f(\eta_t)\eta_t\ = \mu_t^f\ \eta_t + f(\eta_t)\ (a - \iota(g) - (r + \lambda(\sigma_t^p)C(\xi_t) - g)p_t + \mu_t^p + \sigma\sigma_t^p)) + \sigma_t^f\ (\sigma p_t + \sigma_t^p)$$

$$\text{and} \quad a - \iota(g) - (r + \lambda(\sigma_t^p)(C(\xi_t) + (1-\xi_t)C'(\xi_t)) - g)p_t + \mu_t^p + \sigma\sigma_t^p + \sigma_t^f/f(\eta_t)\ \ (\sigma p_t + \sigma_t^p) = 0.$$

As before, in equilibrium η_t evolves on the range $[0, \eta^*]$, with a different boundary η^*. Experts pay themselves bonuses only when η_t is at η^*.

In equilibrium experts borrow at a rate higher than r due to verification costs - they pay the rate $r + \lambda(\sigma_t^p)\,(L(\xi_t) + C(\xi_t))/\xi_t$. This is the promised interest rate - due to limited liability in the event of default the actual cost of borrowing is only $r + \lambda(\sigma_t^p)/\xi_t$. Higher cost of borrowing makes equilibrium leverage relative to our baseline model without idiosyncratic shocks.

Securitization. We model securitization as risk-sharing within the financial sector. Specifically, assume that all shocks, both idiosyncratic J_t^i and aggregate Z_t, are observable and contractible among the experts, but not between experts and households.

Denote by ω_t the risk-premium on aggregate risk and by ω_t^i the risk premium on idiosyncratic risk. A hedging contract for aggregate risk adds

$$\theta_t\,(\omega_t\,dt + dZ_t)$$

to the law of motion of expert *i*'s wealth, where θ_t is the overall risk exposure. A contract on expert *j*'s idiosyncratic risk adds

$$\theta_t^j\,(\omega_t^j\,dt + dJ_t^j)$$

to the law of motion of expert *i*'s wealth, and may affect the verification region and verification costs. The following proposition characterizes the equilibrium when hedging within the financial sector is possible.

Proposition. *If hedging within the financial sector is possible, then in equilibrium experts will fully hedge idiosyncratic risk, which carries the risk premium of* $\omega_t^i = 0$. *Nobody hedges aggregate risk, which carries the risk premium of* $\omega_t = -\sigma_t^f/f_t > 0$.
Since idiosyncratic shocks are fully hedged, the equilibrium is identical to one in a setting without those shocks.

Proof. It is easy to see that the idiosyncratic risks are fully hedged and that the risk premia are zero, since market clears when each expert optimally chooses to offload his own idiosyncratic risk, and take on a little bit of everybody's risks (which cancel out). Once idiosyncratic risks are removed, the law of motion of individual expert's capital is

$$dn_t = \rho\, n_t\, dt + k_t\,((a - (\rho - g)\, p_t + \mu_t^p + \sigma\, \sigma_t^p)\, dt + \alpha_t\,(\sigma p_t + \sigma_t^p)\, dZ_t) - \theta_t\,(\omega_t\, dt + dZ_t),$$

where the optimal choice of θ_t must be zero in order for hedging markets to clear. The appropriate risk premium for aggregate risk can be found from the Bellman equation

$$\rho f_t n_t = \max_{k,\theta}\ \mu_t^f\, n_t + f_t\,(\rho\, n_t + k\,(a - (\rho - g)\, p_t + \mu_t^p + \sigma\sigma_t^p) + \theta\omega_t) + \sigma_t^f\,(k\alpha_t(\sigma p_t + \sigma_t^p) + \theta).$$

In order for $\theta = 0$ to be optimal, we need $\omega_t = -\sigma_t^f/f_t$. QED

Experts fully hedge out idiosyncratic shocks when securitization is allowed, they face the cost of borrowing of only r, instead of $r + \lambda(\sigma_t^p)/\xi_t$. Lower cost of borrowing leads to higher leverage quicker payouts. As a result, the financial system becomes less stable. Thus, even though in principle securitization is a good thing, as it allows financial institutions to share idiosyncratic risks better, it leads to greater leverage and the amplification of systemic risks.

Remark. By varying the verification costs and the loss distribution, our framework can capture several other models. Kiyotaki-Moore assume that financial experts can borrow only up to fraction θ of the market value of assets. Thus, someone with net worth n_t can hold at most $1/(1-\theta)n_t$ worth of assets, by financing $\theta/(1-\theta)n_t$ of the assets with debt and the rest, n_t, with personal wealth. This is captured in our framework by setting the verification costs to zero up to a certain level and infinity afterwards. Alternatively, one can assume that margins are set equal to the value-at-risk (VaR) as in Shin (2010). In Brunnermeier and Pedersen (2009), margins increase with endogenous price volatility. These effects are captured in our model through the dependence of potential losses on price volatility. The framework of BGG, who use the costly state verification model of Townsend (1979), corresponds to the assumptions that $\alpha_t = 1$ and $\lambda(\sigma_t^p)$ is a constant.

5. Conclusions and Regulatory Implications

Events during the great liquidity and credit crunch in 2007-09 have highlighted the importance of financing frictions for macroeconomics. Unlike many existing papers in macroeconomics, our analysis is not restricted to local effects around the steady state. Importantly, we show that non-linear effects in form of adverse feedback loops and liquidity spirals are significantly larger further away from the steady state. Especially volatility effects and behavior due to precautionary motives cause these large effects. Second, we identify and isolate several externalities both within the financial sector and also from the financial sector to the real sector of the economy. Due to these externalities, financial experts leverage and maturity mismatch is excessive. We argue that financial regulation should aim to internalize these externalities. For this purpose co-risk measures have to be developed.

Appendix A: Our contracting space.

Consider a principal-agent environment, in which the agent generates cash flows

$$dX_t = \pi(e_t, k_t, \eta_t)\, dt + \sigma(k_t, \eta_t)\, dZ_t,$$

where $e_t \in \{0, -E\}$ is the agent's effort, k_t is the scale of production and η_t is the state of the economy that evolves according to

$$d\eta_t = \mu_t^\eta\, dt + \sigma_t^\eta\, dZ_t',$$

and cannot be controlled by the agent or the principal. Brownian motions Z and Z′ may be correlated.

In our model

$$\pi(e_t, \eta_t, k_t) = k_t\,(a - \iota(g) - (r - g)p_t + \mu_t^p + \sigma\sigma_t^p + p_t\, e_t) \qquad \text{and} \quad \sigma(\eta_t, k_t) = k_t\,(\sigma p_t + \sigma_t^p).$$

Both the principal and the agent are risk-neutral, and the agent's discount rate ρ is greater than the risk-free rate r. The agent's payoff flow is $dc_t - h(e_t, k_t, \eta_t)\,dt$, where $h(e, k, \eta) = b\, f(\eta)ke$ in our setting. We assume that $b < a/(r + \underline{\delta})$, so it is never optimal to let the agent not put effort (because it is always better to liquidate capital).

There is a standard way to solve these problems using the agent's continuation value as a state variable, but those solutions may be challenging to relate to real world. Much more intuitive is the approach of Fudenberg, Holmström and Milgrom (1990), who propose an implementation of the optimal contract with the agent's wealth as a state variable, in which the principal breaks even at any moment of time. Such an implementation exists whenever the principal's profit function $F(w_t, \eta_t)$ is decreasing in the agent's continuation value W_t, and the agent's continuation payoff as a function of wealth n_t and the state of the economy η_t is

$$w_t = H(n_t, \eta_t), \qquad \text{such that } F(w_t, \eta_t) = -n_t.$$

Agents enter short-term contracts with principals, characterized by variables β_t, k_t and β_t'. Under this contract the agent collects a fraction of output β_t, the principal collects $1 - \beta_t$ and pays the agent the fee $(1 - \beta_t)\,\pi(e_t, k_t, \eta_t)$ (so the principal breaks even) and the agent hedges β_t' of aggregate risk, so that

$$dn_t = \beta_t\, dX_t + (1 - \beta_t)\,\pi(e_t, k_t, \eta_t)\,dt + \beta_t'\, dZ_t' - dc_t$$

Optimal short-term contracts (β_t, k_t, β_t') can be found from the Bellman equation

$$\rho H(n_t, \eta_t)dt = \max_{\beta,\, k,\, \beta',c,e} dc_t - h(e, k, \eta_t) + E\,[\, dH(n_t, \eta_t)\,]$$

subject to the incentive-compatibility constraint that e maximizes

$$\beta\, H_1(n,\eta)\, \pi(e, k, \eta) - h(e, k, \eta).$$

We do not allow the full contracting space in our paper, but limit the hedging of aggregate risk by forcing β' to be 0. With limited instruments, optimization leads to the value functions

$$H(n_t,\eta_t) = f(\eta_t)\, n_t \qquad \text{and} \qquad F(w_t,\eta_t) = -w_t/f(\eta_t),$$

with $F(w_t,\eta_t)$ decreasing in w_t as required by Fudenberg, Holmström and Milgrom (1990). Thus, contracts in our paper are optimal dynamic contracts from a smaller contracting space, so we have contract incompleteness.

In the section on securitization, we allow hedging of aggregate risk through a market mechanism within the financial sector. This leads to a risk premium λ_t on aggregate risk, so the agent's wealth evolve according to

$$dn_t = \beta_t\, dX_t + (1 - \beta_t)\, \pi(e_t, k_t, \eta_t)\, dt + \beta_t'\, (dZ_t' + \lambda_t\, dt) - dc_t.$$

Appendix B: Contracting at k_t

Appendix B analyzes the case in which contracting directly on k_t is possible instead of $k_t p_t$. For simplicity, we focus on the case where α=1. Expert manages capital that follows

$$dk_t = g\, k_t\, dt + \sigma\, k_t\, dZ_t$$

(if he puts effort) and produces output (a-i) dt. Furthermore, suppose that the expert can "divert" capital, and get the marginal benefit of $\lambda \leq 1$ units of capital per unit diverted (note: our baseline model corresponds to $\lambda = 1$; we call it α in the paper). The price of capital is p_t and, because this is an expert, his outside value of funds is f_t per dollar. What is the optimal contract, if k_t is used as the measure of performance? Consider contracts based on the agent's net worth as a state variable

$$dn_t = r\, n_t\, dt + \beta_t\, (dk_t - g\, k_t\, dt) - \sigma_t^f/f_t\, \beta_t\, \sigma\, k_t\, dt,$$

where the incentive constraint is

$$\beta_t \geq \lambda\, p_t,$$

since the expert gets $\lambda\, p_t$ units of net worth (that can be used elsewhere to gain utility $\lambda\, p_t\, f_t$) for one unit of capital diverted.
Just to make sure that $f_t\, n_t$ is a martingale, we have

$$d(n_t f_t) = (r\, n_t\, dt + \beta_t\, \sigma\, k_t\, dZ_t - \sigma_t^f/f_t\, \beta_t\, \sigma\, k_t\, dt)\, f_t + (\mu_t^f\, dt + \sigma_t^f\, dZ_t)\, n_t + \beta_t\, \sigma\, k_t\, \sigma_t^f\, dt =$$

$$d(n_t f_t) = (r\, n_t\, dt + \beta_t\, \sigma\, k_t\, dZ_t)\, f_t + ((\rho - r)\, f_t\, dt + \sigma_t^f\, dZ_t)\, n_t = \rho\, (n_t f_t)\, dt + \text{volatility term},$$

where we used the property that $\mu_t^f = (\rho - r)\, f_t$ (the same as in the main paper).

Now, how about market price of capital? If contracting is based on k_t only, then households hire experts to manage their capital, but households themselves take on the price risk. The market price of capital still depends on the experts' risk-taking capacity. The net worth of a household that holds capital k_t evolves according to

$$(a - i)\, k_t\, dt + d(p_t k_t) - \beta_t\, k_t\, \sigma\, dZ_t + \sigma_t^f/f_t\, \beta_t\, \sigma\, k_t\ dt =$$
$$(a - i)\, k_t\, dt + (p_t k_t)\, [(\mu_t^p + g + \sigma\, \sigma_t^p)\, dt + (\sigma + \sigma_t^p)\, dZ_t] - \ ) - \beta_t\, k_t\, \sigma\, dZ_t + \sigma_t^f/f_t\, \beta_t\, \sigma\, k_t\ dt$$

In expectation households should get the return of r $p_t k_t$, so we need

$$(a - i)/p_t + \mu_t^p + g + \sigma\, \sigma_t^p - r + \sigma_t^f/f_t\, \lambda\, \sigma\ = 0.$$

This equation is different from the one in the paper because the risk premium is based only on exogenous risk (for which households must compensate the experts that manage their capital).
Also, the law of motion of η_t will be different, since (combining the law of motion of n_t and the condition that the households must get return r)

$$dn_t = r\, n_t\, dt + (k_t p_t)\, [((a - i)/p_t + \mu_t^p + g + \sigma\, \sigma_t^p - r)\, dt + \lambda\, \sigma\, dZ_t\,]$$

has a missing endogenous risk term. As a result,

$$dN_t = r\, N_t\, dt + K_t\, [\, (a - i + (g + \mu_t^p + \sigma\sigma_t^p - r)\, p_t)\, dt + \lambda\, \sigma\, p_t\, dZ_t]$$

and in combination with
$dK_t = g\, K_t\, dt + \sigma\, K_t\, dZ_t \Rightarrow d(1/K_t) = - g/K_t\, dt + \sigma^2/K_t\, dt - \sigma/K_t\, dZ_t \Rightarrow$

$$d\eta_t = r\, \eta_t\, dt + (a - i + (g + \mu_t^p + \sigma\sigma_t^p - r)\, p_t)\, dt + \lambda\, \sigma\, p_t\, dZ_t + \eta_t\, (-g\, dt + \sigma^2\, dt - \sigma\, dZ_t) - \lambda\, \sigma^2\, p_t\, dt =$$
$$d\eta_t = (r - g + \sigma^2)\, \eta_t\, dt + (a - i + (g + \mu_t^p + \sigma\sigma_t^p - r - \lambda\sigma^2)\, p_t)\, dt + \ \sigma\, (\lambda p_t - \eta_t)\, dZ_t$$

For the case when $\lambda = 1$ (which is what we assume in our paper),

$$\sigma_t^\eta = \sigma(p_t - \eta_t), \qquad \sigma_t^p = p'(\eta_t)/p_t\, \sigma(p_t - \eta_t),$$

so there's still amplification through leverage (that's the difference between the price of capital p_t and experts' net worth η_t), but no more feedback effect through prices.

$$\mu_t^p\, p_t = p'(\eta_t)\, [(r - g + \sigma^2)\, (\eta_t - p_t) + (a - i + (\mu_t^p + \sigma\sigma_t^p)p_t)\,] + \tfrac{1}{2}\, (\sigma_t^\eta)^2\, p''(\eta_t) \Rightarrow$$
$$\mu_t^p = \frac{p'(\eta_t)[(r - g + \sigma^2)(\eta_t - p_t) + a - i + \sigma\sigma_t^p p_t] + \frac{1}{2}(\sigma_t^\eta)^2 p''(\eta_t)}{p_t(1 - p'(\eta_t))}$$

and the equilibrium can be characterized via ODEs just like in our model. First, $p''(\eta)$ can be found from

$$-\frac{a - i}{p_t} - g - \sigma\sigma_t^p + r - \frac{\sigma_t^f}{f_t}\sigma = \frac{p'(\eta_t)[(r - g + \sigma^2)(\eta - p_t) + a - i + \sigma\sigma_t^p p_t] + \frac{1}{2}(\sigma_t^\eta)^2 p''(\eta)}{p_t(1 - p'(\eta))}$$

and f'', from

$$(\rho - r)\, f_t = f'(\eta_t)\, [(r - g + \sigma^2)\, (\eta_t - p_t) + (a - i + (\mu_t^p + \sigma\sigma_t^p)p_t)\,] + \tfrac{1}{2}\, (\sigma_t^\eta)^2\, f''(\eta_t).$$

Of course, these equations don't look very insightful (but they can be used to compute equilibria) but the volatility formula $\sigma_t^p = p'(\eta_t)/p_t\, \sigma(p_t - \eta_t)$ definitely is.

Appendix C.

Proposition A1. There is a stationary distribution of ω_t only if the system never becomes absorbed at $-a/(r+\underline{\delta})$ because assets are liquidated sufficiently fast when ω_t approaches $-a/(r+\underline{\delta})$. In that case, the stationary density must satisfy the standard equation

$$\tfrac{1}{2}\, d^2/d\omega^2\, (\sigma^\omega(\omega)^2\, d(\omega)) + d/d\omega\, (\mu^\omega(\omega)\, d(\omega)) = 0,$$

where $\mu^\omega = ((r - g + \sigma^2)\, \omega + a - \varphi(g)$ and $\sigma^\omega = - \sigma\omega$. The relevant boundary conditions are $d'(\omega^*) = 0$ (because it is a reflecting boundary) and

$$\int_{-L}^{\omega^*} d(\omega) d\omega = 1.$$

Proof. To be completed.

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