The Formation of Financial Networks

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Abstract

Modern banking systems are highly interconnected. Despite their various benefits, the linkages that exist between banks carry the risk of contagion. In this paper we investigate how banks decide on direct balance sheet linkages and the implications for contagion risk. In particular, we model a network formation process in the banking system. Banks form links order to reduce the risk of contagion. The network is formed endogenously and serves as an insurance mechanism. We show that banks manage to form networks that are resilient to contagion. Thus, in an equilibrium network, the probability of contagion is virtually 0.

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1 Introduction

A notable feature of the modern financial world is its high degree of interdependence. The banking system can be easily described as a collection of banks and links between banks, thus, essentially, as a network. The incentives for linking are driven by the benefits these links bring. For instance, banks can solve their liquidity unbalances without requiring the intervention of a Central Bank simply by transferring funds from the ones that have a cash surplus to those with a cash deficit. The supply and demand for liquidity connect in this way the financial institutions into a network. Are the connections in the financial world, however, limited to the ones created by market forces? There are many instances that suggest this is not the case. The "too big to fail" phenomenon, that receives a lot of attention in the economic literature, is one example. Linking, under the form of lending money, is not motivated here by the intersection of supply and demand. The rational behind bailing out a certain institution is given by the danger of contagion.

In this paper we develop a model where banks form links with each other in order to reduce the risk of contagion. The network is formed endogenously between banks and serves as an insurance mechanism.

Banks and other financial institutions are linked in a variety of ways. Despite their obvious benefit, the linkages come at the cost that small shocks, which initially affect only a few institutions, can propagate through the entire system. The problem of contagion can be conveniently rephrased in terms of networks. In particular, two interesting questions are to be addressed. On the one hand, it is worthwhile investigating what network structures are resilient to contagion. On the other hand, perhaps more importantly is to understand how financial institutions form these connections.

This paper addresses the second question. We investigate how banks form linkages with each other, where a link represents a transfer of funds between two institutions. In particular, we study a network formation process that is mainly driven by the risk of contagion. Identifying what networks emerge in equilibrium and how resilient to contagion these structures is of particular importance for emergent market economies. Lacking the sound regulatory frameworks, that characterize the developed financial systems, the banking systems of developing economies have to rely on the ways banks take decisions. The network formation process can be interpreted as decentralizing an insurance scheme that a central planner would adopt.

The tools advanced by the theory of network formation helps answer this question. A recent and rapidly growing literature on network formation games has developed in the past few years, introducing various approaches to model network formation and proposing several equilibrium concepts (Bala and Goyal, 2000, Bloch and Jackson, 2006, Jackson and Wolinsky, 1996, Dutta et. al, 2005).

Despite the numerous applications of these models in the social science context, the literature on financial networks is still very much incipient. Moreover, most of the research done in financial networks studies network effects. Largely, they investigate how different network structures respond to the breakdown of a single bank in order to identify which ones are resilient to contagion.

The theoretical literature mainly concentrates on contagious effects via direct balance sheet interlinkages. For instance, Freixas et al. (2000) considers the case of banks that face liquidity needs as consumers are uncertain about where they are to consume. In their model the connections between banks are realized through interbank credit lines that enable these institutions to hedge regional liquidity shocks. The authors analyze different market structures and find that a system of credit lines, while it reduces the cost of holding liquidity, makes the banking sector prone to experience gridlocks, even when all banks are solvent. Dasgupta (2004) also discusses how linkages between banks represented by crossholding of deposits can be a source of contagious breakdowns. Fragility arises when depositors, that receive a private signal about banks' fundamentals, may wish to withdraw their deposits if they believe that enough other depositors will do the same. A unique equilibrium is isolated and this depends on the value of the fundamentals. Eisenberg and Noe (2001) take a more technical approach when investigating systemic risk in a network of financial institutions. First the authors show the existence of a clearing payment vector that defines the level of connections between banks. Next, they develop an algorithm that allows them to evaluate the effects small shocks have on the system. Leitner (2005) constructs a model that shows how agents may be willing to bail out other agents, in order to prevent the collapse of the whole network. Parallel to this literature, there is a number of theoretical papers that focus on indirect linkages. Lagunoff and Schreft (2001) construct a model where agents are linked in the sense that the return on an agent's portfolio depends on the portfolio allocations of other agents. Similarly, de Vries (2005) shows that there is dependency between banks' portfolios, given the fat tail property of the underlying assets, and this caries the potential for systemic breakdown.

The empirical papers which study banking contagion paint a more optimistic message. Recently, there has been a substantial interest in looking for evidence of contagious failures of financial institutions resulting from the mutual claims they have on one another. Most of these papers use balance sheet information to estimate bilateral credit relationships for different banking systems. Subsequently, the stability of the interbank market is tested by simulating the breakdown of a single bank. Upper and Worms (2004) analyze the German banking system. Sheldon and Maurer (1998) consider the Swiss system. Cocco et al. (2003) present empirical evidence for lending relationships existent on the Portuguese interbank market. Furfine (2003) studies the interlinkages between the US banks, while Wells (2002) looks at the UK interbank market. Boss et al. (2004) provide an empirical analysis of the network structure of the Austrian interbank market and discuss its stability when a node is eliminated. In the same manner, Degryse and Nguyen (2004) evaluate the risk that a chain reaction of bank failures would occur in the Belgian interbank market. These papers find that the banking systems demonstrate a high resilience, even to large shocks. Simulations of the worst case scenarios show that banks representing less than five percent of total balance sheet assets would be affected by contagion on the Belgian interbank market, while for the German system the failure of a single bank could lead to the breakdown of up to 15% of the banking sector in terms of assets.

The paper that is closest related to ours is by Allen and Gale (2000). Allen and Gale (2000) study how banking system responds to contagion when banks are connected under different network structures. The authors show that incomplete networks are more prone to contagion than complete structures. Specifically, they take the case of an incomplete network where the failure of a bank may trigger the failure of the entire banking system. They prove that, for the same set of parameters, if banks are connected in a complete structure, then the system is resilient to contagious effects.

The main innovation of our paper is to endogenize the network of banks. Although, we use the same framework to motivate interactions on the interbank market, we no longer consider that the network of banks is fixed. We allow the endogenous formation of links and analyze what are the implications for the stability of the banking system. At the base of the link formation process lies the same intuition developed in Allen and Gale (2000): better connected networks are more resilient to contagion. In fact, in our model there is a connectivity threshold above which contagion does not occur. Thus, in order to insure against the risk of contagion, banks form links to rich this threshold. We show that banks manage to form networks that are usually resilient to the propagation of shocks.

This paper is organized as follows. Section 2 presents an illustrative example of how a network formation process develops in a 4 - bank framework. The details of the model are presented in Section 3, while Section 4 constructs a game theoretical structure and introduces the payoffs banks have from forming links. Section 5 models the network formation process and Section 6 analyses the efficiency of banks' link formation decisions. Section 7 concludes.

2 Example

To illustrate what incentives banks have when forming link, we start by constructing a simple example. We consider the banking system consists of 4 banks. Shocks in the liquidity demand differentiate banks into two types in the following way: there will be two banks that have a liquidity surplus $+z$ and two banks with a liquidity shortage $-z$. Although ex-ante banks do not have information on their type, they know how the liquidity shocks are correlated. Thus, they can insure against these shocks simply by transferring funds between banks of a different type. These transfers create links between banks, exposing the system to the risk of contagion. Explicitly, when a bank fails it induces a loss to all the neighboring banks. If this loss is above a threshold, the banks that incur it will go bankrupt as well. This way, the initial shock propagates to other banks via links.

We have learned that the risk of contagion depends on the size of the loss a failed bank induces to its neighbors. The loss has two interesting properties. First, it is increasing in the size of the deposits transferred between banks. The size of the deposits, however, depends on the number of connections with banks of a different type.[1] Hence, the loss a bank incurs to its neighbors decreases with the number of neighbor banks of a different type. Second, the loss is decreasing in the total number of links a bank has. Thus, increasing the connectivity level in the banking system reduces its propensity to contagion. This explains why a complete network is more resilient to the propagation of shocks than

[1] Only transfers between banks of a different type play a role in balacing the liquidity surplus or shortage of each bank. Moreover, when a bank has two neighbors of a different type it can use both connections to gain insurance, thus reducing by half the size of each transfer.

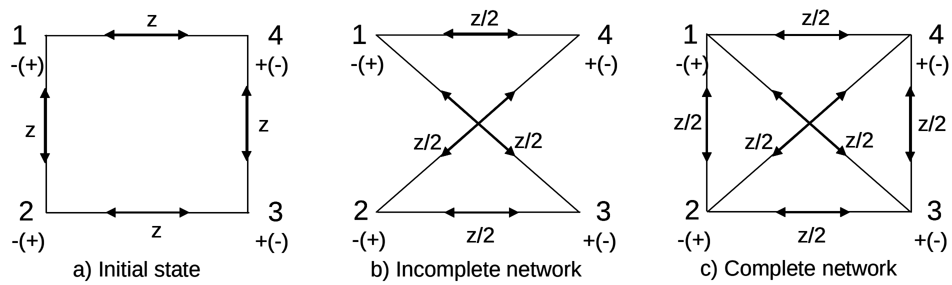

Figure 2.1: Different states of the banking system

incomplete structures.[2]

We take the following initial state of the banking system simply to illustrate how these two properties affect the banks' link formation decisions. Consider a banking system connected as in fig. 2.1(a), and the corresponding transfers of interbank deposits marked by arrows. We assume that the set of parameters is such that a shock in one of the banks will propagate through contagion in the entire system. For the same parameters, however, the complete network[3] is sufficient to prevent contagion. We do not impose necessity, in order to follow the link formation process gradually. We study how a social planner would design a network in order to prevent contagion, in parallel to the decentralized link formation process.

Social planner solution: A social planner that is able to organize transfers between banks faces a trivial problem. Simply redirecting the links in order to move the banking system to the network shown in fig. 2.1(b) reduces the size of interbank deposits by half. If the loss associated to the new level of deposits is below the limit loss threshold, than the failure of a bank will have no significant effects on the other banks in the system. The social planner can achieve the objective of 0 contagion. Otherwise, the social planner can further decrease the loss by increasing the connectivity level in the system. Adding a link between banks 1 and 2 on the one hand, and banks 3 and 4, on the other hand increases the number of connections each bank has and, thus, reduces the loss a bank spreads when it fails. The parameters are assumed to insure that the complete network is resilient to contagion. Thus, the social objective is completed again.

Banks solution: The banking system has the incentive to implement the social planner

[2] We show why the two properties hold in Section 3.

[3] All the concepts are properly defined in Section 4.

solution. To understand what drives the link formation decision of banks, one should note that the payoffs from linking are heterogenous. Linking with banks of the same type is costly, as it exposes banks to the risk of contagion without bringing any direct benefits. While links with banks of a different type provide insurance against liquidity shocks, links with banks of the same type serve only to reduce systemic risk by increasing the connectivity of the system. Since the connectivity of the system depends only on the number of links each bank has, banks will always trade a link with a bank of the same type for a link with a bank of a different type.

Whether the banking system will be shaped as an incomplete network, depends on whether an incomplete network is resilient to contagion. Nonetheless, the initial state of the system is unstable, since banks have the incentive, as reasoned above, to severe the links with neighbors of the same type. Provided that they can coordinate, banks will move the system to the network represented in fig. 2.1(b). If this pattern of connections is yet insufficient to prevent contagion, the banks can and will move the system to the complete network. Consider, for instance, the case of banks 1 and 2 in fig. 2.1(b). Although there is no link between 1 and 2, the failure of either bank will trigger the failure of the other. The failure of bank 1 (2) will determine banks 3 and 4 to go bankrupt and this will cause in turn bank 2 (1) to fail.[4] Assuming that the loss a bank spreads can be further reduced by increasing the number of links per bank, than both banks 1 and 2 are better off in a network where they are directly connected. Hence, both banks 1 and 2 will agree to form the link that will prevent them failing by contagion. Similar incentives determine bank 3 and 4 to link. The complete network is formed.

3 The Model

3.1 Consumers and Liquidity Shocks

We assume that the economy is divided into $2n$ regions, each populated by a continuum of risk averse consumers. There are three time periods $t = 0, 1, 2$. Each agent has an endowment equal to one unit of consumption good at date $t = 0$. Agents are uncertain

[4] Note that bank 2 (1) will fail independently of how many links banks 3 and 4 have. Although banks 3 and 4 might have sufficient links such that a single bank failure will not propagate further in the system, when both fail simultaneously the cumulated loss will be large enough to trigger the failure of 2 (1).

about their liquidity preferences: they are either early consumers, who value consumption only at date 1, or they are late consumers, who value consumption only at date 2. In the aggregate there is no uncertainty about the liquidity demand in period 1. Each region, however, experiences different liquidity shocks, caused by random fluctuations in the fraction of early consumers. Thus, although it is known with certainty that on average the fraction of early consumers is $q = (p_H + p_L)/2$, this is realized only with a small probability π. Each region, however, will face with probability $(1-\pi)/2$ either a high proportion p_H of agents that need to consume at date 1 or a low proportion p_L of agents that value consumption in period 1.

All the uncertainty is resolved at date 1, when the state of the world is realized and commonly known. At date 2, the fraction of late consumers in each region will be $(1-p)$ where the value of p is known at date 1 as either p_H or p_L.

3.2 Banks, Demand Deposits and Asset Investments

We consider that in each region i there is a competitive representative bank. Agents deposit their endowment in the regional bank. In exchange, they receive a deposit contract that guarantees them an amount of consumption depending on the date they choose to withdraw their deposits. In particular, the deposit contract specifies that if they withdraw at date 1, they receive $C_1 > 1$, and if they withdraw at date 2, they receive $C_2 > C_1$.

There are two possibilities to invest. First, banks can invest in a liquid asset with a return of 1 after one period. They can also choose an illiquid asset that pays a return of $r < 1$ after one period, or $R > 1$ after two periods. Let x and y be the per capita amounts invested in the liquid and illiquid asset, respectively. It seems natural that banks will use the liquid asset to pay depositors that need to withdraw in the first period and will reserve the illiquid asset to pay the late consumers. Since the average level of liquidity demand at date 1 is qC_1, we assume that the investment in the liquid asset, x, will equal this amount, while the investment in the illiquid asset, y, will cover $(1-q)C_2/R$.[5] This way at date 1 each bank has with probability $(1-\pi)/2$ either a liquidity shortage of $zC_1 = (p_H - q)C_1$ or a liquidity surplus of $zC_1 = (q - p_L)C_1$.

In addition, banks are subject to idiosyncratic shocks that are not insurable. For

[5] Allen and Gale (2000) show in their paper that both the deposit contract and the investement in liquid and illiquid asset are such that they maximize the expected utility of consumers.

reasons that will become clear in due course, these idiosyncratic shocks are attached to the state of the world $\bar{S}$. In particular, with a small probability π, the failure of a bank will occur in period 1. This implies that each bank has an individual probability of $\pi/2n$ of going bankrupt in period 1. This event is anticipated and the way will affect banks' actions will be explained later in the paper.

The shocks in the regional liquidity demand can now be expressed as shortages or excesses in the liquidity holdings of each bank. Table 1 best describes the states of the world for the banking system and the attached probability distribution.

Probability	State/Bank	1	2	...	n	$n+1$	$n+2$	...	$2n$
$(1-\pi)/2$	S_1	$-z$	$-z$	$-z$	$-z$	$+z$	$+z$	$+z$	$+z$
$(1-\pi)/2$	S_2	$+z$	$+z$	$+z$	$+z$	$-z$	$-z$	$-z$	$-z$
$\pi/2n$	$\bar{S}_1$	$fail$	0	0	0	0	0	0	0
$\pi/2n$	$\bar{S}_2$	0	$fail$	0	0	0	0	0	0
$\pi/2n$	...	...	...	...	...	...	...	...	...
$\pi/2n$	$\bar{S}_{2n}$	0	0	0	0	0	0	0	$fail$

Table 1: Distribution of shocks in the banking system

3.3 Balance Sheet Linkages

The regional liquidity shocks that characterize the states of the world S_1 and S_2 is what motivates the banks in the first place to interact. These interactions create balance sheet linkages between banks. In particular, the links take the form of interbank deposits that banks exchange at date 0. As we explained above, at date 1 each bank has with probability $(1-\pi)/2$ either a liquidity shortage of $zC_1 = (p_H - q)C_1$ or a liquidity surplus of $zC_1 = (q - p_L)C_1$. Since in aggregate the liquidity demand matches exactly the liquidity supply, banks will hedge these liquidity shocks simply by transferring funds as interbank deposits at date 0.[6] This way, a contract would be closed between two banks that gives the right to both parts to withdraw their deposit, at any of the subsequent dates. For the amounts

[6] An important feature of the model is that the swap of deposits occurs ex-ante, before the state of the world is realized. Note, however, that this prevents cases when lenders have some monopoly power to arise. For instance, in an ex-post market for deposits, lenders might take advantage of their position as liquidity providers to extract money from banks with a shortage of liquidity. To avoid this unfavorable situation, banks prefer to close firm contracts that set the price of liquidity ex-ante.

exchanged as deposits, each bank would receive the same return as consumers: C_1, if they withdraw after one period, and C_2 if they withdraw after two periods.

Banks' portfolios consist now of three assets: the liquid asset, the illiquid asset and the interbank deposits. Each of these three assets can be liquidated in any of the last 2 periods. However, the costliest in terms of early liquidation is the illiquid asset. This implies the following ordering of returns:

$$1 < \frac{C_2}{C_1} < \frac{R}{r} \tag{3.1}$$

The states S_1 and S_2 differentiate the banks into two types. Banks are said to be of a different type if they will experience different liquidity shocks in period 1. In particular, a bank is of type H if it will face a high liquidity demand, that is a liquidity shortage, and a bank is of type L if it will face a low liquidity demand, that is a liquidity surplus.

Let a_{ij} denote the amount exchanged as deposits between banks i and j at date 0. We consider that deposit contracts are bilateral, hence we have $a_{ij} = a_{ji}$. Let N_i be the set of banks i is linked to and let N_i^{cross} be a subset of N_i representing the banks of a different type i is linked to. Then, the total amount of deposits i exchanges with its neighbors should balance out its liquidity shortage or excess. Since the insurance against liquidity shocks is provided only through links with banks of a different type, a_{ij} should satisfy the *feasibility constrain*:

$$\sum_{j \in N_i^{cross}} a_{ij} = z \tag{3.2}$$

3.4 Losses Given Default

In the previous section we have argued that regional shocks in the liquidity demand and liquidity supply create incentives for banks of a different type to form links by exchanging deposits. In this section we turn to analyze the effects of the idiosyncratic shocks that occur in states $\bar{S}_1, ..., \bar{S}_{2n}$.

The non insurable idiosyncratic shocks, bare the risk of contagion. That is, the shock that affects initially only one institution can propagate in the entire system. To understand why this is the case, recall that these shocks take the form of an exogenous failure of a bank in period 1. If this event realizes, then it might spill over first, to any bank linked to the failed bank and next, via links, to all the other banks. In order to evaluate contagion risk we need to introduce a measure that quantifies it. The risk of contagion is evaluated in terms of *loss given default* (henceforth *LGD*). *LGD* expresses the excess of nominal liabilities over the value of the assets of the failed bank. In our setting, *LGD* will be given by the loss of value a bank incurs on its deposits when one of its neighbor banks is liquidated.

To calculate *LGD* we need to determine the value of the assets of the failed bank. If a bank i fails, its portfolio of assets is liquidated at the current value and distributed equally among creditors. Now, recall that a bank portfolio consists of three assets. First, banks hold an amount of x per capita invested in a liquid asset that pays a return of 1. Second, banks have invested an amount y per capita in an illiquid asset that pays a return of $r < 1$ if liquidated in the first period. And lastly, there are interbank deposits summing up to $\sum_{k \in N_i} a_{ik}$ that pay a return of C_1 per unit of deposit. On the liability side, a bank will have to pay its depositors, normalized to 1 and at the same time to repay its interbank creditors that also add up to $\sum_{k \in N_i} a_{ik}$. This yields a new return per unit of good deposited in a bank i equal to $\bar{C}_i = \frac{x + ry + \sum_{k \in N_i} a_{ik} C_1}{1 + \sum_{k \in N_i} a_{ik}} < C_1$.[7] The *LGD* of bank j given that bank i has failed is easy now to express as[8]:

$$LGD_{ji} = a_{ji}(C_1 - \bar{C}_i) = a_{ji} \frac{C_1 - x - ry}{1 + \sum_{k \in N_i} a_{ik}} \tag{3.3}$$

LGD has two interesting properties that have important implications on the way banks form links.

- *First*, LGD_{ji} is increasing in amount of deposits a_{ji} exchanged between banks. This gives banks an incentive to exchange the minimum amount of deposits.

- *Second*, LGD_{ji} is decreasing in $(\sum_{k \in N_i - \{j\}} a_{ik})$. This implies that the more connected one bank is, the smaller the loss it induces to its neighbors in case it fails. In other words, the more connected one bank's neighbors are, the better the respective bank is.

[7] Eq. (3.1) ensures that the inequality holds.

[8] In principle $LGD_{ji} \neq LGD_{ij}$ since it may be that $\sum_{k \in N_i} a_{ik} \neq \sum_{k \in N_k} a_{jk}$

3.5 Limit Loss

In the previous section we have introduced losses given default and implied that the failure of a bank might spill over to other banks via links. In this section we explain how this spill over effect realizes.

Recall that the failure of a banks characterizes the state of the world $\bar{S}_1, ..., \bar{S}_{2n}$. When this event realizes, any neighbor bank of the failed bank incurs a loss in the value of the deposits exchanged with the bank that is liquidated (LGD). A closer look at the balance sheet of a bank that incurs a loss, shows that the value of the liquid asset qC_1 is reduced by the size of LGD. Thus, in order to meet its obligations towards early consumers, which rise to qC_1, a bank needs to compensate the difference by liquidating part of the illiquid asset. More precisely, the amount it liquidates from the illiquid asset should match the LGD value. Liquidating the illiquid asset prematurely, however, involves a penalty rate $r < 1$ and has negative consequences for the late consumers. In fact, if too much of the illiquid asset is liquidated, the consumption of late consumers may be reduced to a level below C_1. In this case, the late consumers might gain more by imitating the early consumers and withdrawing their investment from the bank at date 1. This will induce a run on the bank and, subsequently, it will trigger the failure of the bank.

We can determine the maximum amount of the illiquid asset that can be liquidated without causing a run. This can be expressed as a function that depends on the fraction of late consumers.

$$b(q) = r\left[y - \frac{(1-q)C_1}{R}\right] \tag{3.4}$$

The maximum amount of illiquid asset that can be liquidated without causing a run on the bank can be interpreted as a *limit loss*. Thus, any bank that incurs a LGD higher than $b(q)$ will inevitably fail. A value of LGD below the threshold $b(q)$ will not trigger the failure of a bank. It will, however, be costly for the late consumers, given that their consumption is now reduced to $\tilde{C}_2 < C_2$.[9]

[9] The consumption of late consumers at least equals the consumption of the early consumers: $\tilde{C}_2 \geq C_1$.

4 The Game

4.1 Concepts and Notations

Let $N = \{1, 2, ..., 2n\}$ denote the set of banks. A network g on the set N is a collection of g_{ij} pairs, with the interpretation that i and j are linked. Thus, if i and j are linked in the network g, then $g_{ij} \in g$.

The set of *neighbors* of bank i in the network g is $N_i(g) = \{j \in N \mid g_{ij} \in g\}$. Let $\eta_i(g) = |N_i(g)|$, where $|\cdot|$ represents the cardinality of a finite set. The number of neighbors bank i has in the network g is called the *degree* of bank i. In addition, let $N_i^{inner}(g) = \{j \in N \,|g_{ij} \in g \text{ and } i, j \text{ are of the same type}\}$ and $\eta_i^{inner}(g) = \left|N_i^{inner}(g)\right|$. The number of neighbor banks of the same type is defined as the *inner degree* of bank i. An equivalent notation exists for banks of a different type $N_i^{cross}(g) = N_i(g) \setminus N_i^{inner}(g)$ and the number of neighbor banks of a different type is: $\eta_i^{cross}(g) = \left|N_i(g) \setminus N_i^{inner}(g)\right|$.

We use the notation $g + g_{ij}$ to denote the new graph obtained from g by linking i and j, if $g_{ij} \notin g$. Similarly, we consider that $g - g_{ij}$ represents the graph obtained from g by deleting an existent link between i and j, when $g_{ij} \in g$

A network g is connected if there exists a path between any two nodes i and j from N. A path of length k between i and j is a sequence of distinct agents $(i, j_1, ..., j_{k-1}, j)$ such that g_{ij_1}, $g_{j_1 j_2}$,..., $g_{j_{k-1}j} \in g$. A network g is complete if for any node $i \in N$, $\eta_i(g) = n - 1$. A network g is regular of degree k if for any node $i \in N$, $\eta_i(g) = k$.

4.2 Strategies, Objectives and Welfare

The interaction between banks on the interbank market can be modeled as a network formation game, where banks are the nodes and the deposits they exchange are the links. The network is formed as a result of banks' actions, who decide how to form links. Since the deposits are exchanged on bilateral basis (i.e. bank i agrees to pass its deposits to bank j if and only if bank j will pass its deposits to bank i in turn), the network is undirected and the formation of a link requires the consent of both parties involved. However, the severance of a link can be done unilaterally.

The strategy of bank i can be described as a linking vector $s_i = (s_{i1}, s_{i2}, ..., s_{i2n})$ such that $s_{ij} \in \{0, 1\}$ for each $j \in N \setminus \{i\}$ and $s_{ii} = 0$, where $s_{ij} = 1$ means that i intends to form a link with bank j. A link between i and j is formed if and only if $s_{ij} = s_{ji} = 1$.[10] Recall that shocks in the liquidity demand differentiate banks into two types. It becomes apparent banks follow different strategies when linking with banks of a different type than when linking with banks of the same type. In this paper we model explicitly the network formation between banks of the same type for a given pattern of interactions between banks of a different type. In particular we assume that banks of a different type form a complete bipartite graph. Thus, for any bank $i \in \{1, 2, ..., n\}$ the linking vector can be written as $s_i = (s_{i1}, s_{i2}, ..., s_{in}, 1, 1, ..., 1)$, while for any bank $j \in \{n + 1, ..., 2n\}$ the linking vector is represented by $s_j = (1, 1, ..., 1, s_{jn+1}, ..., s_{j2n})$. Given this, we model the interactions between banks of the same type and, furthermore, we study if the bipartite complete graph can always be supported in the equilibrium.

The factor that mainly drives banks when forming links is to insure against the regional liquidity shocks that affect the system when the states S_1 and S_2 realize. Banks can hedge their potential liquidity excess or shortage by linking (i.e. exchanging deposits) with banks of a different type. Thus, any linking pattern between banks of a different type is sufficient for insurance purposes, as long as it allows banks to exchange deposits which satisfy the feasibility constrain (3.2). However, links between banks expose them to contagion risk. The risk of contagion, measured in terms of *LGD*, can be reduced on the one hand by decreasing the size of deposits exchanged or, on the other hand, by increasing the degree of each bank. If we look only at symmetric solutions, the amount of deposits exchanged is minimal when each bank is linked to all the other banks of a different type. This is the rational behind the assumption that banks of a different type form a complete bipartite graph. In this paper we model how banks endogenously reduce contagion risk by increasing the degree of each node in the network.

Proposition 1 *Let Ω be the set of all possible states of the worlds and denote with ω an element of $\Omega \setminus \{\bar{S}_1, ..., \bar{S}_{2n}\}$. Let $N^H(\omega)$ denote the set of banks of type H and $N^L(\omega)$ the set of banks of type L in the state of the world ω. Then the minimization problem for LGD associated to each link has a symmetric solution when any $i \in N^H(\omega)$ and $i' \in N^L(\omega)$ are connected and $a_{ii'} = \frac{z}{n}$.*

Proof. The proof is provided in the Appendix. ■

[10] This condition capture that the formation of a link between two banks requires the consent of both participants.

The following remark extends the result from Proposition 1 to links between any two banks.

Remark 1 *For simplicity, we consider that the amount of deposits exchanged between banks i and j of the same type is also $a_{ij} = \frac{z}{n}$.*

Banks' objective: In a network where banks can fully insure against liquidity shocks, they need only to prevent losses through contagion. Thus, banks' objective function is reduced to minimize the probability of failure through contagion. Secondarily, once the risk of failure through contagion is eliminated, banks minimize the expected loss in states $\bar{S}_1, ..., \bar{S}_{2n}$.

Social planner's objective: The main concern of a social planner is the systemic risk. In our framework, the systemic risk is given by the number of banks that fail, following a shock in a single institution. Thus, a social planner objective function is to minimize the number of banks that fail through contagion.

4.3 Payoffs' Properties

The notion of *limit loss* introduced in Section 3.3 comes in handy for describing the payoffs banks have from linking to other banks. We briefly give the intuition of how the payoffs are constructed. When a failed bank induces a value of LGD higher than the limit loss, then all the the neighboring banks will inevitably fail in turn. By assumption, each bank has at least n neighbors. Thus, the failure of a bank will cause the failure of at least n banks more. Not surprisingly, this chain of failure will reverberate towards the remaining banks, triggering, at the end, the failure of the entire system. If, however, the LGD caused by a failed bank is below the threshold, only the neighboring banks will experience the consequences, since the consumption of their late consumers will be reduced to $\tilde{C}_2 < C_2$. Any other bank will be able to pay C_1 to the early consumers and C_2 to the late consumers.

In expressing the payoffs, we use the second property of LGD. Namely, the LGD a failed bank induces to its neighbors depends on how well connected the respective bank is. Proposition 1 and Remark 1 show that the size of deposits exchanged between any two banks is z/n. Hence, we can infer that the LGD induced by a failed bank depends on how many neighbors the respective bank has.

The limit loss $b(q)$ is identical for each bank and is independent of the number of links a banks has. Thus we can find a number $t \in \mathbb{N}$, in order to bracket the limit loss $b(q)$ as follows:

$$\frac{z}{n}\frac{C_1 - x - ry}{1 + (n+t)\frac{z}{n}} \leq b(q) < \frac{z}{n}\frac{C_1 - x - ry}{1 + (n+t-1)\frac{z}{n}} \tag{4.1}$$

The left hand side of the inequality is exactly the LGD a bank that has $(n+t)$ neighbors induces and the right hand side is the LGD a bank with $(n+t-1)$ neighbors generates.

Inequality 4.1 relates the contagious effects of a bank failure to the number of links banks have. Thus, the failure of a bank that has $(n+t-1)$ links or less will trigger the failure of the entire system, through the mechanism described above. The failure of a bank with at least $(n+t)$ links, however, has only consequences for its neighbors by decreasing the utility of the late consumers.

We discuss in detail the implications of a bank failure for the case $t \in (0, n) \cap \mathbb{N}$. Consider the failure of a bank j in a network g. We distinguish the following cases:

1. $\eta_j^{inner}(g) < t$. In this case, for any $i \in N_j(g)$ we have $LGD_{ij} > b(q)$. Consequently, any bank $k \in N$ will also fail.[11]

2. $\eta_j^{inner}(g) \geq t$. In this case, for any $i \in N_j(g)$ we have $LGD_{ij} \leq b(q)$. Thus, any bank $i \in N_j(g)$ will pay the early consumers C_1. The late consumers, however, will have their consumption reduced to $\tilde{C}_2 < C_2$. Any other non-neighboring bank $k \in N \setminus N_j(g)$ will not be affected in any way and will be able to pay its consumers C_1 at date 1 and C_2 at date 2.

4.4 Payoffs

The arguments presented above are useful in defining the payoff a bank i gains from network g. When the limit loss $b(q)$ satisfies inequality 4.1, then the higher the number of banks with an inner degree larger than t, the higher the benefits to each bank. Linking is nevertheless costly, thus the fewer neighbors with an inner degree larger than t a bank has, the lower the costs for the respective bank.

Formally, we can express the payoff of a bank $i \in N$ as a function u, increasing in the number of nodes with an inner degree higher than t and decreasing in the number of

[11] With this discussion, we are able to conclude that the existence of a single bank with insufficient link may trigger the failure of the entire system.

neighbors with an inner degree higher than t.

$$u_i(g) = f(|T|, |N_i(g) \cap T|) \tag{4.2}$$

where $T = \{j \in N \,\middle|\, \eta_j^{inner}(g) \geq t\}$ and $|\cdot|$ represents the cardinal of a set.

In addition, if the limit loss $b(q)$ satisfies the inequality 4.1, then for any node i and any inner degree η_i^{inner}, the payoff u_i has the following *properties*:

1. $u_i(g + g_{ij}) = u_i(g)$ and $u_i(g - g_{ij}) = u_i(g)$, $\forall j \in N$ s.t. $\eta_j^{inner}(g) < t - 1$

   The explanation for this indifference relies on the fact that the failure of a node with an inner degree below t will trigger the failure of the entire system. The failure of j leads to the failure of i, regardless of i creating a link or severing an existent link with j.

2. $u_i(g + g_{ij}) > u_i(g)$, $\forall j \in N$ s.t. $\eta_j^{inner}(g) = t - 1$

   If $\eta_j^{inner}(g) = t - 1$ and i creates a link with j, then the inner degree of j becomes $\eta_j^{inner}(g) = t$. Thus, i trades a situation when the failure of j induces its own failure, for a situation when the failure of j results in merely a lower utility for i's late consumers.

3. $u_i(g + g_{ij}) < u_i(g)$, $\forall j \in N$ s.t. $\eta_j^{inner}(g) \geq t$

   When j has already an inner degree sufficiently high, its failure will have only effects for the neighbor banks. Linking with j does not bring i any benefits, but it comes at the cost represented by the loss i might incur if j fails.

4. $u_i(g - g_{ij}) > u_i(g)$, $\forall j \in N$ s.t. $\eta_j^{inner}(g) \geq t + 1$

   Severing an existent link with j, will leave j with an inner degree still sufficiently high. It will, however, spare i from experiencing a loss in case j fails.

5 Network Formation

The main goal of this paper is to understand what networks are more likely to arise in a banking system where financial players' incentives to form links are shaped by the effects of two types of shocks. On the one hand, the need for insurance against regional liquidity shocks determines the formation of links between banks that have negatively correlated shocks. For this case, however, we assume that the pattern of interactions that permits the proper transfer of funds takes the form of a bipartite complete graph. On the other hand, the same links that allow banks to hedge the liquidity shocks expose them to the risk of contagion. Nevertheless, we have showed that the risk of contagion may be reduced by increasing the number of links each bank has. We will focus thus on developing a model of endogenous link formation between banks driven by the risk of failure by contagion.

In order to identify what networks are stable, we introduce the following concept of equilibrium due to Jackson and Wolinsky (1996).

Criterion 1 *Let $g_{ij} = \min(s_{ij}, s_{ji})$ and consider that $g_{ij} \in g$ when $g_{ij} = 1$. A network g is pairwise stable if*

1. *for all $g_{ij} \in g$, $u_i(g) \geq u_i(g - g_{ij})$ and $u_j(g) \geq u_j(g - g_{ij})$ and*

2. *for all $g_{ij} \notin g$, if $u_i(g) < u_i(g + g_{ij})$ then $u_j(g) > u_j(g + g_{ij})$.*

where $u_i(g)$ is the payoff of bank i in the network g.[12]

The first condition of the stability criterion establishes that in equilibrium there is no bank that wishes to severe a link it is involved in. At the same time, the second condition requires that in a stable network there should not exist two unconnected banks that would both benefit by forming a link between themselves. In other words, a network is an equilibrium if there are no banks that wish to deviate neither unilaterally (by severing existent links), nor bilaterally (by adding a link between two banks).

All the following results hold under two assumptions:

1. the limit loss $b(q)$ satisfies the inequality 4.1;

2. for any bank $i \in N$, in the network g the crossing degree is $\eta_i^{cross}(g) = n$.

The first result gives a necessary condition for a stable network to exist.

Proposition 2 *Let g be a pairwise stable network. Then any bank $i \in N$ must have an inner degree $\eta_i^{inner}(g) \leq t$.*

[12] We use the standard notations: $g - ij$ denotes the graph obtained by deleting link ij from the existing graph g, while $g + ij$ is the graph obtained by adding the link ij to the graph g.

Proof. The proof follows immediately from properties 3 and 4 described in the previous section, properties that characterize the payoffs a bank i gains from the network g. Suppose that there exists a bank i such that $\eta_i^{inner}(g) > t$. Than, any neighbor $j \in N_i(g)$ has an incentive to severe the link that connects it with i. This way, g is no longer an equilibrium. ■

This result provides only a partial characterization of stable networks. In fact, under payoffs that respect properties $1-4$, there exists a broad range of networks that are pairwise stables. It is easy to check that any network for which each node i has an inner degree $\eta_i^{inner}(g) \leq t-2$ constitutes an equilibrium. This multiplicity of equilibria is mainly driven by the indifference in forming or severing links expressed in property 1. In what it follows we alter property 1 in order to restrict the set of equilibria. Namely, we consider that banks have a slight preference to forming links with other banks. The remaining properties are unchanged.

Formally, when the limit loss $b(q)$ satisfies the inequality 4.1, then for any node i and any inner degree η_i^{in}, the payoff u_i has the following *properties*:

1'. $u_i(g+g_{ij}) = u_i(g)+\varepsilon$ and $u_i(g-g_{ij}) = u_i(g)-\varepsilon$, $\forall j \in N$ s.t. $\eta_j^{inner} < t-1$;

1. $u_i(g+g_{ij}) > u_i(g)$, $\forall j \in N$ s.t. $\eta_j^{inner} = t-1$;

2. $u_i(g+g_{ij}) < u_i(g)$, $\forall j \in N$ s.t. $\eta_j^{inner} \geq t$;

3. $u_i(g-g_{ij}) > u_i(g)$, $\forall j \in N$ s.t. $\eta_j^{inner} \geq t+1$.

The new set of payoffs' properties allows us to characterize comprehensively the set of equilibria and give prediction for the stability of the banking system.

Proposition 3 *Let g be a pairwise stable network and $T = \{i \in N \left| \eta_i^{inner}(g) = t\right.\}$. Then $|T| \geq 2(n-t)$.*

Proof. The proof is provided in the Appendix. ■

Proposition 3 has strong implications for the stability of the banking system, especially when t is small. If t is small, proposition 3 shows that in equilibrium most of the banks have sufficient links to prevent a shock in one of the institutions spreading through contagion. For t large, however, the predictions are weaker. We need thus a refinement for large values of t. The following proposition provides such a refinement.

Proposition 4 *Let g be a pairwise stable network and $T = \{i \in N \mid \eta_i^{inner}(g) = t\}$. If $t \geq n/2$, then $|T| \geq n$.*

Proof. The proof is provided in the Appendix. ■

Proposition 4 establishes that in an equilibrium network, at least half the banks will have a sufficiently large number of links to insure losses that are small enough. The following two results relate these findings to implications for the stability of the system.

Corollary 1 *Let g be a pairwise stable network and $T = \{i \in N \mid \eta_i^{inner}(g) = t\}$. If $t < n/2$, then the probability that the failure of a bank will spread through contagion is at most $t\pi/n$.*

Proof. The proof follows simply from Proposition 3. ■

Corollary 2 *Let g be a pairwise stable network and $T = \{i \in N \mid \eta_i^{inner}(g) = t\}$. If $t \geq n/2$, then the probability that the failure of a bank will spread through contagion is at most $\pi/2$.*

Proof. The proof follows simply from Proposition 4. ■

The first result implicitly insures that for high levels of the limit loss the probability of contagion is significantly low. The intuition for this result relies on the fact that the higher the limit loss is, the lower the number of links banks need in order prevent contagion. Proposition 3 indicates that a lower connectivity of the banking system is easier to obtain. A low level of the limit loss, however, requires a high connectivity in the banking system. Hence, the increased probability of contagion.

6 Efficiency

When the limit loss $b(q)$ satisfies the inequality 4.1, the failure of a bank with at least $(n+t)$ links, although it has consequences for its neighbors by decreasing the utility of the late consumers, will not propagate by contagion. Thus, there is a complete range of efficient networks a social planner can design in order to prevent contagion in the banking system. The set of efficient networks is characterized in the following proposition.

Proposition 5 *Let g be a network such that $\eta_i^{cross}(g) = n$ and $\eta_i^{inner}(g) \geq t$, $\forall\ i \in N$. Then g is efficient.*

Proof. Since $\eta_i^{inner} \geq t$, $\forall\ i \in N$ then it follows immediately that $\eta_i \geq n + t$, $\forall\ i \in N$. Thus, for any pair ij it must be that $LGD_{ij} \leq \frac{z}{n} \frac{C_1 - x - ry}{1 + (n+t)\frac{z}{n}}$. As the limit loss $b(q)$ satisfies inequality 4.1, then $LGD_{ij} \leq b(q)$ for any pair of banks ij. Hence, in the network g the failure of a bank will not trigger the failure of other banks in the system. ■

The conflict between efficient outcomes and individual incentives is a classical theme in economics. In this model, however, the incentives are partially aligned. Indeed, the set of equilibrium networks, described by proposition 3 and 4, includes the efficient equilibrium. It is easy to check that a network g such that $\eta_i^{inner}(g) = t$, $\forall\ i \in N$ is pairwise stable and, by proposition 5, is also efficient.[13]

The set of equilibrium networks, nevertheless, incorporates many non-efficient equilibria, especially for large values of t. We show that we can restrict the set of equilibria to the efficient one if we extend the pairwise stability concept previously used, to allow for deviations in which a pair of players each can delete one or more links *and/or* add a link in a coordinated manner. For this purpose, we introduce the notion of *bilateral equilibrium.*

Criterion 2 *Let $g_{ij} = \min(s_{ij}, s_{ji})$ and consider that $g_{ij} \in g$ when $g_{ij} = 1$. A network g^* is a bilateral equilibrium if:*

1. *There is a Nash equilibrium strategy profile s^* which yields g^*.*

2. *For any pair of players $i, j \in N$, and every strategy pair (s_i, s_j),*

$$u_i(g(s_i, s_j, s^*_{-i-j})) > u_i(g(s^*_i, s^*_j, s^*_{-i-j})) \Rightarrow u_j(g(s_i, s_j, s^*_{-i-j})) < u_j(g(s^*_i, s^*_j, s^*_{-i-j}))$$

[13]The existance of such an equilibrium efficient network is conditional on the existance of a t-regular network. Lovasz (1979) discusses in detail conditiond for the existance of a t-regular network with n nodes.

A network g can be, thus, supported in a 'bilateral equilibrium' if no player or pair of players can deviate and benefit from the deviation (at least one of them strictly). This equilibrium concept allows a pair of players to deviate by creating a link between themselves, if the link did not exist before, and, at the same time, by severing other links they are involved in. The terminology of 'bilateral equilibrium' was introduced in Goyal and Vega-Redondo (2004). Note that any bilateral equilibrium network is also pairwise stable.

Proposition 6 *Let g be a bilateral equilibrium network. Then the probability that the failure of a bank will spread through contagion is at most $\pi/2n$.*

Proof. The proof is provided in the Appendix. ■

The bilateral equilibrium concept rules out all the inefficient equilibria, except for one. For instance, a network such that $\eta_j^{cross} = n$, where a single node i that has an insufficient number of link, $\eta_i^{inner} < t$ and $\eta_{j \in N-\{i\}}^{inner} = t$ can be sustained in equilibrium. In this network, although the probability of contagion is very low, $\pi/2n$, the failure of bank i triggers the failure of the entire system. Thus, with probability $\pi/2n$ there will be $(2n-1)$ banks that will fail due to contagious effects.

In this paper we have discussed a particular set of equilibrium networks. Namely, we have assumed that $\eta_i^{cross} = n$, $\forall\ i \in N$ and we have modeled the link formation process that takes place between banks of the same type. In other words, we have studied the set of equilibrium networks for which each bank is linked to all the banks of the other type. It is important to note, however, that the set of pairwise stable networks is, by no means, restricted the set of networks for which $\eta_i^{cross} = n$, $\forall\ i \in N$. Due to the limitations of the pairwise stability concept, networks where there exist nodes such that the crossing degree is smaller than n can be sustained in equilibrium. The full characterization of the set of pairwise stable networks is possible, however not of much interest since there is no efficient equilibrium that can emerge when there exist nodes such that the crossing degree is smaller than n, for the given set of parameters.

The bilateral equilibrium concept solves this problem and proposition 6 holds without any prior assumption about the crossing degree of banks.

7 Conclusions

The problem of contagion within the banking system is a fairly debated issue. The main contribution this paper brings to the existent literature is endogenizing the degree of interdependence that exists between banks. In particular, we develop a model of network formation for the banking system. We investigate how banks form links with each other, when the banking system is exposed to contagion risk. The question we address is wether banks form networks that are resilient to the propagation of small idiosyncratic shocks.

The message this paper transmits is rather optimistic. Banks respond to contagion risk by forming links. The stable network architectures that emerge are very likely to support systemic stability. For instance, when the probability of a shock is π, then the probability that it will spread by contagion is at most $\pi/2n$. For large values of the limit loss, the probability of contagion is virtually 0.

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A Appendix

In what it follows we will prove some results advanced in the main text.

Proposition 1 *Let Ω be the set of all possible states of the worlds and denote with ω an element of $\Omega - \{\bar{S}\}$. Let $N^H(\omega)$ denote the set of banks of type H and $N^L(\omega)$ the set of banks of type L in the state of the world ω. Then the solution for minimizing LGD associated to each link is for any $i \in N^H(\omega)$ and $i' \in N^L(\omega)$ to be connected and, hence, $a_{ii'} = \frac{z}{n}$.*

Proof. The optimization problem is:

$$\forall\ i \in N, i' \in N_i^{cross},\ \min_{a_{ii'}} LGD_{ii'}, \tag{A.1}$$

$$\text{s.t.} \sum_{i' \in N_i^{cross}} a_{ii'} = z \tag{A.2}$$

First we show that $LGD_{ii'}$ is decreasing in $a_{ii'}$. For this it is useful to express LGD as

$$LGD_{ii'} = a_{ii'} \frac{C_1 - x - ry}{1 + a_{ii'} + \sum_{\substack{k \in N_i(g) \\ k \neq i'}} a_{ik}} \tag{A.3}$$

The derivative of $LGD_{ii'}$ with respect to $a_{ii'}$ is given by

$$\frac{\partial LGD_{ii'}}{\partial a_{ii'}} = \frac{(C_1 - x - ry)(1 + \sum_{\substack{k \in N_i(g) \\ k \neq i'}} a_{ik})}{(1 + a_{ii'} + \sum_{\substack{k \in N_i(g) \\ k \neq i'}} a_{ik})^2} > 0 \tag{A.4}$$

A positive sign for the derivative implies that $LGD_{ii'}$ is increasing in $a_{ii'}$.

The only restriction in minimizing LGD_{ij} is the feasibility constraint (3.2). According to it, any bank i needs to insure that the amount of deposits exchanged with banks of a different type sums up to z.

We impose that the solution is symmetric. That is $a_{ii'} = \frac{z}{\gamma}$. Since there are n banks of a different type and $LGD_{ii'}$ is increasing in the amount of deposits a_{ij}, the solution to the minimization problem dictates that each bank creates links to all the other banks of a different type. Subsequently, the amount exchanged on each link is $a_{ij} = \frac{z}{n}$. ■

Proposition 3 *Let g be a pairwise stable network and $T = \{i \in N \mid \eta_i^{inner}(g) = t\}$. Then $|T| \geq 2(n - t)$.*

Proof. Let Ω be the set of all possible states of the worlds and denote with ω an element of $\Omega - \{\bar{S}\}$. Let $N^H(\omega)$ ($N^L(\omega)$) be the set of banks of type H (L) when the state ω realizes. And let $T^H(\omega)$ ($T^L(\omega)$) be the set of banks of type H (L) that have an inner degree $\eta_i^{inner}(g) = t$ when the state ω realizes. Clearly we have $|T| = \left|T^H(\omega)\right| + \left|T^L(\omega)\right|$. In order to prove that $|T| \geq 2(n-t)$, we show that $\left|T^H(\omega)\right| \geq n-t$ and $\left|T^L(\omega)\right| \geq n-t$. Since the cases are symmetric, we prove only that $\left|T^H(\omega)\right| \geq n-t$. For this we assume the contrary in order to arrive to a contradiction.

Suppose that $\left|T^H(\omega)\right| < n-t$. This implies that the set $T^H(\omega)$ has at most $n-t-1$ elements. Further, this implies that $\left|N^H(\omega) - T^H(\omega)\right| \geq n-(n-t-1)$. In other words, the set of banks with an inner degree $\eta_i^{inner}(g) < t$ has at least $t+1$ elements. By property 1' and 2 above we know that in a stable network all the banks such that $\eta_i^{inner} \leq t-1$ must be directly linked with each other. Since the set of banks with this property is at least $t+1$, it must be that each bank in $N^H(\omega) - T^H(\omega)$ has an inner degree $\eta_i^{inner} \geq t$. We arrived thus to a contradiction. ■

Proposition 4 *Let g be a pairwise stable network and $T = \{i \in N \left|\eta_i^{inner}(g) = t\right.\}$. If $t \geq n/2$, then $|T| \geq n$.*

Proof. The proof follows similar steps as the proof for the previous result. Adopting the same notations, we prove only that $\left|T^H(\omega)\right| \geq n/2$.

Let $\left|T^H(\omega)\right| = \tau$. If $\tau \geq t$, the proof is complete.

Consider the case when $\tau < t$. By property 1' and 2 above we know that in a stable network all the banks in the set $N^H(\omega) - T^H(\omega)$ must be directly linked with each other. This implies that the total number of links[14] between banks of the same type with an inner degree $\eta_i^{inner} < t$ must be $(n-\tau)(n-\tau-1)$. In addition, since $\tau < t$, it must be that each bank in $T^H(\omega)$ has some links with banks in $N^H(\omega) - T^H(\omega)$. Assuming that all banks in $T^H(\omega)$ are directly linked with each other, there must be at least $\tau(t-\tau+1)$ links with banks in $N^H(\omega) - T^H(\omega)$.

Since all the banks in $N^H(\omega) - T^H(\omega)$ have an inner degree $\eta_i^{inner} < t$, the total amount of links these banks have should not exceed $(n-\tau)t$. Thus, the following inequality must hold:

$$(n-\tau)(n-\tau-1) + \tau(t-\tau+1) < (n-\tau)t$$

This inequality can be rewritten as

[14] Links here are counted twice for each node. However, we maintain the same double counting for the rest of the proof, such that in the end it cancels out.

$$(t - \tau + 1)(2\tau - n) < (n - \tau)(2\tau - n)$$

Since $t - \tau + 1 < n - \tau$, it must be that $2\tau - n > 0 \Leftrightarrow \tau > n/2$. This concludes the proof. ■

Proposition 5 *Let g be a bilateral equilibrium network. Then the probability that the failure of a bank will spread through contagion is at most $\pi/2n$.*

Proof. We show that in a bilateral equilibrium networks there exists at most one node i such that $\eta_i^{inner} < t$.

Suppose that in an equilibrium network there exist at least two nodes i and j such that $\eta_i^{inner} < t$, $\eta_j^{inner} < t$. In a network that there are at least two nodes with an insufficient number of links, there are two sources of contagious failure. Thus the probability a bank associates to failing by contagion is at least $2\pi/2n$.

Let $\tilde{g}$ be such a network. Then there exist a pair ij of nodes of a different type (i.e. $i \in N^H(\omega)$ and $j \in N^L(\omega)$) such that it pays off to severe the links they are involved in and form the link $\tilde{g}_{ij}$, if $\tilde{g}_{ij} \notin \tilde{g}$. Formally, let $\tilde{s}_i$ and $\tilde{s}_j$ be the strategy profile bank i and bank j follow, respectively, in network $\tilde{g}$.And let $s_i^* = (\underbrace{0, 0, ..., 0}_{1,...,j-1}, 1, \underbrace{0, 0, ..., 0}_{j+1,...,n})$ and $s_j^* = (\underbrace{0, 0, ..., 0}_{1,...,i-1}, 1, \underbrace{0, 0, ..., 0}_{i+1,...,n})$. Then

$$u_i(\tilde{g}(s_i^*, s_j^*, \tilde{s}_{-i-j})) > u_i(g(\tilde{s}_i, \tilde{s}_j, \tilde{s}_{-i-j})) \text{ and } u_j(\tilde{g}(s_i^*, s_j^*, \tilde{s}_{-i-j})) > u_j(\tilde{g}(\tilde{s}_i, \tilde{s}_j, \tilde{s}_{-i-j})) \tag{A.5}$$

In the new network, the only link i and j have is $\tilde{g}_{ij}$ and thus they are exposed to contagion stemming from one source. If $\tilde{g}_{ij}$ is the only link banks i and j have, thus this link will bear the entire amount of deposits necessary to provide insurance against liquidity shocks: z. Thus, if one of the banks fails, the other one fails by necessity, since the loss it incurs is much above the limit loss threshold. However, the probability that one of the two banks will fail is $\pi/2n$ and smaller than in the network $\tilde{g}$. Hence, $\tilde{g}$ cannot be an equilibrium.

Since, in a bilateral equilibrium there exists at most one node i such that $\eta_i^{inner} < t$, it follows that the probability that the failure of a bank will spread through contagion is at most $\pi/2n$. ■