Financial innovation and the transactions demand for cash*

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Abstract

Several studies report interest rate elasticities for money demand below one half, the value predicted by the Baumol-Tobin model. It is also widely documented that money holdings decrease as new technologies to economize in the use of cash are introduced. We also document other patterns that differ from the predictions of the Baumol-Tobin model, as well as from the Miller and Orr model. One is that interest rate elasticity of the average number of withdrawals is smaller than 1/2. The other is that the average ratio of withdrawal to average cash holdings is between 1.2 and 1.4. This paper develops a dynamic inventory model for cash balances that is consistent with the features mentioned above. The key new ingredient of our model is the assumption that agents have random opportunities to withdraw cash at no costs.

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1 Introduction

Several studies report interest rate elasticities for money demand below one half, the value predicted by the Baumol-Tobin model. It is also widely documented that money holdings decrease as new technologies to economize in the use of cash are introduced. We also document other patterns that differ from the predictions of the Baumol-Tobin model, as well as from the Miller and Orr model. One is that interest rate elasticity of the average number of withdrawals is smaller than $1/2$. The other is that the average ratio of withdrawal to average cash holdings is between 1.2 and 1.4. This paper develops a dynamic inventory model for cash balances that is consistent with the features mentioned above.

In Section 2 we discuss some patterns of currency holding behavior based on a panel data of Italian households. Section 3 analyzes the effects of financial diffusion using a deterministic steady-state model that allows a close comparison with the well known results of Baumol and Tobin. The core of this section is a simple model where, as opposed to the case in Baumol and Tobin, agents have a deterministic number of free withdrawals per period. We show that both the level of money demand and the interest rate elasticity decrease as the number of free withdrawals increases. Section 4 introduces our benchmark stochastic dynamic inventory model. In this model agents have random meetings with a financial intermediary in which they can withdraw money at no cost. This is a dynamic version of the model of Section 3. The implications of this model concerning the distribution of currency holdings, the aggregate money demand, the average number of withdrawals and the average size of withdrawals are presented in Section 5. We show that, qualitatively, the model reproduces the features of the data that we highlight in Section 2. A comparison between the money demand implied by this model and the one used in the steady state analysis of Section 3 is developed in Section 6 (To be completed). Section 7 generalizes the model of Section 4 to a more realistic set up. Section 8 presents a calibration of the model to the Italian household data.

Before describing the data we briefly discuss two related models in the literature. These models provide a rationale for an interest rate elasticity smaller than $1/2$, the value obtained in the Baumol and Tobin model. The explanation we propose is complementary to the ones in those papers because it focuses on the level and interest rate elasticity of individual households demand for money.

Miller and Orr (1966) study the optimal inventory policy of cash for an agent subject to stochastic cash inflows and outflows, and obtain an interest rate elasticity of $1/3$. Their model is more suitable for the problem faced by firms, given the nature of stochastic cash inflows and outflows. Instead, our paper focuses on a problem that better describes individual consumers problem, since we study the optimal inventory policy of cash for an agent that faces deterministic cash outflows (consumption expenditure) and no cash inflows. To be consistent with our model, when we analyze micro-level household data, we exclude households headed by self-employed.

Mulligan and Sala-i-Martin (2000) also study a model where the aggregate money demand can feature interest rate elasticity smaller than 1/2. In their model agents must pay a fixed cost to have a deposit account. Agents who face a low value for the total benefit of investing their wealth (either because wealth is low or because its return is low) will not pay the fixed cost and hence locally they will show a zero elasticity to changes in interest rates. Their model offers an explanation for a low interest elasticity of *aggregate* money demand. Instead, we concentrate on the interest rate elasticity of individual demands, by using micro-level household data and conditioning on the agents who do possess an interest bearing deposit account.

2 Cash Holdings Patterns of Italian Households

This section presents summary statistics of the cash holdings patters of Italian households from the *Survey of Household Income and Wealth.*[1] We focus on the surveys conducted from 1989 to 2004 because they include a section dedicated to the household cash management. Table 1 reports cross section averages of some key money holdings statistics, normalized by daily cash expenditures.[2]

Two statistics of Table 1 are at odds with the simplest versions of two classic money demand models: the one by Baumol and Tobin (BT henceforth) and the one by Miller and Orr (MO henceforth). First, households withdraw even if their cash balances are not zero, as they report that the minimum cash balances that triggers a withdrawal is about one third of their average cash balances. Second, the average ratio between the bank (and ATM) withdrawal and the currency holdings is just below 1.4 for households without an ATM card and below 1.2 for those with an ATM card. For comparison, this ratio is 2 in the BT model and 3/4 in the MO model.

Table 2 reports summary statistics on the supply of bank services, such as the diffusion of bank branches and ATMs, and on the interest rate paid on deposits.[3] Differences in nominal interest rates across provinces (witnessed by the standard deviations reported in parenthesis) are the result of segmentation in banking mar-

[1] This is a periodic survey of the Bank of Italy that collects information on several social and economic characteristics of household members, such as age, gender, education, employment, income, real and financial wealth, consumption and saving behavior. Each survey is conducted on a sample of about 8,000 households.

[2] Cash consumption is only available since 1993. In Appendix F we display simlar information deflating the nominal quantities by daily consumption of non-durable and services, which is available since 1989.

[3] These data are drawn from the Supervisory Reports to the Bank of Italy and the Italian Central Credit Register. Elementary data on ATMs and interest rates are available at the province/year level (the sample covers about 100 provinces; the size of a province is broadly comparable to that of a U.S. county). Elementary data for bank branches are available at the city/year level (the sample covers about 400 cities).

Table 1: Households' currency holdings

Variable	1993	1995	1998	2000	2002	2004
Average currency[a]						
Household w/o account	16.8	20.0	19.9	21.6	26.7	25.2
Household w. account						
w/o ATM	15.6	17.1	19.0	17.4	16.9	17.7
w. ATM	10.8	11.4	13.2	12.5	13.0	13.9
Average withdrawal[a]						
Household w/o ATM	23.8	20.4	23.9	21.1	21.2	21.5
Household w. ATM	10.9	9.5	12.5	11.6	11.4	12.4
Withdrawal to Currency Ratio[e]						
Household w/o ATM	1.4	1.3	1.3	1.3	1.4	1.3
Household w. ATM	1.2	1.1	1.1	1.2	1.2	1.1
Minimum currency[a,b]						
Household w/o ATM	5.4	4.2	7.7	6.6	6.3	na
Household w. ATM	3.9	2.9	4.2	4.7	4.6	na
Number of withdrawals[c]						
Household w/o ATM	12.3	13.1	19.8	16.5	17.5	17.9
Household w. ATM	48.0	49.5	58.6	61.7	56.7	63.1
Non durable consumption and services[d]						
Household w/o ATM	12,839	14,867	13,494	14,149	15,141	16,606
Household w. ATM	17,651	20,007	19,784	21,530	22,599	24,090
Share of cash expenditures[f]						
Household w/o ATM	0.69	0.68	0.66	0.68	0.67	0.64
Household w. ATM	0.65	0.63	0.60	0.57	0.54	0.50
N. of Observations	8,089	8,135	7,147	8,001	8,011	8,012

Notes: [a]Ratio to daily expenditures done in cash. - [b]Reported level of currency holdings that triggers a withdrawal. - [c]Per year. - [d]In euros, in year 2000 prices. - [e]Computed excluding households with a ratio bigger than 4. - [f]Ratio of cash expenditure to consumption of nondurables and services. Source: Bank of Italy - *Survey of Household Income and Wealth*; entries computed using sample weights.

kets.[4] Until the early nineties commercial banks faced restrictions to open new bank branches in other provinces. A gradual process of liberalization has occurred since then, which has led to a sharp increase in the number of bank branches and a reduction of the interest rate differentials (see Casolaro, Gambacorta and Guiso (2006) for a review of the main developments in the banking industry during the past two decades).

Table 3 presents least square regressions of the currency to consumption ratio for two groups of households: those with a deposit account but without an ATM card (first and second column) and those with a deposit account and an ATM card (third and fourth column). Since the model we present below focuses on the currency

[4]They do not reflect differences in the services or features of the underlying checking account (these statistics are built with the main objective of ensuring comparability and thus focus on a highly homogenous type of service).

Table 2: Financial development and interest rates

Variable	1989	1991	1993	1995	1998	2000	2002	2004
Bank branches[a,b]	na	0.34	0.38	0.42	0.47	0.50	0.53	0.55
	(na)	(0.13)	(0.13)	(0.14)	(0.16)	(0.17)	(0.18)	(0.18)
ATM[a,c]	na	0.22	0.31	0.39	0.50	0.57	0.65	0.65
	(na)	(0.13)	(0.18)	(0.19)	(0.22)	(0.22)	(0.23)	(0.22)
Interest rate[c,d]	6.96	6.74	6.10	5.23	2.15	1.16	0.77	0.32
	(0.49)	(0.52)	(0.42)	(0.32)	(0.23)	(0.22)	(0.15)	(0.11)

Notes: Cross-section mean (standard deviation in parenthesis). [a] Per thousand residents. - [b] Elementary data available at the city / year level. - [c] Elementary data available at the province / year level. [d] Net nominal interest rates expressed in percentages (Source: Central credit register).

demand of consumers we remove from the sample the data of households whose head is a self employed. [5]

Table 3: The household (HH) demand for currency

Dependent variable: $log(M/c)$	HH w/o ATM		HH w. ATM	
log(R)	-0.02	-0.08	-0.06	-0.09
	(0.02)	(0.09)	(0.02)	(0.12)
log(R) $\cdot$ bank-branches		0.15		0.07
		(0.24)		(0.29)
bank-branches	-0.84	-0.94	-0.96	-0.99
	(0.21)	(0.27)	(0.22)	(0.26)
Sample size	13,032	13,032	15,292	15,292

Notes: OLS regressions based on 1993-2002 surveys; the dependent variable is the (log of) household average cash holdings relative to cash expenditures. Robust standard errors (in parenthesis) are computed by clustering observations at the province*year, the finest level of disaggregation at which the interest rate is available. Bank-branches is defined as bank branches per capita at the city level; The net nominal interest rate is measured in percent (see Table 2).

The estimates display a systematic negative correlation between the (log of the) level of currency holdings and the diffusion of bank branches. The correlation of cash holdings with the interest rate is small and imprecisely estimated. We think that the identification of the effects of interest rate on cash holdings is complicated by the fact that, as Table 2 shows, both variables display a time trend during the short time period covered by this data set. The regressions in Table 4 use year and province

[5]Keeping these households, who make a great number of visits to the bank most probably for business reasons, does not alter the regressions results substantially.

dummies in an attempt to remove unobserved time and regional effects affecting money demand, e.g. differences in the incidence of small crime, a factor which likely reduces currency holdings. The point estimate of the interest rate elasticity is about 1/4 in absolute value, which are much larger than the ones estimated in Table 3. The coefficient for bank branches remains negative, although it is about one third as large. The regressions with an interaction term display the same patterns, with interest rate elasticities that are decreasing (in absolute value) in the level of financial diffusion. Thus these regressions imply that more financial development imply less response of currency holdings to interest rate variations. The size of the standard errors for the regressions that include year and province dummies imply that these effects are imprecisely measured.

Table 4: The household (HH) demand for currency - year and province dummies

Dependent variable: $log(M/c)$	HH w/o ATM		HH w. ATM	
log(R)	-0.20	-0.36	-0.22	-0.35
	(0.15)	(0.17)	(0.14)	(0.18)
log(R) · bank-branches		0.36		0.28
		(0.18)		(0.21)
bank-branches	-0.26	-0.46	-0.37	-0.45
	(0.25)	(0.28)	(0.22)	(0.24)
Sample size	13,032	13,032	15,292	15,292

Notes: OLS regressions based on 1993-2002 surveys; the dependent variable is the (log of) average cash holdings relative to cash expenditures. Robust standard errors (in parenthesis) are computed by clustering observations at the province*year, the finest level of disaggregation at which the interest rate is available. The regressors also include a constant, year dummies and 103 province dummies. Bank-branches is defined as bank branches per capita at the city level; The net nominal interest rate is measured in percent (see Table 2).

Table 5 and 6 report analogous estimates for the (log of the) average number of withdrawals per year for households with and without ATM cards. The regressions without year and province dummies in Table 5 display a systematic positive correlation between the level of currency holdings and the diffusion of bank branches. The correlation with the interest rate is small and imprecisely estimated. The regressions in Table 6 introduce year and province dummies, and obtain interest rate elasticities of about 1/3, which are much larger than the ones estimated in Table 5. The coefficient for bank branches remains positive. The regressions with an interaction term display the same patterns, with interest rate elasticities that are increasing in the level of financial diffusion. Thus these regressions imply that more financial development imply less response of the average number of withdrawals to interest rate variations. As in the case of the average currency holdings, the standard errors for the regression with year and province dummies show that these effects are imprecisely measured.

The results in tables 3-6 are robust both to including a battery of demographics

Table 5: Number of withdrawals

Dependent variable: $log(n)$	HH w/o ATM		HH w. ATM	
log(R)	-0.08	-0.02	-0.07	-0.13
	(0.02)	(0.11)	(0.01)	(0.06)
log(R) · bank-branches		-0.16		0.17
		(0.27)		(0.16)
bank-branches	0.80	0.94	0.69	0.60
	(0.29)	(0.26)	(0.13)	(0.16)
Sample size	12,901	12,901	17,054	17,054

Notes: OLS regressions based on 1993-2002 surveys; the the dependent variable is the (log of) number of withdrawals. Robust standard errors (in parenthesis) are computed by clustering observations at the province*year, the finest level of disaggregation at which the interest rate is available. Bank-branches is defined as bank branches per capita at the city level; The net nominal interest rate is measured in percent (see Table 2).

Table 6: Number of withdrawals - year and province dummies

Dependent variable: $log(n)$	HH w/o ATM		HH w. ATM	
log(R)	0.29	0.14	0.34	0.34
	(0.12)	(0.15)	(0.11)	(0.12)
log(R) · bank-branches		0.31		0.01
		(0.19)		(0.13)
bank-branches	0.44	0.23	0.37	0.37
	(0.23)	(0.26)	(0.16)	(0.17)
Sample size	12,901	12,901	17,054	17,054

Notes: OLS regressions based on 1993-2002 surveys; the the dependent variable is the (log of) number of withdrawals. Robust standard errors (in parenthesis) are computed by clustering observations at the province*year, the finest level of disaggregation at which the interest rate is available. The regressors also include a constant, year dummies and 103 province dummies. Bank-branches is defined as bank branches per capita at the city level; The net nominal interest rate is measured in percent (see Table 2).

as well as adding the (log of) consumption cash expenditures to the regressors.[6]

Finally we compare the estimated interest rate elasticities with the predictions of the BT and MO model. In the BT model, the interest rate elasticity of currency holdings is -1/2 and the elasticity of the average number of withdrawals is 1/2, which are larger in absolute value relative to the ones reported in the regression in Tables 3-6. In the MO model the interest rate elasticity of currency holdings is -1/3, a value close to one estimated in the regressions in Table 4. Nevertheless, the MO model predicts an interest rate elasticity of the average of number of withdrawals of 2/3, which is even larger than the one predicted by the BT model, and larger than

[6] The following demographic controls were considered: the number of children, adults and income earners in the household and the age, education and occupational status of the household head.

the one estimated in the regressions above.

3 Money demand : a deterministic steady state problem

In this section we model a form of technological progress on the withdrawal technology and discuss its implications for money demand. We conduct the analysis by focusing on steady state calculations. We minimize the steady state cost of attaining a given constant flow of consumption, as opposed to minimizing the expected discounted cost. We do this to increase the comparability with the standard derivation of the Baumol-Tobin money demand and to simplify the exposition of the effect of progress on technology for withdrawals from banks. In particular, this calculation helps understand why the level of the money demand and its interest rate elasticity are smaller for better withdrawal technologies.

Consider the following steady state problem. We let M be the average money balances, and $T(M, c)$ be the average (steady state) number of costly withdrawals from the bank per unit of time required to finance a consumption flow c when the average money balances are M. The function T depends on the withdrawal technology available to agents. We assume that T is decreasing in M, so that fewer withdrawals require higher average balances, and that T is convex, so the minimization problem is well behaved. We let R be the net nominal interest rate, and b the cost of each withdrawal. The average money demand solves the minimization problem

$$\min_{M} \; R\ M + b\ T(M, c) \tag{1}$$

The optimal choice of M must balance the impact on the cost due to forgone interest, $R\ M$ with the effect on the cost of withdrawals, $T(M, c)$. The formulation of this problem, as in the traditional BT model, uses three simplifying assumptions:

(A1) Steady state assumptions

(i) average (steady state) money balances times the interest rate is used to measure the cost, instead of the discounted interest rate cost, and

(ii) the average (steady state) number of withdrawals from the bank is used as opposed to the discounted (expected) cost of withdrawals,

(iii) R is not an argument of the function T.

The assumptions behind this formulation make the comparative statics analysis of the optimal M simple and intuitive. In particular the combination of iii) and the Fisher equation (say that $R = r + \pi$ for a fixed interest rate r), implies that the inflation rate π is not an argument of T. This is not completely satisfactory because if c and M denote real variables then the inflation rate should appear as an argument of T, as inflation erodes the real value of money holdings.[7] We will remove these simplifying assumptions in the analysis of Section 4.

The first order condition for problem (1) is:

$$R + b\, T'\left(M\left(R\right), c\right) = 0.$$

In this first order condition $T'\left(M\right) \equiv \partial T\left(M, c\right)/\partial M$ is the decrease in the cost of withdrawals due to an extra unit of money holdings, and R is the marginal cost of forgone interest due to an extra unit of money. We will refer to T' as the marginal cost of a withdrawal and to $-R$ as the marginal benefit of an increase in M.

We turn to the analysis of the elasticities of M with respect to c, b and R. We consider withdrawal technologies $T\left(M, c\right)$ which are homogeneous of degree zero on $\left(M, c\right)$. The idea is that if an agent needs to finance twice as much expenditure, it is feasible to double the size of each withdrawal (leaving all other features of her plan unaltered), hence keeping the average number of withdrawals T fixed. Using this homogeneity, the problem can be rewritten so the object of choice the ratio M/c as follows

$$\min_{M/c} \left(\frac{cR}{b}\right)\left(\frac{M}{c}\right) + T\left(\frac{M}{c}, 1\right)$$

The next remark summarizes some key features of the money demand.

Remark 1 *Assume that* T *is homogenous of degree zero in* $\left(M, c\right)$, *then* $M\left(R, c, b\right)$ *is homogenous of degree one in* $\left(b, c\right)$, *which implies that* $\frac{M}{c}$ *is a function of* $\frac{cR}{b}$, *namely* $\frac{M}{c} = \left(T'\right)^{-1}\left(-\frac{cR}{b}\right)$. *This, in turn, implies the following result:*

$$\begin{aligned} \frac{b}{M}\frac{\partial M}{\partial b} + \frac{c}{M}\frac{\partial M}{\partial c} &= 1, \\ -\frac{R}{M}\frac{\partial M}{\partial R} &= \frac{b}{M}\frac{\partial M}{\partial b}. \end{aligned}$$

Moreover, if T *is convex in the sense that:*

$$\frac{\partial^2 T}{\partial M^2} > 0 \text{ and } \frac{\partial^2 T}{\partial c^2} \geq 0$$

then

$$0 \leq -\frac{R}{M}\frac{\partial M}{\partial R} \leq \frac{1}{2} \leq \frac{c}{M}\frac{\partial M}{\partial c} \leq 1 .$$

Proof. *See appendix B.*

[7] Alternatively, one might take c and M to denote nominal quantities, which is an unsatisfactory characterazion of the consumption behavior. Yet another (not so satisfactory) interpretation is that the inflation rate does not change as R changes, which means that the model comparative statics concern changes in the real rate r.

The assumption that $\partial^2 T/\partial M^2 > 0$ implies that minimization problem is well posed. The quantity $\partial T/\partial c$ measures the additional average number of withdrawals that the agent needs to finance an increase in the flow of consumption c, keeping the same average cash balances M. The assumption that $\partial^2 T/\partial c^2 \geq 0$ implies that there are decreasing returns to scale relative to withdrawals. In Sections 3.1 and 4 we present technologies that parametrize the degree of decreasing returns, as related to rate at which agents can meet with a financial intermediary. Notice that the upper bound of 1/2 on the interest rate elasticity is attained in the classical BT specification, since there $\partial^2 T/\partial c^2 = 0$.

A central point of our inquiry is to analyze the effect of technological changes in T on the money demand M. To do so we present two comparative static results, one about the level of money demand, and the other about its interest rate elasticity. Consider two withdrawal technologies T_i for $i = 1, 2$ and their corresponding money demands, M_i for $i = 1, 2$.

Remark 2 *If the marginal cost of withdrawals is higher, then the money demand is lower. Formally, if*

$$T_2'(M) \geq T_1'(M) \text{ for all } M$$

then

$$M_2 \leq M_1 \text{ for all } R \geq 0.$$

This remark follows from the foc of the problem using that T is convex in M. Before stating the second result it is useful to understand the determinants of the interest rate elasticity of the money demand. In this model the interest rate elasticity is inversely related to the curvature of the cost function T. In particular,

$$-\frac{R}{M}\frac{\partial M}{\partial R} = 1 \;/\; \left(M \frac{T''}{-T'} \right) . \tag{2}$$

This follows from differentiating the foc from the optimal money demand w.r.t. R :

$$1 + bT'' \frac{\partial M}{\partial R} = 0$$

using the foc again to replace R, and rearranging. The expression $-M\; T''/T' \geq 0$ is a measure of the local curvature of the cost function T. It is also the elasticity of the marginal cost T'. Thus, (2) says that if the marginal cost is more sensitive to M, then the money demand is less sensitive to interest rate changes. The next result follows directly from (2)

Remark 3 *If the interest rate elasticity of the marginal cost of withdrawals is higher, the interest rate elasticity of the money demand is smaller. Formally, assume* *that at a given R*

$$M_2 \frac{T_2''(M_2)}{-T_2'(M_2)} \geq M_1 \frac{T_1''(M_1)}{-T_1'(M_1)}$$

then

$$-\frac{R}{M_2}\frac{\partial M_2}{\partial R} \leq -\frac{R}{M_1}\frac{\partial M_1}{\partial R}.$$

Notice that the assumption in Remark 3 is that the curvature of T_2 is greater than the one of T_1 evaluated at the, potentially different values, of the money demands M_2 and M_1.

3.1 A technology with (exactly) p free withdrawals

Now we use these results to analyze the effect on money demand of a simple form of technological progress in T. We consider

$$T_p(M, c) = \max\{\frac{(c/2)}{M} - p, 0\} . \tag{3}$$

The parameter p index the level of technology T, in particular it has the interpretation of the average number of free withdrawals per unit of time.

Setting $p = 0$ in (3) all the trips are costly, and we obtain as a baseline case the classical Baumol-Tobin,

$$T_0(M, c) = \frac{(c/2)}{M}$$

An agent with consumption flow c, withdraws 2 M, which last $2M/c$ periods, and hence has average balances M and makes $(c/2M)$ trips to the bank. Notice that T_0 has a marginal cost function T_0' has a constant elasticity equal to 2, which implies the well known result that the money demand elasticity is 1/2.

The interpretation of the case of $p > 0$ is that the agent has p free withdrawals, so that if the total number of withdrawals is $(c/2)/M$, then she pays only for the excess of $(c/2)/M$ over p, which gives the expression (3).

Throughout the analysis in this section we allow T to take any real value. However, the specification of the technology in (3) essentially puts a lower bound of p on T. This is similar to the seminal analysis of Tobin (1956) where the integer constraint on the number of transactions is carefully taken into account. Of course the integer constraint puts a lower bound equal to zero on the number of transactions. Our specification of T_p can be thought of as allowing the lower bound on the transactions to be a parameter that indexes technological change.

The following is a concrete set-up that gives rise to the assumption of p free withdrawals used for the specification of the technology T_p. Assume that the cost b represents the opportunity cost of the time of a trip to a bank branch or an ATM.

Think of an agent who, on her way to the ball game, passes by a bank branch or an ATM, say once a week. In this case we can represent the technology T_p as saying that she has one free withdrawal a week, or $p = 1$ per week. Now imagine that an ATM is installed on the way of her job, and assume that she works 6 days a week. This "technological improvement" can be represented by an increase in p, so that she gets 7 free withdrawals a week, or $p = 7$ per week.

In Section 4 we discuss a model where agents get a possibility of a free withdrawal at random times, a specification that we think better captures the availability of bank branches and ATM machines. This alternative technology is more difficult to analyze because the optimal pattern of cash holdings is not longer sawtooth (i.e. one of withdrawing only when cash attains zero value). While we think that the random withdrawal model is more realistic, the simple model using the T_p technology captures the key features of the effect of technological innovation on individual money demand.

The money demand for technology $p \geq 0$ is given by

$$M_p(R) = \begin{cases} \sqrt{\frac{b}{2}\frac{c}{R}} & \text{for } R \geq R^* \\ \sqrt{\frac{b}{2}\frac{c}{R^*}} & \text{for } R < R^* \end{cases} \tag{4}$$

where

$$R^* \equiv (p)^2 \, 2b/c \; . \tag{5}$$

Consider the case where $p = 0$, so that it is the BT set-up. In this case, for low R the forgone interest cost is small, so that agents decide to economize in costly withdrawals, and hence choose a large value of M. Now consider the case of $p > 0$. In this case there is no reason to have less than p withdrawals per unit of time, since these are, by assumption, free. Hence, for all $R < R^*$ agents will choose the same level money holdings, namely, $M_p(R) = M_p(R^*)$, since they are not paying for any withdrawal but they are subject to positive forgone interest rate costs (hence the interest elasticity is zero for $R < R^*$). Since for $p > 0$ the money demand is constant for $R < R^*$, it is both lower in its level and it has a lower interest rate elasticity than the money demand from the BT model. Indeed, (4)-(5) implies that the range of interest rate R for which the money demand is lower and has lower interest rate elasticity is increasing in p.

Improvements in the particular technology described in (3) produce a money demand that is lower in level and has a smaller interest rate elasticity because it indeed satisfies the assumptions for remarks 2 and 3 presented above. To see this, consider two technologies indexed by $0 \leq p_1 < p_2$. These technologies satisfy the following three properties:

i) A greater value of p represents technological progress, because T_p is decreasing in p, formally

$$T_{p_2}(M,c) \leq T_{p_1}(M,c)$$

with strict inequality for $M < (c/2)/p_1$.

ii) While a greater value of p gives a lower total cost, a higher value of p increases the marginal cost T'_p, at least for some values of M. In particular,

$$0 = T'_{p_2}(M,c) > T'_{p_1}(M,c) \text{ for } (c/2)/p_2 < M < (c/2)/p_1$$

and equal otherwise.

iii) A greater value of p increases the curvature of T_p. To see this, notice that T_{p_2} can be obtained by applying an increasing and convex transformation to T_{p_1}. Formally,

$$T_{p_2}(M,c) = g\left(T_{p_1}(M,c)\right)$$

for

$$g(\tau) = \max\{\tau - (p_2 - p_1), 0\},$$

which is increasing and convex in τ.

Let M_{p_1} and M_{p_2} be the money demand corresponding to the two technologies. Given Property *ii*), applying remark 2, we have that a better technology yields a lower money demand. Using Property *iii*) one can verify the conditions for remark 3, and hence that a better technology yields a lower interest rate elasticity of the money demand.

The analysis of the money demand and remarks 2 and 3 were obtained assuming that T was differentiable as a function of M. The technology T_p is not differentiable at one point, but the analysis goes through with minor modifications. Notice also that changes in this technology, i.e. changes in p, produce stark changes in the money demand. In the rest of the paper we consider the case of random withdrawals where the changes in p produce smooth changes in the money demand.

4 Money demand: a stochastic dynamic problem

This section extends the analysis along two dimensions. First, it takes an explicit account of the dynamic nature of the cash inventory problem, as opposed to the steady state analysis of Section 3. In doing so it also relaxes the steady state assumptions in (A1). Second, it introduces a variation on the withdrawal technology considered in Section 3.1. In particular, the technology considered here is one where agents have a Poisson arrival of free opportunities to withdraw cash , as opposed to the assumption of Section 3.1 of having a deterministic number of free withdrawals per period. We think that, relative to the deterministic number of free withdrawals, this assumption is a more realistic depiction of reality. Our maintained assumption is that the main component of the cost for a withdrawal is the opportunity cost of the households. We imagine that, for a given density of ATMs and bank desk, an agent bumps into them at certain rate per unit of time – denoted by p in the model. These are chance meetings with an intermediary that involves zero cost of withdrawal[8]. We argue that random meetings with a financial intermediary is a more realistic depiction of the opportunities faced by households. Our hypothesis is that, as the density of bank branches and ATMs increases, then households get more of these free opportunities to withdraw.

This model has several advantages, besides realism in the modeling of the search technology, over the one with a fixed deterministic number of withdrawals per period. First, a piece of evidence in favor of the random meeting model is that households withdraw much before their cash balances reach zero (see the statistics on the minimum currency in Table 1). A related feature, is that the model with random meetings implies, as shown in the data of Table 1 –and contrary to the implication of the basic BT model and of the model with exactly p free withdrawals– that the ratio of the average withdrawal to the average cash balances is below 2. Second, the model with random meetings smooths out some of the stark features of the model with exactly p free withdrawals. For instance, it turns out that its interest elasticity is lower than 1/2 for the whole range of interest rates, as opposed to be either 1/2 or 0. Third, the random meeting model implies that the interest rate elasticity of the average number of withdrawals is, in absolute value, smaller than the interest rate elasticity of the money demand, a feature that finds some support in the Italian households data (Tables 3 and 4).

Additionally, we think that the explicit dynamic nature of the model will allow us to use it in future work as a building block of a more complete model of cash management, where the decision of paying with cash is formally introduced. Finally, it turns out that the cost of introducing random meetings in an explicitly dynamic model is small, in the sense that the agent decision problem turns out to be very tractable, with an almost close form solution, a feature that we plan to use in a structural estimation of the model.

We turn now to the description of the agent problem. She faces a cash-in-advance constraint and can withdraw or deposit from an interest bearing account. The sequence problem is to choose an increasing sequence of stopping times $\{\tau_j\}$ at which to withdraw (or deposit) money in an interest bearing account, and the amounts to withdraw at each time, so as to minimize the expected discounted cost of financing a given constant real consumption flow c, denoted by TC_0:

$$TC_0(\tau, m) = E_0\left[\sum_{j=0}^{\infty} e^{-r\ \tau_j}\left\{b\ I_{\tau_j} + \left(m\left(\tau_j^+\right) - m\left(\tau_j^-\right)\right)\right\}\right] \tag{6}$$

where we use $m(t)$ to denote the real value of the stock of currency. The stock of currency jumps discontinuously up at the time of a withdrawal, so use $m(t^+)$ and

[8]In section 7 we extend the model by assuming that withdrawals that occur upon these chance meetings, rather than being free, are subject to a small fixed cost.

$m(t^-)$ to denote the right and left limits of m. Thus the amount of a withdrawal at τ_j is $m\left(\tau_j^+\right) - m\left(\tau_j^-\right)$. The law of motion of the real value of the stock of money between withdrawals is given by

$$\frac{dm(t)}{dt} = -c - m(t)\,\pi \tag{7}$$

where π is the inflation rate and c the real consumption flow . We assume that the agent contacts a financial institution with an exogenous probability p per unit of time. More precisely, contacts with the financial intermediary follow a Poisson process with arrival rate p. In the case of a contact the agent can withdraw (or deposit) money in an interest bearing account without incurring a cost. If the agent wants to withdraw (or deposit) in the financial institution in any other time, it must pay a real cost b. The indicator I_{τ_j} takes the value of zero if the withdrawal (or deposit) takes place at the time $t = \tau_j$ of a contact with a financial intermediary, and takes the value of one otherwise. The agent chooses stopping times and withdrawals as function of the history of contacts with the intermediary. We use r for the real rate at which cash flows are discounted. The initial conditions for the problem are the real cash balances, $m(0) = m_0$ and whether at time $t = 0$ the agent is matched with a financial intermediary or not.

We define the shadow cost of a policy $\{\tau_j, m\}$ as the expected discounted cost of the withdrawals plus the expected discounted opportunity cost of the cash balances held by the agent. We denote the shadow cost as SC_0, which is given by:

$$SC_0(\tau, m) = E_0\left[\sum_{j=0}^{\infty} e^{-r\ \tau_j}\left\{b\ I_{\tau_j} + \int_0^{\tau_{j+1}-\tau_j} R\ m(\tau_j + t)\ e^{-rt}\ dt\right\}\right] \tag{8}$$

where R is the nominal interest rate and m follows the law of motion (7). The shadow costs is defined in terms of the opportunity cost R and the parameters used to define the total cost, (r, p, π, b). In the next Proposition we show that, provided the Fisher equation $R = r + \pi$ holds, then the total cost can be written as the shadow cost plus the present value of c.

Proposition 1 *Assume that $R = r + \pi$. For any policy $\{\tau, m\}$ the total cost equals the shadow cost plus the present value of c, or*

$$TC_0 = \frac{c}{r} + SC_0\ .$$

Proof. See appendix B.

Proposition 1 implies that minimizing the shadow cost is equivalent to minimizing the total cost only when $R = r + \pi$. Nevertheless, below we consider the shadow cost problem for the general case of arbitrary values for R, r and π. We keep this general case for two reasons. One is to accommodate other costs and benefits of holding cash (such as the costs of petty crime). The second relates to the literature, such as the classic papers by Baumol and Tobin, that does not impose the Fisher equation as discussed above.

We use $V_s(m)$ for the value function corresponding to the minimization of the shadow cost:

$$V_s(m_0) = \min_{\tau, m} SC_0(\tau, m) \tag{9}$$

subject to $m(0) = m_0$ and where $s = f$ denotes that the agent is matched to a financial intermediary and $s = u$ that she is not. The next section solves for V.

Finally, Proposition 1 also helps linking the dynamic model with the steady state analysis done in Section 3. Each of the terms

$$b\ I_{\tau_j} + R \int_0^{\tau_{j+1}-\tau_j} m(\tau_j + t)\ e^{-rt}\ dt$$

in the summation of the shadow cost is similar to cost $b\,T(M, c) + RM$ in the steady state formulation of Section 3. The difference is that here $\int_0^{\tau_{j+1}-\tau_j} m(\tau_j + t)\ e^{-rt}\,dt$ are real balances, as opposed to the nominal M, and that the consumption flow c in (7) that is to be maintained constant is also real, as opposed to nominal, as discussed above.

4.1 Bellman equation for V and optimal policies

We now describe the Bellman equation for $V_s(\cdot)$, find an analytical solution for it and the associated optimal policy. We first write down the Bellman equation for an agent unmatched with a financial intermediary and holding a real value of cash m. The only decision that this agent must make is whether to remain unmatched, or to pay the fixed cost b and be matched with a financial intermediary. If the agent chooses not to contact the intermediary then, as standard, the Bellman equation states that the return on the value function $rV_u(m)$ must equal the flow cost, given by the opportunity cost Rm, plus the expected change per unit of time. There are two sources of expected changes per unit of time: The first is that she finds a financial intermediary with probability p, upon which she incurs in a change in value $V_f(m) - V_u(m)$. The second is that in the next instant of time the real value of cash balances decreases by the amount $c + m\pi$ due, respectively, to her consumption and the effect of inflation. Thus, denoting by $V_u'(m)$ the derivative of $V_u(m)$ with respect to m, the Bellman equation satisfies:

$$rV_u(m) = Rm + p(V_f(m) - V_u(m)) + V_u'(m)(-c - m\pi) \tag{10}$$

On the other hand, if the agent chooses to contact the intermediary, the Bellman equation satisfies

$$V_u(m) = b + V_f(m) \tag{11}$$

Notice that an agent can end up being matched with a financial intermediary either because it exogenously "bumps" into it with probability p, or because she pays the cost b. Regardless of how she is matched, an agent matched with a financial intermediary chooses the optimal withdrawal, which we denote by w, as follows

$$V_f(m) = \min_w V_u(m+w) \tag{12}$$

subject to

$$w + m \geq 0 \tag{13}$$

where the constraint stipulates that after the withdrawal, or deposit, the cash balances are non-negative. Inspection of (12) reveals that $V_f(\cdot)$ does not depend on m, so from now we denote this value as V^*.

We now turn to the characterization of the Bellman equations and its optimal policy. We will guess and later verify that the optimal policy is described by two parameters, $0 < m^* < m^{**}$. The threshold m^* is the value of cash that agents choose at a financial intermediary; we refer to it as the cash replenishment level. The threshold m^{**} is a value of cash beyond which agents will pay the cost b, contact the intermediary, and make a deposit so as to leave her cash balances at m^*.

Given our guesses, m^*, m^{**}, $V_u(m)$ and V^* we will assume, and later verify, that

$$V_u(m) < V^* + b \;\; \text{for } m \in (0, m^{**})$$

so that for $m \in (0, m^{**})$ is not optimal to pay the cost and contact the intermediary. We have that

$$V_u(0) = V^* + b$$

This equality follows since at $m = 0$ the agent must withdraw, since if she does not in the next instance either $m(t)$ becomes negative or she will not be able to finance her consumption. Similarly,

$$V_u(m) = V^* + b \;\; \text{for } m \geq m^{**}$$

which follows from the assumption that agents contact the intermediary for $m \geq m^{**}$. Inserting these guesses into (10), (11), (12) implies that a solution of (9) is given by numbers V^*, m^*, m^{**} and the function $V_u(m)$, satisfying:

$$V^* = V_u(m^*) = \min_z V_u(z) \tag{14}$$

$$V_u(m) = \begin{cases} V^* + b & \text{if } m = 0 \\ \dfrac{Rm + pV^* - V_u'(m)(c + m\pi)}{r + p} & \text{if } m \in (0, m^{**}) \\ V^* + b & \text{if } m \geq m^{**} \end{cases} \tag{15}$$

In Appendix A we display the Bellman equations for the a discrete time version of the model. The appendix provides an alternative derivation of the continuous time Bellman equations (14) and (15) by taking limits of the discrete time case as the length of the time interval goes to zero.

The next proposition gives one non-linear equation whose unique solution determines the cash replenishment value m^* as a function of the parameters of the model: $R,\ \pi,\ r,\ p,\ c$ and b.

Proposition 2 *Assume that $r+\pi+p>0$. The optimal return point m^* is given by the unique positive solution to*

$$\left(\frac{m^*}{c}\pi+1\right)^{1+\frac{r+p}{\pi}}=\frac{m^*}{c}\left(r+p+\pi\right)+1\ +\left(r+p\right)\left(r+p+\pi\right)\frac{b}{cR} \tag{16}$$

for $\pi\neq 0$. See appendix B for a proof, and appendix C for the $\pi=0$ case.

Note that, keeping r and π fixed, the solution for m^*/c is a function of (b/cR) , as it is in the steady state derivation of money demand of Section 3. The next proposition gives a closed form solution for the function $V_u(\cdot)$, and the scalar V^* in terms of m^*.

Proposition 3 *Assume that $r+\pi+p>0$. Let m^* be the solution of (16).*
(i) The value for the agents not matched with a financial institution, for $m\in(0,m^{**})$, is given by the convex function:

$$V_u(m)=\left[\frac{pV^*-Rc/\left(r+p+\pi\right)}{r+p}\right]+\left[\frac{R}{r+p+\pi}\right]m+\left(\frac{c}{r+p}\right)^2A\ \left[1+\frac{\pi}{c}m\right]^{-\frac{r+p}{\pi}}$$

where the constant A is given by:

$$A=\frac{r+p}{c^2}\left(R\ m^*+\left(r+p\right)b+\frac{Rc}{r+p+\pi}\right)>0.$$

For $m\geq m^{**}$

$$V_u(m)=V^*+b$$

(ii) The value for the agents matched with a financial institution, V^*, is given by:

$$V^*=\frac{R}{r}m^*$$

See appendix B for a proof and appendix C for the $\pi=0$ case.

The following picture displays an example value function:

The next proposition uses the characterization of the solution for m^* to conduct some comparative statics.

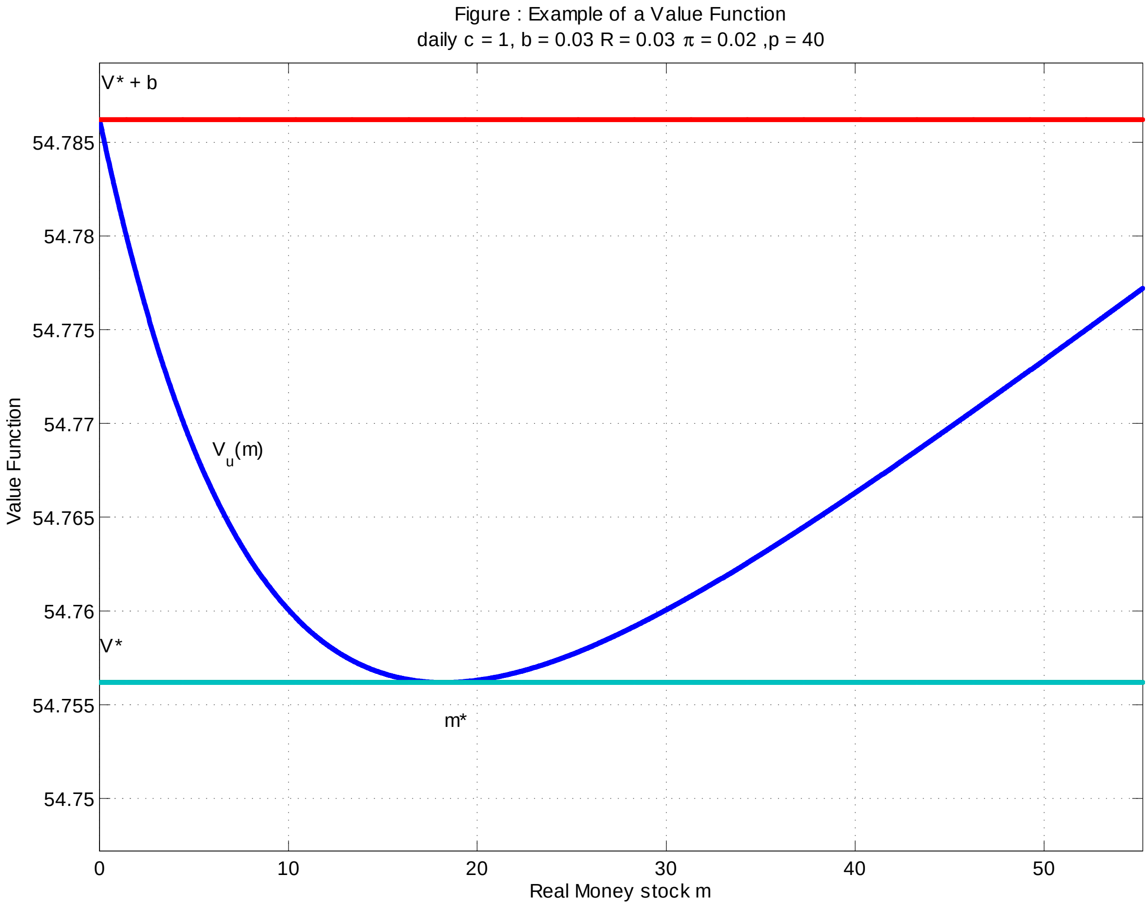

Figure 1: An example Value function

Proposition 4 *The optimal return point* $m^*(R, r, \pi, c, b, p)$ *has the following properties:*
1. m^* *is homogenous of degree one in* (c, b).
2. The elasticity of m^* *with respect to* b

$$0 \leq \frac{b}{m^*}\frac{dm^*}{db} \leq \frac{1}{2}$$

is decreasing in p, *moreover* $m^* \to 0$ *as* $b \to 0$
3. m^* *is increasing in* c, *and*

$$\frac{c}{m^*}\frac{dm^*}{dc} = 1 - \frac{b}{m^*}\frac{dm^*}{db}.$$

4. The interest rate elasticity satisfies

$$0 \leq -\frac{R}{m^*}\frac{dm^*}{dR} = \frac{b}{m^*}\frac{dm^*}{db} \leq \frac{1}{2}$$

and hence it is decreasing in p.
5. For small b/c, we can approximate m^* by the the solution in BT model, or

$$m^*/c = \sqrt{2\ \frac{b}{cR}} + o\left(\sqrt{\frac{b}{c}}\right)$$

where $o\left(\sqrt{\frac{b}{c}}\right)/\sqrt{\frac{b}{c}} \to 0$ *as* $\sqrt{\frac{b}{c}} \to 0$.
6. Assuming that the Fisher equation holds, in that $\pi = R - r$, the elasticity of m^* evaluated at zero inflation, i.e. at $R = r$, satisfies

$$0 \leq -\frac{p}{m^*}\frac{dm^*}{dp}|_{R=r} \leq \frac{p}{p+r}\ .$$

7. Assuming that the Fisher equation holds, in that $\pi = R - r$, the elasticity of m^* evaluated at zero inflation, i.e. at $R = r$, satisfies

$$-\frac{R}{m^*}\frac{dm^*}{dR}|_{R=r} \leq \frac{1}{2}\ .$$

with strict inequality iff $r + p > 0$.
Proof. See appendix B.

Properties 1-4 are the same as in the steady state money demand derived in Section 3. Property 5. says that when b is small relative to c, the resulting money demand is well approximated by the one for the BT model. Property 6. has its analog in the model with p free trips of Section 3.1. The elasticities in 6. and 7. are computed imposing the Fisher equation $R = r + \pi$, in particular we replace inflation using $\pi = R - r$. Instead in the elasticity computed in property 4, as R changes, the inflation rate π and the real rate r are kept constant. The fact that the interest rate elasticity is smaller than $1/2$ and decreasing (in absolute value) on p is is one the main results of the model.

5 Distribution of cash balances, and average number and size of cash withdrawals

This section derives the distribution of real cash holdings when the policy characterized by the parameters (m^*, p, c) is followed and the inflation rate is π. The policy is to replenish cash holdings up to the return value m^*, either when a match with a financial intermediary occurs, which happens at a rate p per unit of time, or when the agent runs out of money (i.e. real balances hit zero). In the previous section we showed that this is the nature of the optimal policy and we characterized how m^* depends on the fundamental parameters (R, r, π, p, c, b).

Our first result is to compute the expected number of withdrawals per unit of time, denoted by n. This includes both the withdrawals that occur upon an exogenous contact with the financial intermediary and the ones initiated by the agent when her cash balances reach zero.[9]

Proposition 5 *The number of cash withdrawals per unit of time when $\pi \neq 0$ is*

$$n\left(m^*; c,\ \pi, p\right) = \frac{p}{1 - \left(1 + m^* \pi / c\right)^{-\frac{p}{\pi}}} \tag{17}$$

See appendix B for a proof and appendix C for the $\pi = 0$ case.

For future reference we notice that n is homogenous of degree zero in (m^*, c) .As can be seen from the expression the ratio $n/p \geq 1$ since in addition to the p free withdrawals it includes the costly withdrawals that agents do when they exhaust their cash. Notice that n/p is decreasing in m^*, indicating that a greater value for the return point allows the agent to finance consumption over a longer time-span. The reciprocal of n gives the expected time between withdrawals. We can see that $1/n$ is a concave and increasing function of $m^*\pi/c$. A second order approximation of this function gives:

$$\frac{1}{n\left(m^*; c\ \pi, p\right)} = \frac{m^*}{c} - \frac{1}{2}\left(\pi + p\right)\left(\frac{m^*}{c}\right)^2 \tag{18}$$

Note how this formula yields exactly the expression in the BT model when $p = \pi = 0$. The formula shows, moreover, that the expected time between withdrawals is decreasing in π and in p.

The next figures displays the average number of withdrawals against the level of interest rates R for different values of the parameter p. All the flow variables are expressed annually, except consumption which is expressed daily, so n is the average number of withdrawals per year.

We use $b = 0.03$ as implying a cost of about 3 percent of daily cash consumption, which is equivalent to less than 2 percent of consumption of non-durables an services (see Table 2), or about 1 percent of daily income. Comparing the numbers in this plot with the ones of Table 1, it seems that a value of p of about 40 is reasonable for those households with an ATM card and one p about 10 may be appropriate for those without an ATM card.

[9] For instance if $n = 52$ when all the parameters are measured per annum or equivalently if $n_{day} = 52/365 = 1/7$ if measured per day, then the agent withdraws 52 times in a year or, equivalently, she withdraws every 7 days ($1/n_{day} = 7$).

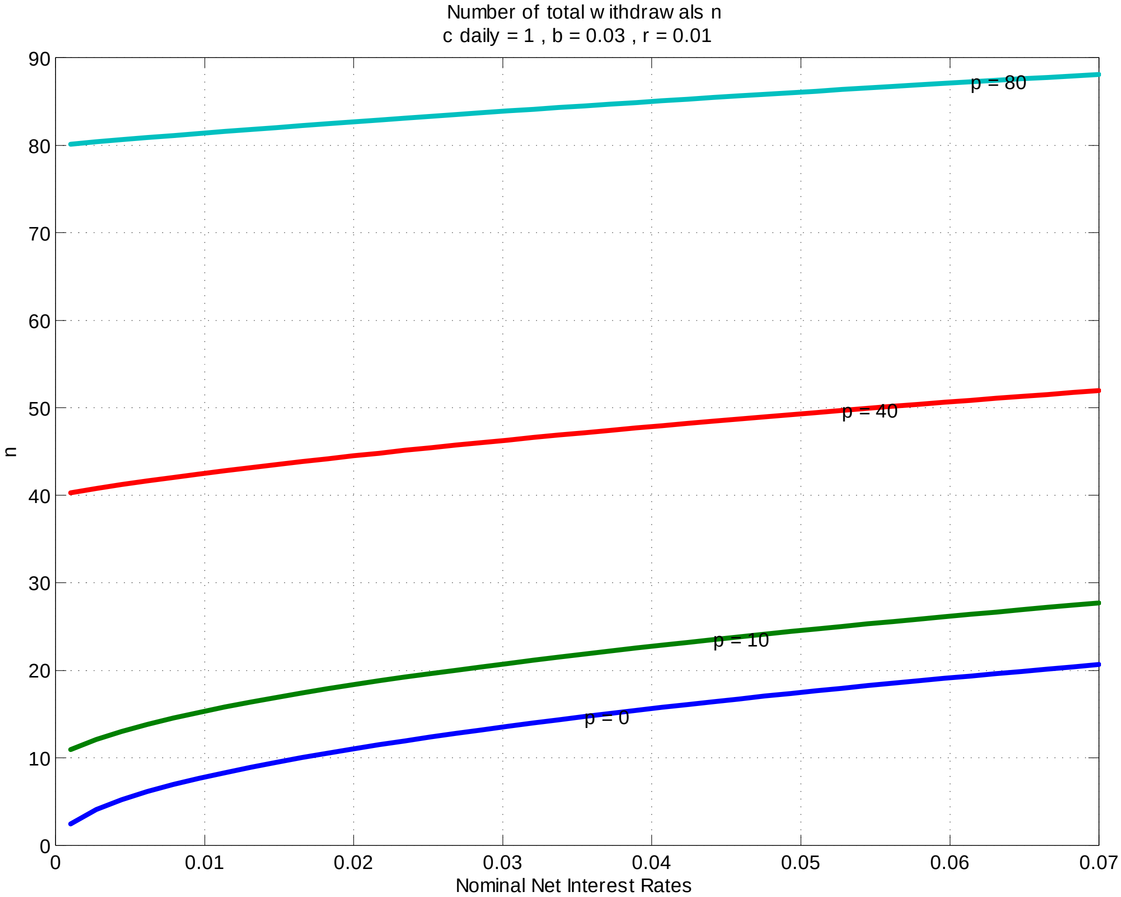

Figure 2: Number of withdrawals

The next Proposition derives the density of the distribution of real cash balances as a function of p, π, c and m^*.

Proposition 6 *(i) The density for the real balances m when $\pi \neq 0$ is*

$$h(m) = \left(\frac{p}{c}\right)\left(\frac{\left[1+\frac{\pi}{c}m\right]^{\frac{p}{\pi}-1}}{\left[1+\frac{\pi}{c}m^*\right]^{\frac{p}{\pi}}-1}\right) \tag{19}$$

(ii) Let $H(m, m_1^*)$ be the cumulative distribution of m for a given m^*. Let $m_1^* < m_2^*$, then $H(m, m_2^*) \leq H(m, m_1^*)$, i.e. $H(\cdot, m_2^*)$ first order stochastically dominates $H(\cdot, m_1^*)$.
See appendix B for a proof and appendix C for the $\pi = 0$ case.

In the proof of Proposition 6 we show that the density of m solves the following ODE:

$$\frac{\partial h(m)}{\partial m} = \frac{(p-\pi)}{(\pi m + c)} h(m)$$

for any $m \in (0, m^*)$. There are two forces determining how the mass is spread out, i.e. determining the shape of this density. One force is that agents meet a financial intermediary at a rate p, where they replenish their cash balances. The other is that inflation eats away the real value of their nominal balances. Notice that if $p = \pi$ these two effects cancel and the density is uniformly constant. If $p < \pi$, the density is downward sloping, with more agents at low values of real balances due to the greater pull of the inflation effect. If $p > \pi$, the density is upward sloping due the the greater effect of the replenishing of cash balances. To see this notice that, as shown in the proof of Proposition 2 (in appendix B), $\pi m^* + c > 0$, thus $\pi m + c > 0$ for all m in the invariant support $(0, m^*)$ and the sign of $\partial h(m)/\partial m$ is given by the sign of $(p - \pi)$.

We can now define the aggregate money demand as

$$M = \int_0^{m^*} m h(m)\, dm.$$

The next proposition gives a formula for M as a function of p, π, c, and m^*.

Proposition 7 *(i) For a given m^*, the aggregate money demand is given by :*

$$M = \mu(m^*; c, \pi, p) \equiv c\, \frac{\left(1+\frac{\pi}{c}m^*\right)^{\frac{p}{\pi}}\left[\frac{m^*}{c} - \frac{\left(1+\frac{\pi}{c}m^*\right)}{p+\pi}\right] + \frac{1}{p+\pi}}{\left[1+\frac{\pi}{c}m^*\right]^{\frac{p}{\pi}}-1} \tag{20}$$

for $\pi \neq 0$.
(ii) M is increasing in m^*.
See appendix B for a proof and appendix C for the $\pi = 0$ case.

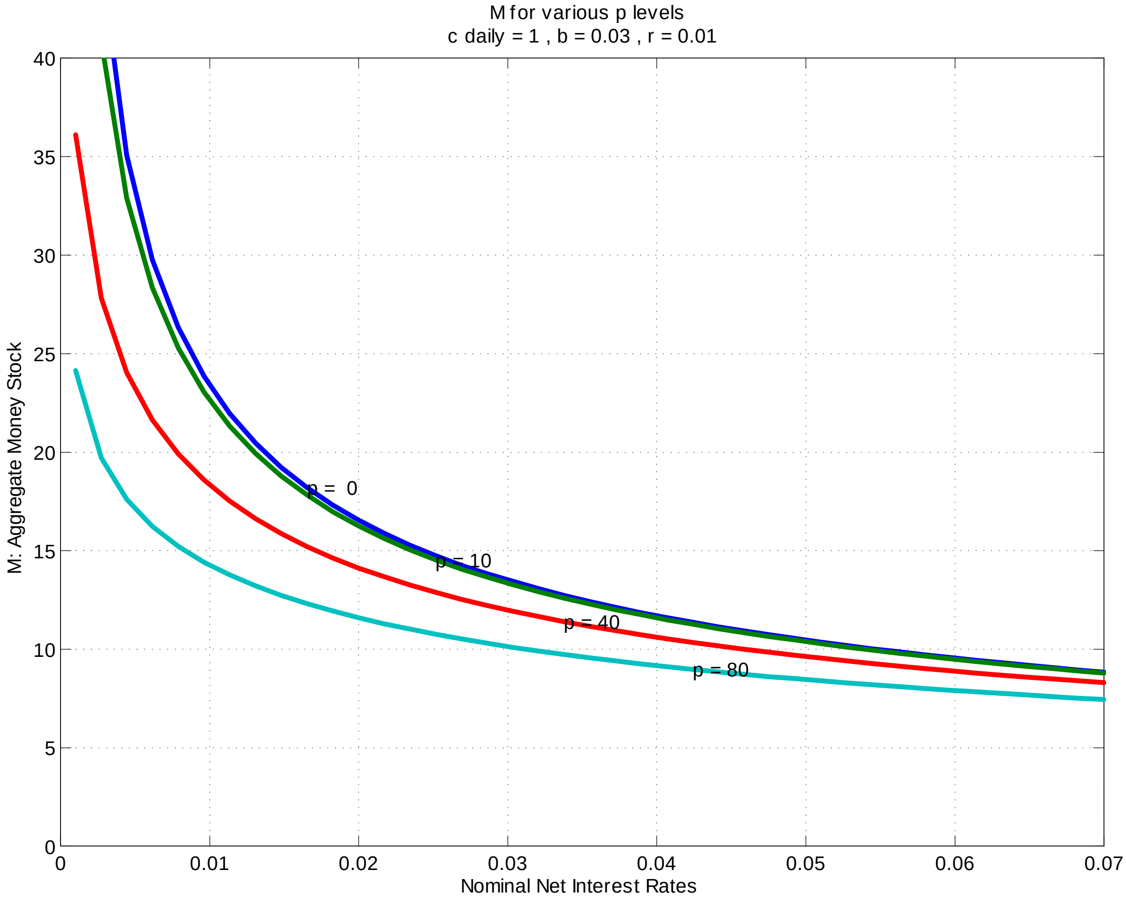

Figure 3: Money demand curves

The next figure displays a plot of the aggregate money demand M as a function of the nominal interest rate R at various levels of financial diffusion p.

Very roughly, the numbers in this figure suggest that $p = 40$ produces cash balances of similar magnitudes that those of ATM card holders in our Italian data set. On the other hand, matching the cash balances of those households without an ATM, requires a much lower values of p, closer to 10.

The next proposition compares the interest rate elasticity of the aggregate money demand, with the one for the average number of withdrawals. As a benchmark, recall that in the Baumol Tobin model these two elasticities are $-1/2$ and $1/2$ respectively. In our model, for $p > 0$, the elasticity of the money demand is higher in absolute value than the elasticity of the average number of withdrawals. The intuition for this result is that the average money demand depends on both the target level for cash replenishment m^* and the average number of withdrawals, n. Indeed we have that if the replenishment policy described above is followed then:

$$\frac{M}{c} = \frac{1}{p+\pi}\left[n \ \ (m^*/c) - 1\right], \tag{21}$$

which can be verified by inserting the expression for n given by (17) into the formula for M in (20). But, since for $p > 0$ some withdrawals entail no cost, the households always makes p withdrawals on average. Notice that this is different from the deterministic steady state model with p free withdrawals (Section 3.1), where the two interest rate elasticities were the same. This is also different from the evidence in Tables 4 and 6 for Italian households, where we find similar interest rate elasticities, in absolute values, for M/c and n.

Proposition 8 *The interest rate elasticity of the average cash balances is larger in absolute value than the interest rate elasticity of the average number of withdrawals, evaluated at* $\pi = 0$.

$$-\frac{M\left(R,r,\pi,p\right)}{R}\frac{\partial n\left(R,r,\pi,p\right)}{\partial R}|_{\pi=0} \geq \frac{n\left(R,r,\pi,p\right)}{R}\frac{\partial n\left(R,r,\pi,p\right)}{\partial R}|_{\pi=0}$$

See the appendix for the proof.

The next two figures display the interest rate elasticities of the average money balances and the average number of withdrawals for different values of the parameter p.

These figures make clear that the interest rate elasticity of the money demand is higher, in absolute value than the one for the average number of withdrawals. These figure also makes clear that to obtain an interest rate elasticity of the money demand as low as 1/3 in absolute value the model requires a relatively large value of p. For instance, with $p = 80$ and the same parameteres, the interest rate elasticity is 1/3 when evaluated at an interest of 3 percent.

For future reference, the next proposition studies the relationship between M and m^* :

Proposition 9 *The ratio* M/m^* *is increasing in* p *with*

$$\begin{aligned} M/m^* &= \frac{1}{2} \text{ for } p = 0 \\ M/m^* &\to 1 \text{ as } p \to \infty. \end{aligned}$$

Proof. To be done.

For the case of $\pi = 0$, the ratio M/m^* has the following a simple expression:

$$M/m^* = \frac{1}{1-\exp\left(-pm^*/c\right)} - \frac{1}{p\left(m^*/c\right)}$$

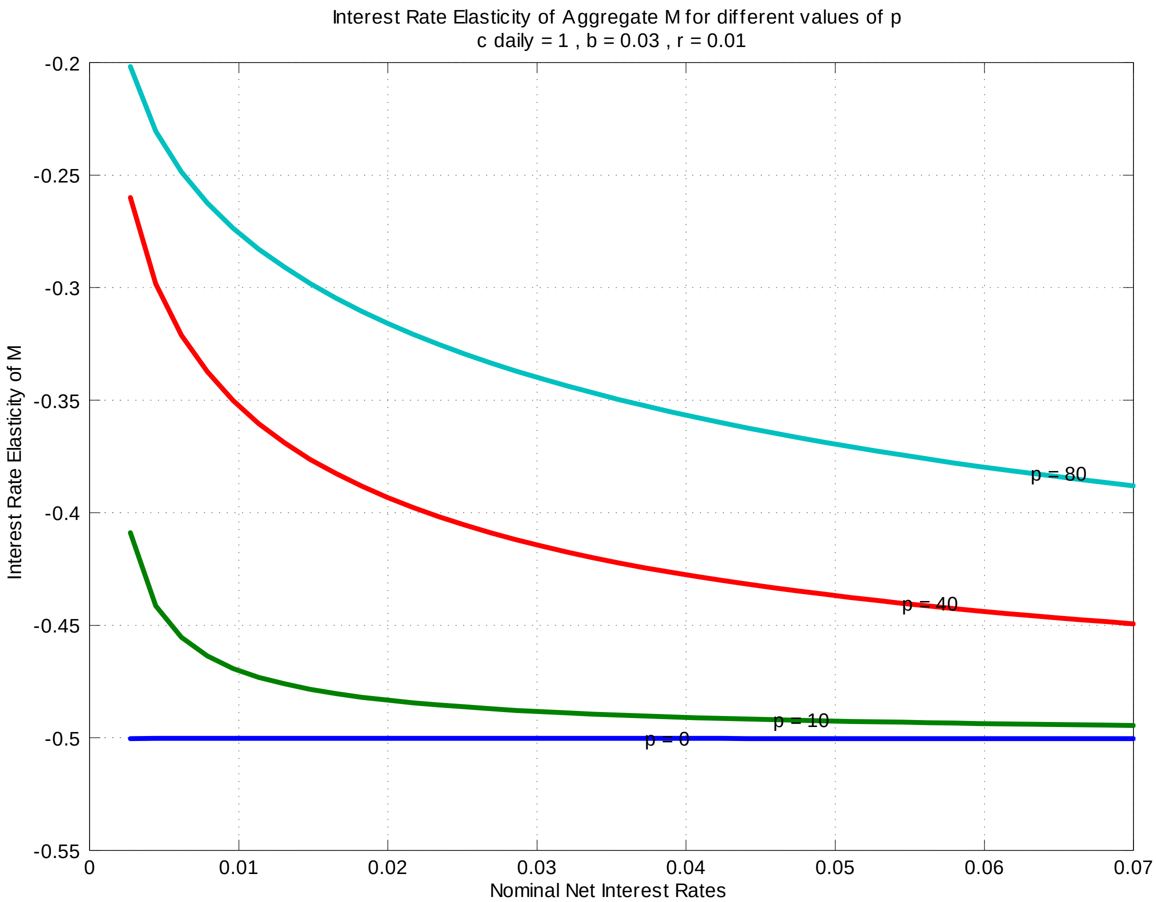

Figure 4: Elasticity of money demand

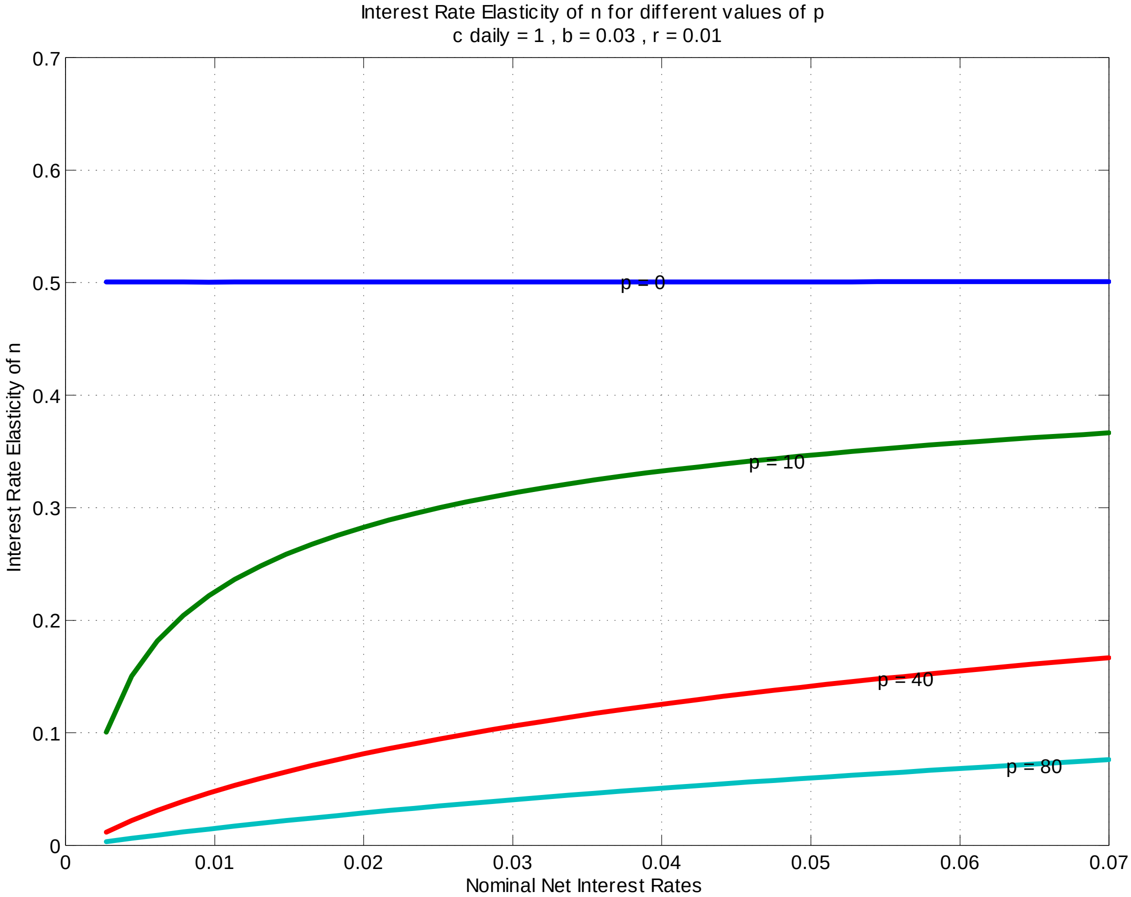

Figure 5: Interest elasticity for the number of withdrawals

Since the right hand side is a decreasing function of $(m^*/c)\,p$, and since we had shown that the elasticity of m^*/c is (in absolute value) smaller than $p/\,(p+r)$, then it implies that M^*/c is decreasing in p.

Now we turn to the analysis of the average number of withdrawals, which we denote by W.

Proposition 10 *The average withdrawal is given by:*

$$W = m^* \left[1 - \frac{p}{n}\right] + \left[\frac{p}{n}\right] \int_0^{m^*} (m^* - m)\, h(m)\, dm \tag{22}$$

where

$$\int_0^{m^*} (m^* - m)\, h(m)\, dm = \frac{\frac{\left(1+\frac{\pi}{c}m^*\right)^{\frac{p}{\pi}+1} - 1}{(p+\pi)/c} - m^*}{\left(1+\frac{\pi}{c}m^*\right)^{\frac{p}{\pi}} - 1}$$

Proof. Follows from Proposition 19 below setting $f = 0$.

To understand the expression for W notice that $n-p$ is the number of withdrawals in a unit of time that occur because agents reach zero balances, so if we divide it by the total number of withdrawals per unit of time (n) we obtain the fraction of withdrawals that occur the agent reaches zero balances. Each of these withdrawals is of size m^*. The complementary fraction gives the withdrawals that occur due to a chance meeting with the intermediary. A withdrawal of size $m^* - m$ happens with frequency $h(m)$.

Combining the previous results we can see that for $p > 0$, the ratio of withdrawals to average cash holdings is less than 2. To see this, using the definition of W we can write

$$\frac{W}{M} = \frac{m^*}{M} - \frac{p}{n}. \tag{23}$$

Since $M/m^* \geq 1/2$, then it follows that $W/M \leq 2$. Indeed notice that for p large enough this ratio can be smaller than one. We mention this property because for the Baumol -Tobin model the ratio W/M is exactly two, while in the data of Table 1 the average ratio is below 1.4 for those households without an ATM card and about 1.2 for those with an ATM card. The intuition for this result in our model is clear: agents take advantage of the free random withdrawals regardless of their cash balances, hence the withdrawals are distributed on $[0, m^*]$, as opposed to be concentrated on m^*, as in the BT model.

The average amount of money that an agent has at the time of withdrawal, $\underline{M}$, is

$$\underline{M} = \left[\frac{p}{n}\right] \int_0^{m^*} m\; h(m)\, dm$$

Simple algebra shows that:

$$\underline{M} = m^* - W \tag{24}$$

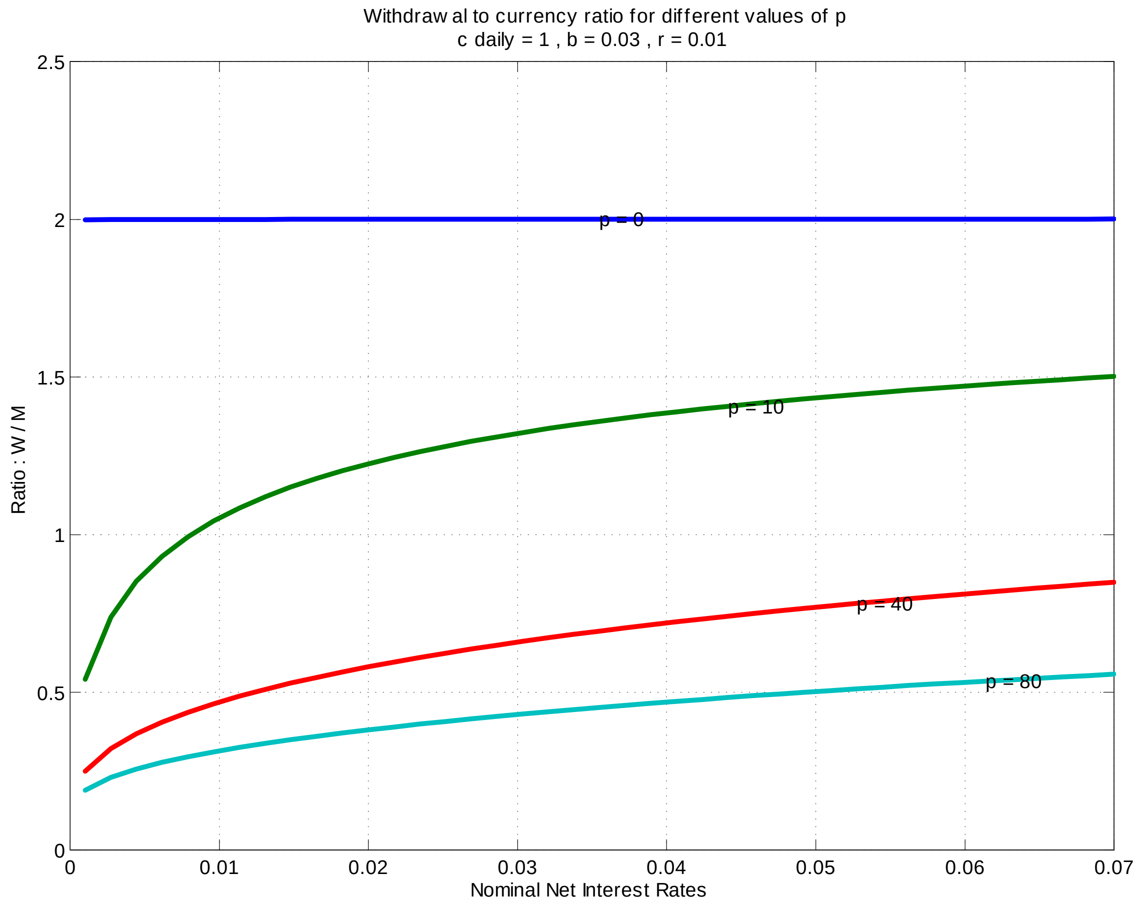

Figure 6: W/M Ratio

The relevance of this statistic is that in the survey of Italian households there is a question about the amount of money that triggers a withdrawal (see Table 1). Depending on the interpretation of this question by the survey respondents, $\underline{M}$ is the theoretical counterpart of this minimum.

6 The steady state money demand (To Be Completed)

In this section we derive the steady state money demand for the dynamic problem of Section 4. This involves computing the model analogue of the T function used in Section 3, from which a steady state money demand can be derived. This function gives the expected number of costly withdrawals per unit of time when the real money balances are M and the real consumption flow is c. The function is derived under the assumption that the policy followed by the agents is of the form analyzed in Section 5. Using μ^{-1} for the inverse of the function defined in (20), and using (17) we have:

$$T(M, c;\ p, \pi) = n\left(\ \mu^{-1}(M; c, p, \pi), c, \pi, p\right) - p. \tag{25}$$

In the previous expression we subtract p to obtain the number of costly withdrawals per unit of time. Since μ is homogenous of degree one in (c, m^*), then μ^{-1} is homogeneous of degree one in (c, M), and hence T is homogenous of degree zero in (M, c). We define the steady state money demand $\tilde{M}(R; c, p, \pi)$ corresponding to the technology (25), as the solution of the problem (1), analyzed in Section 3.

The next proposition compares $\tilde{M}$ with the money demand for the dynamic model.

Proposition 11 *Let $M(R; c, p, \pi, r)$ be the solution of the aggregate money demand for the dynamic problem analyzed in Sections (4) and (5). Then*

$$\tilde{M}(R; c, p, \pi) = M(R; c, p, \pi, 0).$$

See appendix B for a proof.

This result is intuitive, since as $r \to 0$ all the periods in the discounted shadow cost are assigned the same weight, hence the shadow costs tends to the steady state cost.

The point of this comparison is to isolate the features that makes our analysis different from the standard Baumol-Tobin model. Quantitatively, given the values for r that we use, the solution of the money demand for the dynamic model and the steady state one, are virtually the same.

7 Costly random withdrawals

The dynamic model discussed above has the unrealistic feature that agents withdraw every time a match with a financial intermediary occurs, thus making as many withdrawals as contact with the financial intermediary, many of which of a very small size. In this section we extend the model to the case where the withdrawals (deposits) done upon the random contacts with the financial intermediary are subject to a fixed cost f. We assume that $0 < f < b$.

As mentioned above, this model has a more realistic depiction of the distribution of withdrawals, by limiting the minimum withdrawal size. On the other hand, if f is large relative to b, the prediction of the model gets closer to the ones of the Baumol-Tobin model. Indeed, if as f goes to b, then then there is no advantage of a chance meeting with the financial intermediary, and hence the model is identical to the one of the previous section, but with $p = 0$.

Agents face a cash-in-advance constraint, and they can withdraw or deposit from an interest bearing account. The sequence problem is to choose an increasing sequence of stopping times $\{\tau_j\}$ at which to withdraw (or deposit) money in an interest bearing account, and the amounts to withdraw at each time, so as to minimize the expected discounted cost of financing a given constant real consumption flow c. The expected discounted total cost, denoted by TC_0 is:

$$TC_0(\tau, m) = E_0 \left[\sum_{j=0}^{\infty} e^{-r\ \tau_j} \left\{ b\ I_{\tau_j} + f\ \hat{I}_{\tau_j} + \left(m\left(\tau_j^+\right) - m\left(\tau_j^-\right) \right) \right\} \right] \qquad (26)$$

where we use $m(t)$ to denote the real value of the stock of currency. As before, the stock of currency jumps discontinuously up at the time of a withdrawal (so the amount of a withdrawal at τ_j is $m\left(\tau_j^+\right) - m\left(\tau_j^-\right)$) and the law of motion of the real value of the stock of money between withdrawals is given by equation (7).

As before we assume that contacts with the financial intermediary follow a Poisson process with arrival rate p. In the case of a contact the agent can withdraw (or deposit) money in an interest bearing account at a real cost f . If the agent wants to withdraw (or deposit) in the financial institution in any other time, it must pay a real cost b. The indicator I_{τ_j} takes the value of zero if the withdrawal (or deposit) takes place at the time $t = \tau_j$ of a contact with a financial intermediary, and takes the value of one otherwise. The indicator $\hat{I}_{\tau_j}$ takes the value of one if the withdrawal (or deposit) takes place at the time $t = \tau_j$ of a contact with a financial intermediary, and takes the value of one otherwise. The agent chooses stopping times and withdrawals as function of the history of contacts with the intermediary.

As before, we define the shadow cost of a policy $\{\tau_j, m\}$ as the expected discounted cost of the withdrawals plus the expected discounted opportunity cost of the cash balances held by the agent. We denote the shadow cost as SC_0, which is given by:

$$SC_0(\tau,m) = E_0\left[\sum_{j=0}^{\infty} e^{-r\ \tau_j}\left\{b\ I_{\tau_j} + f\ \hat{I}_{\tau_j} + \int_0^{\tau_{j+1}-\tau_j} R\ m(\tau_j+t)\ e^{-rt}\ dt\right\}\right] \tag{27}$$

The next Proposition is the analogous of Proposition 1.

Proposition 12 *Assume that $R = r+\pi$. For any policy $\{\tau, m\}$ the total cost equals the shadow cost plus the present value of c, or*

$$TC_0 = \frac{c}{r} + SC_0\ .$$

Proof. The proof is completely analogous to the one for Proposition 1.

We use $V_s(m)$ for the value function corresponding to the minimization of the shadow cost:

$$V_s(m_0) = \min_{\tau,m} SC_0(\tau,m)$$

subject to $m(0) = m_0$ and where $s = f$ denotes that the agent is matched to a financial intermediary and $s = u$ that she is not. Let V^* be the minimum attained by the value function, i.e. $V^* \equiv V(m^*) = \min_z V(z)$, which is the value attained at the optimal return point m^* and is independent of the state s.

Using notation that is analogous to the one that was used above, the Bellman equation for this problem when the agent is not matched is given by:

$$rV_u(m) = Rm + p\min\{V^* + f - V_u(m), 0\} + V_u'(m)(-c - m\pi) \tag{28}$$

where $\min\{V^* + f - V_u(m)\ ,\ 0\}$ takes into account that it may not be optimal to withdraw/deposit for all contacts with a financial intermediary. Indeed, whether the agent chooses to do so will depend on her level of cash balances.

We will guess, and later verify, a shape for $V_u(\cdot)$ that implies a simple threshold rule for the optimal policy. Our guess is that $V_u(\cdot)$ is strictly decreasing at $m = 0$ and single peaked attaining a minimum at a finite value of m. Then we guess that there will be two thresholds, $\underline{m}$ and $\bar{m}$, that satisfy:

$$V^* + f = V_u(\underline{m}) = V_u(\bar{m}) \tag{29}$$

Under these assumptions the minimized cost takes the form:

$$\min\{V^* + f - V_u(m), 0\} = \begin{cases} V^* + f - V_u(m) < 0 & \text{if } m < \underline{m} \\ 0 & \text{if } m \in (\underline{m}, \bar{m}) \\ V^* + f - V_u(m) < 0 & \text{if } m > \bar{m} \end{cases}$$

Thus solving the Bellman equation is equivalent to finding 5 numbers $m^*, m^{**}, \underline{m}, \bar{m},\ V^*$ and a function $V_u(\cdot)$ such that:

$$V^* = V_u(m^*) = \min_z V_u(z)$$

which, given the convexity of V_u, we can write as the following two equations:

$$V^* = V_u(m^*) \quad (30)$$
$$0 = V_u'(m^*) \quad (31)$$

and

$$V_u(m) = \begin{cases} \dfrac{Rm + p(V^* + f) - V_u'(m)(c + m\pi)}{r + p} & \text{if } m \in (0,\ \underline{m}) \\ \dfrac{Rm - V_u'(m)(c + m\pi)}{r} & \text{if } m \in (\underline{m}, \bar{m}) \\ \dfrac{Rm + p(V^* + f) - V_u'(m)(c + m\pi)}{r + p} & \text{if } m \in (\bar{m},\ m^{**}) \end{cases} \quad (32)$$

and the conditions:

$$V_u(0) = V^* + b \quad (33)$$
$$V_u(m) = V^* + b \text{ for } m > m^{**} \quad (34)$$

Hence the optimal policy in this model is to pay the fixed cost f and withdraw cash when the agent contact the financial intermediary, if her cash balance are in $(0, \underline{m})$ or to deposit if the cash balances are larger than $\bar{m}$. In either case the withdrawal or deposits is such that the post transfer cash balances are set equal to m^*. If the agent contacts a financial intermediary when her cash balances are in $(\underline{m},\ \bar{m})$ then, no action is taken. If the agent cash balances get to zero, then the fixed cost b is paid, and after the withdraw the cash balances are set to m^*. Notice that $m^* \in (\underline{m},\ \bar{m})$. Hence, in this version the withdrawals will have minimum size, namely $m^* - \underline{m}$. This is a more realistic depiction of actual management of cash.

Now we turn to the characterization and solution of the Bellman equation. The solution of the model is similar to the one in the body of the paper, in Propositions 2 and 3. By using the analogous of lemma 1 we obtain the following:

Proposition 13 *For a given* $V^*, \underline{m}, \bar{m}, m^{**}$ *satisfying* $0 < \underline{m} < \bar{m} < m^{**}$:
The solution of (32) for $m \in (\underline{m}, \bar{m})$ *is given by:*

$$\begin{aligned} V_u(m) &= \varphi(m, A_\varphi) \equiv \\ &\equiv \frac{-Rc/(r+\pi)}{r} + \left(\frac{R}{r+\pi}\right) m + \left(\frac{c}{r}\right)^2 A_\varphi \left[1 + \frac{\pi}{c} m\right]^{-\frac{r}{\pi}} \end{aligned} \quad (35)$$

for an arbitrary constant A_φ

Likewise, the solution of (32) for $m \in (0,\ \underline{m})$ or $m \in (\bar{m},\ m^{**})$ is given by:

$$V_u(m) = \eta(m, V^*, A_\eta) \equiv \qquad (36)$$
$$\equiv \frac{p(V^*+f) - Rc/(r+p+\pi)}{r+p} + \left(\frac{R}{r+p+\pi}\right)m + \left(\frac{c}{r+p}\right)^2 A_\eta \left[1+\frac{\pi}{c}m\right]^{-\frac{r+p}{\pi}}$$

for an arbitrary constant A_η.
Proof. See appendix B.

Next we are going to list a system of 5 equations in 5 unknowns that describes a C^1 solution of $V_u(m)$ on the range $[0, m^*]$. The unknowns in the system are $V^*, A_\eta, A_\varphi, \underline{m}, m^*$. Using proposition 13, and the boundary conditions (29),(30),(31) and (33), the system is given by:

$$\varphi_m(m^*, A_\varphi) = 0 \qquad (37)$$
$$\varphi(m^*, A_\varphi) = V^* \qquad (38)$$
$$\eta(\underline{m}, V^*, A_\eta) = V^* + f \qquad (39)$$
$$\eta(0, V^*, A_\eta) = V^* + b \qquad (40)$$
$$\varphi(\underline{m}, A_\varphi) = V^* + f \qquad (41)$$

In the proof of proposition 14 we show that the solution of this system can be found by solving one non-linear equation in one unknown, namely $\underline{m}$. Once the system is solved it is straightforward to extend the solution to the range: (m^*, ∞).

Proposition 14 *There is a unique solution for the system (37)-(41). The solution characterizes a C^1 function that is strictly decreasing on $(0, m^*)$, convex on $(0, \bar{m})$ and strictly increasing on (m^*, m^{**}). This function solves the Bellman equations described above.*
Proof. See appendix B.

The following picture displays an example value function

Next we present a proposition about the determinants of the range of inaction $m^* - \underline{m}$.

Proposition 15 *The range of inaction $(m^* - \underline{m})$ relative to the drift of cash balances, $c + \pi m^*$, solves:*

$$\frac{f}{R(c+m^*\pi)} = \left(\frac{m^*-\underline{m}}{c+m^*\pi}\right)^2 \left[\frac{1}{2} + \sum_{k=1} \frac{1}{(k+2)!}\left(\frac{m^*-\underline{m}}{c+m^*\pi}\right)^k \Pi_{j=2}^{k+1}(r+j\pi)\right] \qquad (42)$$

Hence $(m^* - \underline{m})/(c+m^*\pi)$ is increasing in f/R (with elasticity smaller than $1/2$)and decreasing in r. Moreover it is decreasing (increasing) in π if $\pi > 0$ ($\pi < 0$).

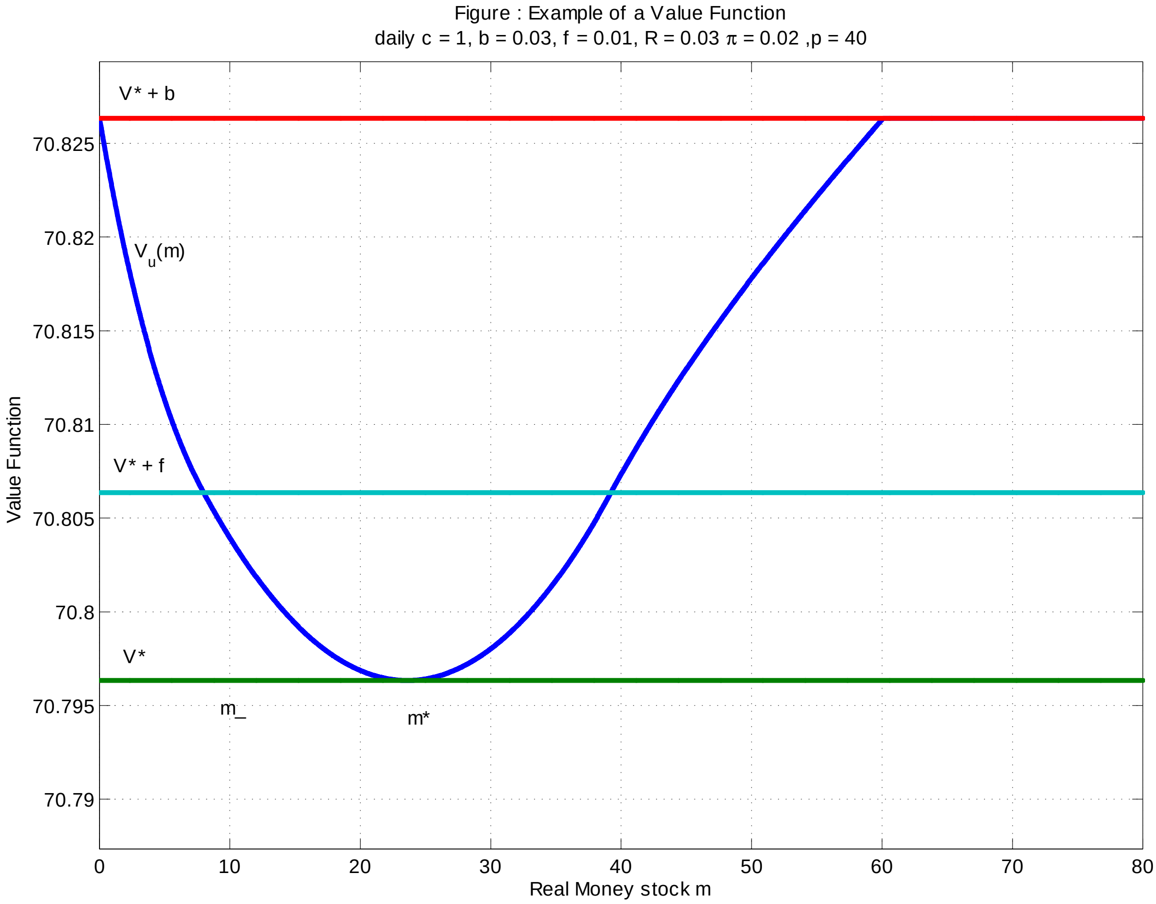

Figure 7: An example value function (for f>0)

Finally, for small $f \ / \ [R(c+\pi m^*)]$ *we have*

$$\frac{m^* - \underline{m}}{c + m^*\pi} = \sqrt{\frac{f}{R\ (c+\pi m^*)}} + o\left(\left(\frac{f}{R\ (c+\pi m^*)}\right)^2\right) . \tag{43}$$

Proof. See appendix B.

Importantly, this proposition says that the scaled range of inaction $(m^* - \underline{m}) / (c + m^*\pi)$ is NOT a function of p or b. Its approximation implies that

$$\frac{R}{(m^* - \underline{m})} \frac{\partial (m^* - \underline{m})}{\partial R}|_{\pi=0} = -\frac{1}{2} + \frac{1}{2}\left(R\frac{m^*}{c}\right)\frac{\partial \pi}{\partial R}.$$

The next proposition examines the expected number of withdrawals n.

Proposition 16 *The expected number of withdrawals per unit of time, n is given by*

$$n = \frac{p}{(p/\pi)\log(1 + (m^* - \underline{m})\pi/c) + 1 - (1 + \underline{m}\pi/c)^{-\frac{p}{\pi}}} \tag{44}$$

and the fraction of agents with cash balances below $\underline{m}$ *is given by*

$$H(\underline{m}) = \frac{1 - (1 + \underline{m}\pi/c)^{-\frac{p}{\pi}}}{(p/\pi)\log(1 + (m^* - \underline{m})\pi/c) + 1 - (1 + \underline{m}\pi/c)^{-\frac{p}{\pi}}} \tag{45}$$

Proof. See appendix B.

Inspection of equation (44) confirms that when $m^* > \underline{m}$ the expected number of withdrawals (n) is no longer bounded below by p. Indeed, as $p \to \infty$ then $n \to [(1/\pi)\log(1 + (m^* - \underline{m})\pi/c)]^{-1}$, which is the reciprocal of the time that it takes for an agent that starts with money holding m^* (and consuming at rate c when the inflation rate is π) to reach real money holdings $\underline{m}$.

Notice that the expected time between withdrawals can be approximated as:

$$\frac{1}{n} = \frac{m^*}{c} - \frac{\pi}{2}\left(\left[\frac{m^*}{c} - \frac{\underline{m}}{c}\right]^2 + \left[\frac{\underline{m}}{c}\right]^2\right) - \frac{1}{2}p\left(\frac{\underline{m}}{c}\right)^2 + o\left\|\left(\frac{\underline{m}}{c}, \frac{m^*}{c}\right)\right\|^2$$

The next figure plots n against the nominal interest rate for several values of p. To highlight the role of $f > 0$ all the subsequent figures have the same parameter values for b, c and r, as the ones for the figures presented above for the case where $f = 0$.

Compare this figure with the one obtained for $f = 0$. Notice that the number of trips associated to the same values for p and R are much smaller; in particular notice than in this case $n < p$. Given the parameters in this figure, a value of p of about 200 is required for the number of withdrawals to be similar to the ones in the

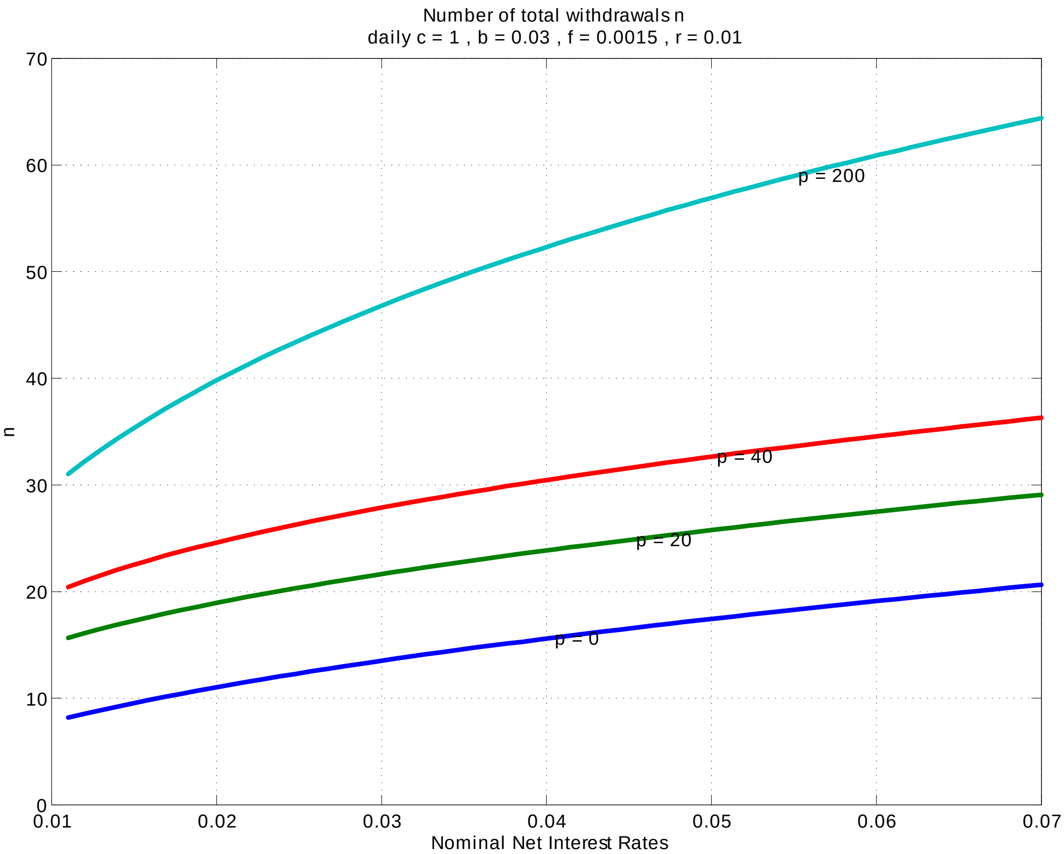

Figure 8: Number of withdrawals (for $f > 0$)

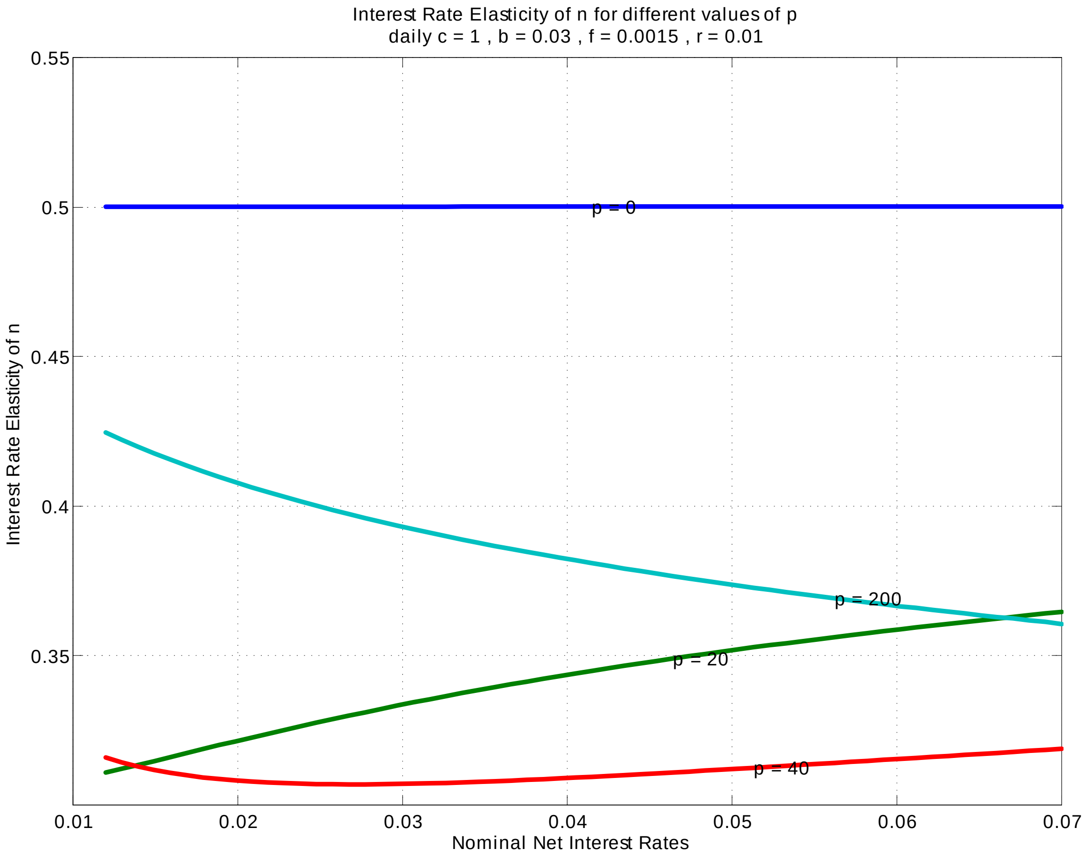

Figure 9: n - elasticity (for $f > 0$)

case of $f = 0$ and $p = 40$, which are similar to the ones of households with an ATM card in our data set.

This figure shows that, interestingly, the interest rate elasticity of n is not monotone on p. While this may not be obvious at first sight, it is to be expected since the model with $0 < f < b$ is in between the BT model, when $f = b$, which has interest rate elasticity of n equal to $1/2$, and our previous model with $f = 0$, which has much lower interest rate elasticity of n. Recall that, at least for $\pi = 0$, the range $m^* - \underline{m}$ is independent of p, and has an interest rate elasticity close to $1/2$. But notice that as $p \to \infty$, $\underline{m} \to 0$, since at at cost $f < b$ agents can, with probability one, find an ATM. Hence, for large enough p the interest rate elasticity of n goes back to $1/2$, the value corresponding to the BT model. We find this feature interesting, because according to the regressions in Table 6, the interest rate elasticity of n is increasing in the empirical measures of p, as it will be for this model if we consider values of p higher than 40. Notice that this is different for the case where

$f = 0$ where the interest rate elasticity of n was decreasing in p.

As before, for any $m \in [0, \underline{m}]$ the density $h(m)$ solves the following ODE:

$$\frac{\partial h(m)}{\partial m} = \frac{(p - \pi)}{(\pi m + c)} h(m)$$

The reason for this is that in this interval the behavior of the system is the same as the one for $f = 0$. On the interval $m \in [\underline{m}, m^*]$ the density $h(m)$ solves the following ODE:

$$\frac{\partial h(m)}{\partial m} = \frac{-\pi}{(\pi m + c)} h(m)$$

The reason for this is that locally in this interval the chance meetings with the intermediary do not trigger a withdrawal, and hence it is as if $p = 0$.

Proposition 17 *The density $h(m)$ and CDF $H(m)$ for $m \in [0, \underline{m}]$ are given by:*

$$h(m) = A_0 \left(1 + \frac{\pi}{c} m\right)^{\frac{p}{\pi} - 1} \tag{46}$$

$$H(m) = \frac{c}{p} A_0 \left[\left(1 + \frac{\pi}{c} m\right)^{\frac{p}{\pi}} - 1\right] \tag{47}$$

where

$$A_0 = \frac{p}{c} \frac{H(\underline{m})}{\left(1 + \frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}} - 1} \tag{48}$$

.

The density $h(m)$ and CDF $H(m)$ for $m \in [\underline{m}, m^*]$ are given by:

$$h(m) = A_1 \left(1 + \frac{\pi}{c} m\right)^{-1} \tag{49}$$

$$H(m) = \frac{c}{\pi} A_1 \log\left(\frac{1 + \frac{\pi}{c} m}{1 + \frac{\pi}{c} m^*}\right) + 1 \tag{50}$$

where

$$A_1 = \frac{\pi}{c} \frac{1 - H(\underline{m})}{\log\left(1 + \frac{\pi}{c} m^*\right) - \log\left(1 + \frac{\pi}{c}\underline{m}\right)} \tag{51}$$

Proof. *See appendix B.*

Using the previous density we compute average money holdings.

Proposition 18 *The average (expected) real money holdings are:*

$$M = \int_0^{\underline{m}} m h(m)\, dm + \int_{\underline{m}}^{m^*} m h(m)\, dm$$

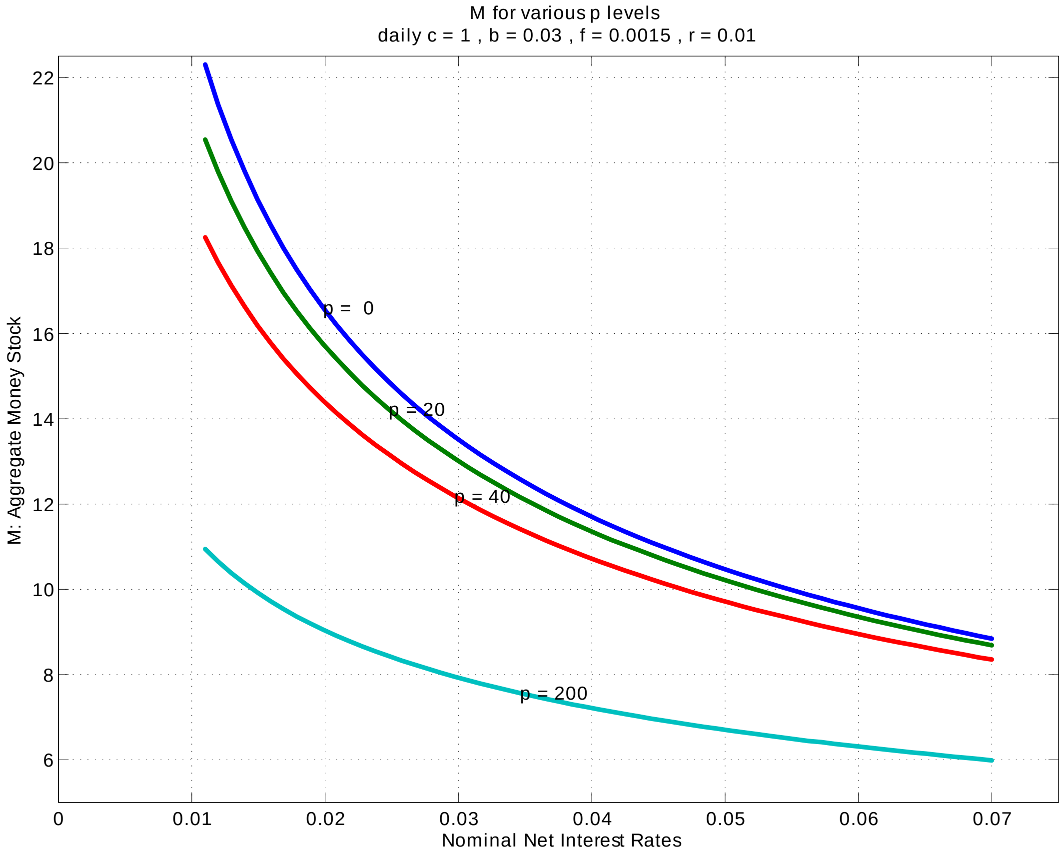

Figure 10: Money demand (when $f > 0$)

or

$$M = m^* - \frac{c}{p}A_0\left[\frac{\left(1+\frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}+1}-1}{(p+\pi)/c} - \underline{m}\right] \tag{52}$$
$$-A_1\left(\frac{c}{\pi}\right)^2\left\{\left(1+\frac{\pi}{c}m^*\right)\left[\log\left(1+\frac{\pi}{c}m^*\right)-1\right]-\left(1+\frac{\pi}{c}\underline{m}\right)\left[\log\left(1+\frac{\pi}{c}\underline{m}\right)-1\right]\right\}$$
$$+\left(m^*-\underline{m}\right)\left(\frac{c}{\pi}A_1\log\left(1+\frac{\pi}{c}m^*\right)-1\right)$$

where A_0 and A_1 are given in (48) and (51).
Proof. See appendix B.

The next figure plots the level and elasticity of money demand for the same parameter values.

While, as shown in the previous figures, the introduction of $f > 0$ has a large

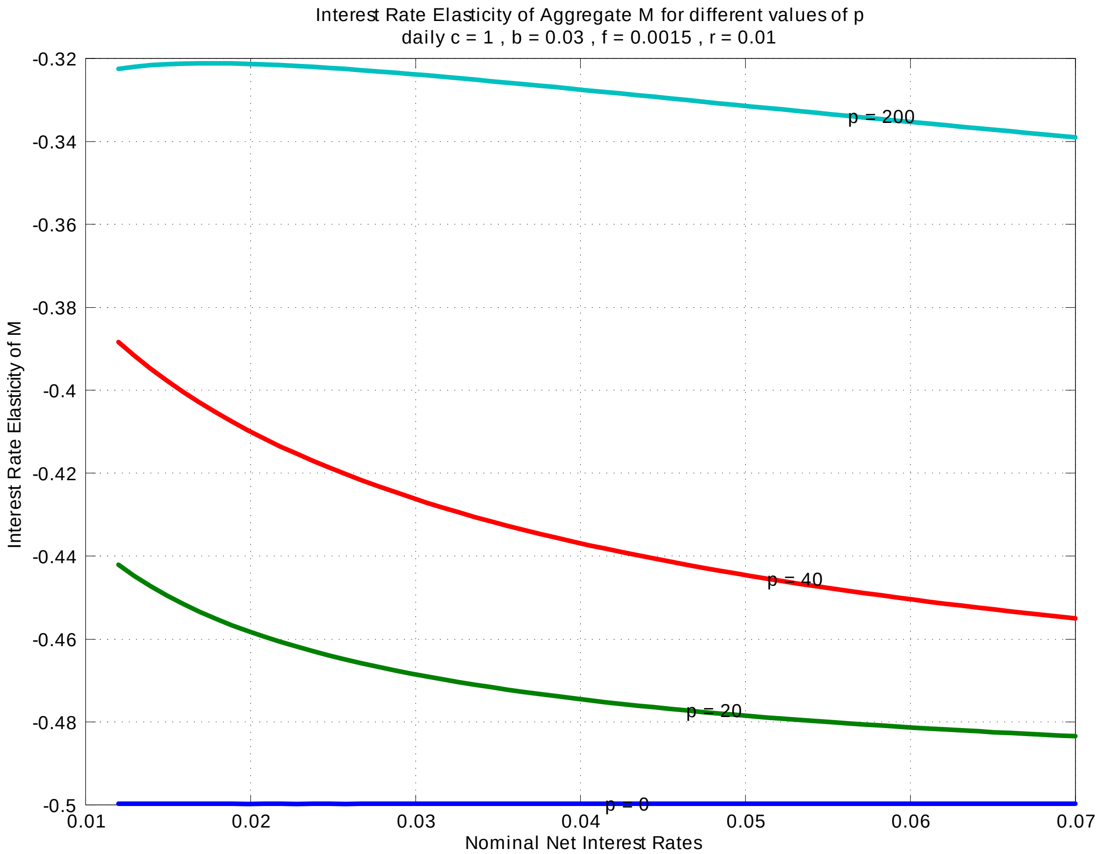

Figure 11: Interest elasticity of money demand (for $f > 0$)

effect on the average number of withdrawals, it has a much smaller effect on the level and on the interest rate elasticity of money demand. This is quite natural, since the effect of the fixed cost f on the number of withdrawals comes from eliminating the ones that are small in size.

As done in Section 5, we use the density to compute the average withdrawal:

Proposition 19 *The average withdrawal* W *is given by:*

$$W = m^* \left[1 - \frac{p}{n} H\left(\underline{m}\right)\right] + \left[\frac{p}{n} H\left(\underline{m}\right)\right] \frac{\int_0^{\underline{m}} \left(m^* - m\right) h\left(m\right) dm}{H\left(\underline{m}\right)} \tag{53}$$

where

$$\frac{\int_0^{\underline{m}} \left(m^* - m\right) h\left(m\right) dm}{H\left(\underline{m}\right)} = m^* - \underline{m} + \frac{\frac{\left(1+\frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}+1} - 1}{(p+\pi)/c} - \underline{m}}{\left(1 + \frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}} - 1}$$

To understand this expressions notice that $n - pH\left(\underline{m}\right)$ is the number of withdrawals in a unit of time that occur because agents reach zero balances, so if we divided it by the total number of withdrawals per unit of time, n, we obtain

$$\left[\frac{n - pH\left(\underline{m}\right)}{n}\right] = 1 - \frac{p}{n} H\left(\underline{m}\right)$$

i.e. the fraction of withdrawals that occur when agent reach zero balances. Each of these withdrawals is of size m^*. The complementary fraction gives the withdrawals that occur due to a chance meeting with the intermediary. Conditional on having money balances in $\left(0, \underline{m}\right)$ then a withdrawal of size $\left(m^* - m\right)$ happens with frequency $h\left(m\right) / H\left(\underline{m}\right)$.

The average amount of money that an agent has at the time of withdrawal, $\underline{M}$, is

$$\underline{M} = 0 \left[1 - \frac{p}{n} H\left(\underline{m}\right)\right] + \left[\frac{p}{n} H\left(\underline{m}\right)\right] \frac{\int_0^{\underline{m}} m \ h\left(m\right) dm}{H\left(\underline{m}\right)}$$

Simple algebra shows that:

$$\underline{M} = m^* - W \tag{54}$$

8 A calibration of the model

This section presents a calibration of the theoretical model derived above to the household data set described in Section 2. We analyze what model parametrization, i.e. values of $\left(b, f, p\right)$, produce values for $\left(M/c, W/c, n\right)$ that are closer to the analogous quantities in the data discussed in Section 2, and relate them to the empirical measures of financial innovations presented in Table 2.

More in detail, by calibration we mean the following. For a given observation (say the average of all household of a given type during a year in a province) we have values for $\left(\hat{M}, \hat{W}, \hat{n}, \hat{\pi}, \hat{R}, \hat{c}\right)$. Our objective is to find the parameters, b, f and p so that the model reproduces the values of $\left(\hat{M}/\hat{c},\ \hat{W}/\hat{c},\ \hat{n}\right)$, the average cash balances, the average withdrawal, and the average number of withdrawals per year. Our procedure consists on two steps. The first step finds a solution to a system of 3 equations in 3 unknowns. The equations are given by the model expressions for n, (44), M (52) and W (53) which are set equal to the corresponding observations in the data. The three unknowns are $(m^*, \underline{m}, p)$. The second step consists on using the values of $(m^*, \underline{m}, p)$ together with the data for $\left(\hat{R}, \hat{\pi}\right)$ and an assumed value for r, to solve for the parameters b and f, so that $m^*, \underline{m}$ are the optimal solution given $\left(\hat{R}, \hat{\pi}, r, p\right)$.

One issue is whether this procedure can always obtain estimates to fit an arbitrary array in the data. In other words, are there always parameters (b, f, p) that can rationalize such a choice of $\left(\hat{M}/\hat{c}, \hat{W}/\hat{c}, \hat{n}\right)$? The answer to this question is, in general, no. In Appendix D we show that given $(m^*,\ \underline{m},\ r,\ \pi,\ R,\ p)$ one can always solve for the required b and f. The problem is in the first step. To better understand this consider an extreme case of trying to fit the model to an observation with $n = 1$ withdrawal per year, $c = 365$ (this households withdraws in average once per year, but has a daily consumption of one dollar). Furthermore, assume that this households says that her average withdrawal is one dollar, i.e. $W = 1$. Clearly, such low values of W and n make impossible to finance the annual consumption flow.

To explore this issue in more detail, let us consider first the simpler version of the model with $f = 0$. In this case we imagine that in the first step we calibrate to M and n only, as opposed to (M, W, n). To further simplify the exposition, assume that inflation is zero, so that $\pi = 0$. For the BT model, i.e. for $p = 0$, we have

$$W = m^*,\ c\ =\ m^*\ n,\ \text{and}\ M = m^*\ \ (1/2)$$

which implies

$$M/c = (1/2)\ \ /\ \ n\ .$$

Hence, if the data were generated by the BT model, M/c and n have to satisfy this equation. Now consider the average cash balances generated by a policy like the one of the model of Section 4 with zero inflation, i.e. with $f = \pi = 0$, for an arbitrary value of p. We have:

$$M/c = \frac{1}{p}\left[n\ m^*/c - 1\right]\ \text{and}\ n = \frac{p}{1 - \exp\left(-pm^*/c\right)} \tag{55}$$

or, solving for M/c as a function of n :

$$M/c = \xi(n,p) \equiv \frac{1}{p}\left[-\frac{n}{p}\log\left(1-\frac{p}{n}\right)-1\right] .$$

We stress that while for a given (R, b, p, r) there is a unique optimal value for m^* and n (and hence M), here we are considering all the combinations of M/c and n that are consistent with a cash management policy of replenishing cash either when the balances reach zero, or when there is chance meeting with an intermediary (which occurs with probability p per unit of time), and suffices to finance a consumption flow c. Notice first that setting $p = 0$ in this equation we obtain BT, i.e.

$$\xi(n,0) = (1/2) \ / \ n$$

Second, notice that this function is defined only for $n \geq p$. Furthermore, note that for $p > 0$ (see Appendix E for details):

$$\begin{aligned}
\frac{\partial \xi}{\partial n} &= \left(\frac{1}{p}\right)^2\left[\log\left(\frac{n}{n-p}\right)-\frac{p}{n-p}\right] \leq 0 \\
\frac{\partial^2 \xi}{\partial n^2} &= \left(\frac{1}{p}\right)^2 \frac{p}{(n-p)^2}\frac{p}{n} > 0 \\
\frac{\partial \xi}{\partial p} &= \frac{1}{p^2}\frac{n}{p}\left[2\log\left(1-\frac{p}{n}\right)+1+\frac{p/n}{1-p/n}\right] > 0
\end{aligned}$$

Think about plotting the data on the $(n,\ M/c)$ plane. For a given M/c, there is a minimum n that the model can generate, namely the value $(1/2)/(M/c)$. Given that $\partial\xi/\partial p > 0$, any value of n smaller than the one implied by the BT model cannot be made consistent with our model, regardless of the values for the rest of the parameters. By the same reason, any value of n higher than $(1/2)/(M/c)$ can be accommodated by an appropriate choice of p. This is quite intuitive: relative to the BT model, our model can generate a larger number of withdrawals for the same M/c if the agent meets an intermediary often enough, i.e. if p is large enough. On the other hand there is a minimum number of expected chance meetings, namely $p = 0$.

Specifically, fix a province-year-type of household combination, with its corresponding values for the averages of M/c and n. Then solving $M/c = \xi(n, p)$ for p gives an estimate of p. Then, taking this value of p and the corresponding values of R and c for this province-year-type of household combination, we use (21) to find the corresponding m^*/c as follows:

$$m^*/c = \frac{p\ M/c + 1}{n}.$$

Finally, we use (83) to find the value of b that rationalizes this choice. In particular, we specialize the expression in Appendix D to the case of $\pi = f = 0$ to obtain:

$$b/c = \left(\frac{R}{(r+p)^2}\right)(\exp((r+p)\, m^*/c) - [1 + (r+p)\,(m^*/c)]\ ) \qquad (56)$$

(see the appendix E for details). To understand this expression, consider two pairs $(M/c, n)$, both pairs in the locus defined by $\xi(\cdot, p)$ for a given value of p. The pair with higher M/c and lower n corresponds to a higher value of b/R. This is quite intuitive: agents will economize on trips to the financial intermediary if b/R is high, i.e. if these trips are expensive relative to the opportunity cost of cash.

Figure 11 plots the function $\xi(\cdot, p)$ for several values of p, as well as the average value of M/c and n for all households of a given type (i.e. with and without ATM cards) for each province year in our data. Notice that 31 percent of province year pairs for households without an ATM card are below the $\xi(\cdot, 0)$ line, so no parameters in our model can rationalize those choices. The corresponding value for those with an ATM card is only 1 percent of the pairs. The values of p required to rationalize the average choice for most province year pairs for those households without ATM cards are in the range $p = 1$ to $p = 20$. The corresponding range for those with ATM cards is between $p = 20$ and $p = 80$.

So far we have consider the model with $f = 0$, or equivalently we set $\underline{m} = m^*$. It can be shown that freeing up this margin does not increase the set of values that the model can fit, since it makes the model closer to BT. Considering the case of $\pi > 0$ makes the expressions more complex, but, at least qualitatively, does not change any of the properties discussed above. Quantitatively, since the inflation rate in our data set is quite low, the expressions of the model for $\pi = 0$, approximates the relevant range for $\pi > 0$ very well.

Now we turn to the analysis of the ratio of the average withdrawal to the average cash balances, W/M. As before, consider first the case of an agent that follows an arbitrary policy of replenishing her cash to a return level m^*, either as her cash balances gets to zero, or at the time of chance meeting with the intermediary. Again, to simplify consider the case of $f = \pi = 0$. Using the expression for for W/M (23), and replacing m^* from (55) we can define the function ζ as follows

$$\frac{W}{M} = \zeta(n,p) \equiv \left[\frac{1}{p/n} + \frac{1}{\log(1 - p/n)}\right]^{-1} - \frac{p}{n}$$

for $n \geq p$, and $p \geq 0$ (see appendix E for details). After some algebra one can show that

$$\begin{aligned} \zeta(n,0) &= 2,\ \zeta(n,n) = 0, \\ \frac{\partial \zeta(n;p)}{\partial p} &< 0 \text{ and } \frac{\partial \zeta(n;p)}{\partial n} > 0 \end{aligned}$$

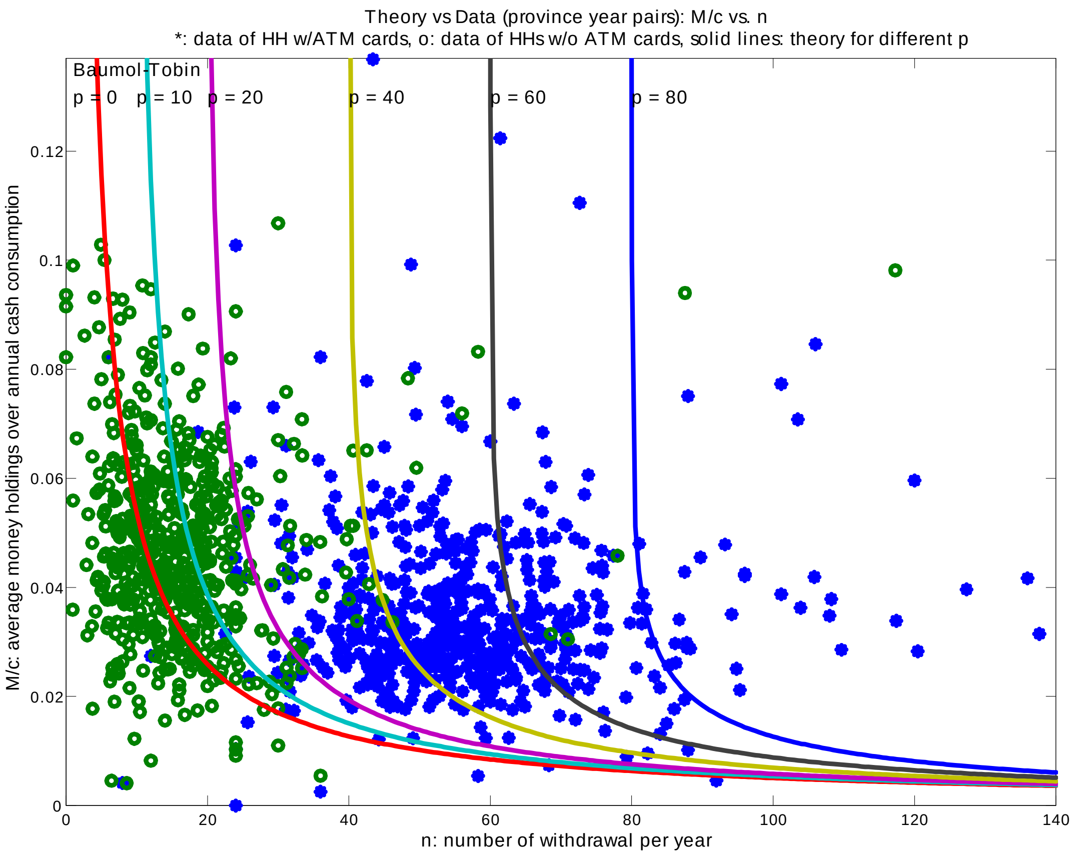

Figure 12: Figure. M/c versus n

(see appendix for details). Notice that the ratio W/M is a function of the ratio p/n.The interpretation of this is clear: for $p = 0$ we have $W/M = 2$, as in the BT model. This is the highest value that can can be achieved of the ratio W/M. As p increases for a fixed n, the replenishing level of cash m^*/c must be smaller, and hence the average withdrawal becomes smaller relative the average cash holdings M/c. Indeed, as n converges to p – a case where almost all the withdrawals are due to chance meetings with the intermediary–, then W/M goes zero. As in the previous case, given values of W/M and n, we can use ζ to solve for the corresponding p. Then, using the values of $(W, M\ p, n)$ we can find a value of b to rationalize the choice of W/M. To see how, notice that given W/M, M, n and p,we can find the value of m^*/c using

$$\frac{W}{M} = \frac{m^*}{M} - \frac{p}{n}.$$

With the values of $(m^*/c,\ p,\ R)$ we can find the unique value of b that rationalize this choice, using (56).

Figure 12 plots the function $\zeta\,(n, p)$ for several values of p, as well as the average values of n and W/M for the different province-year-household type type combinations for our data set. The implied values of p needed to rationalize these data is similar to the one found using the information of M/c and n displayed in Figure 11. We note that this case, as opposed to the experiment displayed in Figure 11, no data on c, the average consumption flow, is used. We also note that about 2 percent of the year province pairs of households with an ATM cards have W/M above 2, while for those without ATM card the corresponding value is 11 percent.

Now we turn to describe the procedure that we use to calibrate the model. We aggregate the data of $\left(\hat{M}/c, \hat{W}/c, \hat{n}\right)$ into the average for combination of province-years, both for households with an ATM card and for those without one, We use nominal interest rate $\hat{R}$ measured at the province-year level as reported in Table 2, and inflation rates π common for each year for all provinces. This gives us about $103 * 6 = 618$ observations to be fitted for each type of household (a bit fewer since there are some missing values for some province years). As explained above, even aggregating at this level,there are many observations for which the product of $\hat{n}$ and $\hat{M}/\hat{c}$ are too low (below 1/2) or for which the ratio of $\hat{W}/\hat{M}$ is too high (above 2), for which there are no parameters for which we can fit the model exactly. Additionally, in some cases both conditions can be met, but still may not be a triplet (b, f, p) which can simultaneously solve for M/c, W/M and n. As an example of this last case, consider an example where in the data $\left(\hat{M}/\hat{c}\right)\ \hat{n}\ >\ (1/2)$, but $\hat{W}/\hat{M} = 2$. Because of these features, instead of solving exactly for (b, f, p) to match $(M/c,\ W/c, n)$ for each observation, we minimize the distance between the model

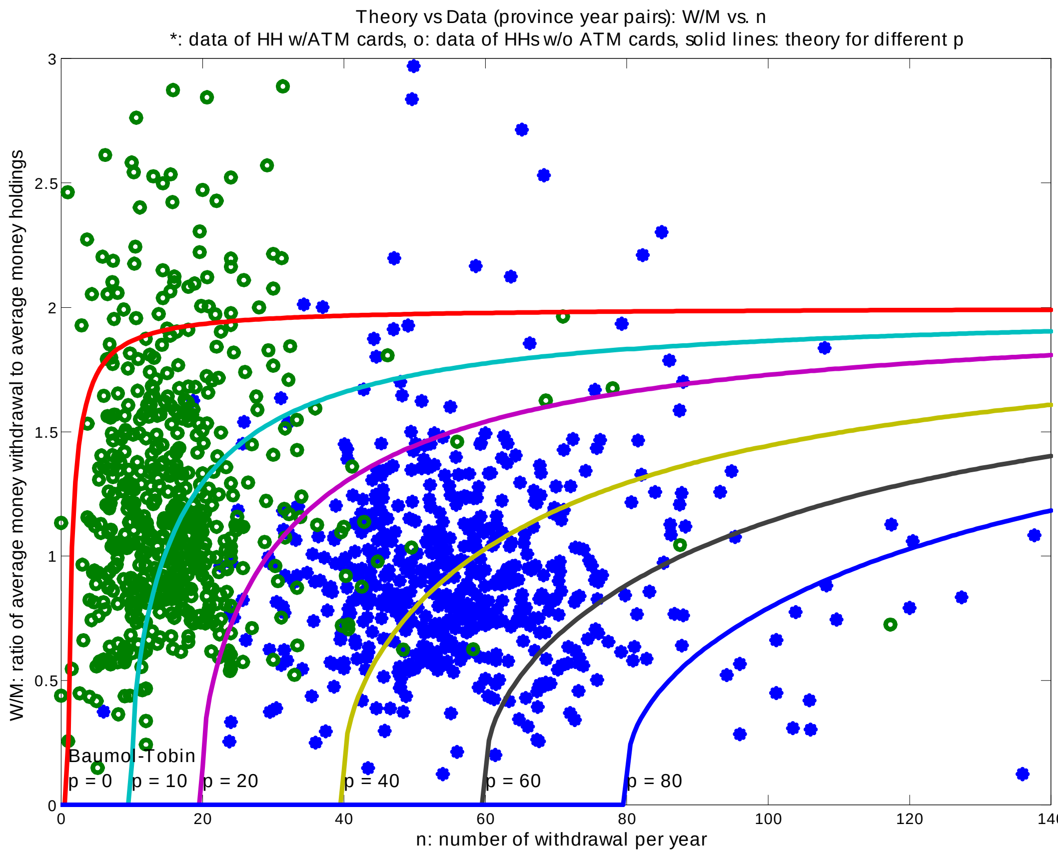

Figure 13: Figure. M/W versus n .

and the data:

$$(b, f, p) = \arg\min \left[\left(\frac{M}{c} - \frac{\widehat{M}}{c} \right)^2 + w_1 \cdot \left(\frac{W}{c} - \frac{\widehat{W}}{c} \right)^2 + w_2 \cdot (n - \widehat{n})^2 \right] \quad (57)$$

for some weights w_1, w_2. In future version we plan to formally model measurement error in our variables as a better way to handle these issues. [10] Table 7 displays statistics for the fitted parameters $(b,\ f,\ p)$ for the model with costly random withdrawals; Table 8 displays the corresponding statistics for (p, b) using the model where the random withdrawals are costless (i.e. setting $f = 0.$). In each table we display the statistic for the parameters fitted separately for those households with and without ATM cards. We use weights $w_1 = w_2 = 1$, and discard those observations with very high values of $\hat{M}/\hat{c}$ for the statistics, which we think are due to measurement error[11]. The statistics in Tables 7 and 8 confirm the information displayed in Figures 11 and 12: there are clear differences between households types, those with ATM cards have much higher average fitted values of p. The values of b and f are similar for both type of households, only slightly higher for those without ATM cards. The large standard deviations in Tables 7 and 8 imply a huge amount of heterogeneity in the implied values of the parameters across province-years, a fact consist with the large dispersion of the data displayed in Figures 11 and 12. Finally, notice that the fitted values of f are very small, which suggest that the simplified model of Section 4 captures most of the relevant variation of the data.

[10] Besides classical measurement error, which is probably important in this type of survey, there is also the issue of whether households have an alternatively source of cash. An example of such as source if if households were paid in cash. This will imply that they do require fewer withdrawals to finance the same flow of consumption, or alternatively, that they effectively have more trips per periods.

[11] Depending of the case these are, at most 7 observations, for which average money holdings are higher than a month of cash consumption.

Table 7: Calibrated parameters for p, b and f

	Household w. ATM		
	p	b	f
Mean	57	0.05	0.0004
Median	45	0.02	0.0000
STD	47	0.11	0.0011
	Household w/o ATM		
Mean	17	0.06	0.006
Median	9	0.03	0.001
STD	21	0.10	0.024

Notes: Descriptive statistics drawn from the distribution of calibrated parameters at the year*province level (the sample size is 542 for HH w. ATM and 578 for HH without ATM).

Table 8: Calibrated parameters for p and b when $f = 0$

	Household w. ATM	
	p	b
Mean	37	0.05
Median	36	0.02
STD	16	0.16
	Household w/o ATM	
Mean	8.1	0.05
Median	7.6	0.03
STD	6.9	0.06

Notes: Descriptive statistics drawn from the distribution of calibrated parameters at the year*province level (the sample size is 587 for HH w. ATM and 590 for HH without ATM).

Table 9: Correlation between (p, b, f) and measures of financial development

	Household w. ATM		
	p	b	f
bank-branch diffusion	0.10	-0.12	-0.11
ATM diffusion	0.05	-0.18	-0.17
	Household w/o ATM		
bank-branch diffusion	0.03	-0.13	-0.11
ATM diffusion	0.03	-0.24	-0.16

Notes: The table reports the correlation between the calibrated parameters and diffusion measures of, respecitvely, bank branches and ATMs, at the year*province level (the sample size if 542 for HH w. ATM and 578 for HH without ATM).

Table 10: Correlation between (p, b) and measures of financial development

	Household w. ATM	
	p	b
bank-branch diffusion	0.13	-0.13
ATM diffusion	0.07	-0.19
	Household w/o ATM	
bank-branch diffusion	0.02	-0.3
ATM diffusion	0.06	-0.4

Notes: The table reports the correlation between the calibrated parameters and diffusion measures of, respecitvely, bank branches and ATMs, at the year*province level (the sample size is 587 for HH w. ATM and 590 for HH without ATM).

Tables 9 and 10 compute correlations between the fitted values of $(b,\ f,\ p)$ for each year-province-household type with each of the two empirical measures of financial innovations presented in Table 2. The correlations between p and the density of bank branches or ATM are positive, while correlation between b and f and these measures of financial innovation are negative. On one hand, we view the signs of these correlations as an encouraging feature of our model, since no data on the densities of bank branches or ATMs was used in the calibration exercise. On the other hand, the correlation between the financial diffusion measures and the fitted values of p while positive, is very low for those households without ATM cards.

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A Appendix: Discrete Time Bellman Equation

Let Δ be the length of the time period, so that the probabilities of matching a financial intermediary, the net nominal interest rate, the discount rate, the inflation rate and consumption are given by Δp, ΔR, Δr, $\Delta\pi$ and Δc.

In this case the Bellman equation for an agent matched with the intermediary with $m \geq 0$ is

$$V_f(m) =$$

$$\min_w \left\{ \Delta R(m+w) + \frac{1}{1+r\Delta}\left[p\Delta V_f\left(\frac{m+w}{1+\Delta\pi} - \Delta c\right) + (1-p\Delta)V_u\left(\frac{m+w}{1+\Delta\pi} - \Delta c\right)\right]\right\}$$

subject to $m + w - \Delta c \geq 0$.

The following are the Bellman equations for an agent not matched with an intermediary. First we consider the case of $m \geq \Delta c$, so that the agent has the option to stay unmatched.

$$V_u(m) =$$

$$\min\left\{V_f(m) + b,\ \Delta R(m) + \frac{1}{1+r\Delta}\left[p\Delta V_f\left(\frac{m}{1+\Delta\pi} - \Delta c\right) + (1-p\Delta)V_u\left(\frac{m}{1+\Delta\pi} - \Delta c\right)\right]\right\}$$

If $m < \Delta c$, then the agent has no option but to pay the fixed cost, since otherwise it cannot finance her consumption. In this case we have:

$$V_u(m) = V_f(m) + b\ .$$

Proposition 20 *As $\Delta \to 0$, the discrete time Bellman equation becomes:*

$$\begin{aligned} V^* &= V_u(m^*) = \min_z V_u(z) \\ V_u(m) &= \min\left\{V^* + b,\ \frac{Rm + pV^* - V_u'(m)(c + m\pi)}{r+p}\right\} \end{aligned}$$

Proof of Proposition 20. Let assume that in a neighborhood of m it is not optimal to go to the bank so

$$\begin{aligned} V_u(m) &= \min\left\{V^* + b,\ \Delta R(m) + \frac{1}{1+r\Delta}\left[p\Delta V^* + (1-p\Delta)V_u\left(\frac{m}{1+\Delta\pi} - \Delta c\right)\right]\right\} \\ &= \Delta R(m) + \frac{1}{1+r\Delta}\left[p\Delta V^* + (1-p\Delta)V_u\left(\frac{m}{1+\Delta\pi} - \Delta c\right)\right] \end{aligned}$$

for an interval around m. Rearranging

$$(1+r\Delta)V_u(m) = \Delta R(m)(1+r\Delta) + p\Delta V^* + (1-p\Delta)V_u\left(\frac{m}{1+\Delta\pi} - \Delta c\right)$$

assuming that V_u is differentiable

$$(1+r\Delta)\,V_u(m) = \Delta R(m)(1+r\Delta) + p\Delta V^* + (1-p\Delta)\left[V_u(m) - V_u'(m)\left(m\frac{\Delta\pi}{1+\Delta\pi} + \Delta c\right) + o(\Delta)\right]$$

cancelling terms and rearranging and dividing by Δ

$$(r+p)\,V_u(m) = Rm(1+r\Delta) + pV^* + (1-p\Delta)\left[-V_u'(m)\left(c + m\frac{\pi}{1+\Delta\pi}\right) + \frac{o(\Delta)}{\Delta}\right]$$

taking the limit as $\Delta \to 0$

$$(r+p)\,V_u(m) = Rm + pV^* - V_u'(m)(c+m\pi)$$

or

$$V_u(m) = \frac{Rm + pV^* - V_u'(m)(c+\pi m)}{r+p}$$

Combining both branches we have:

$$V_u(m) = \min\left\{V^* + b, \frac{Rm + pV^* - V_u'(m)(c+\pi m)}{r+p}\right\}.$$

Taking the limit as $\Delta \to 0$ on

$$\begin{aligned} V^* &= \lim_{\Delta\to 0}\min_w \left\{\Delta R(m+w) + \frac{1}{1+r\Delta}\left[p\Delta V^* + (1-p\Delta)\,V_u\left(\frac{m+w}{1+\Delta\pi} - \Delta c\right)\right]\right\} \\ &= \min_w \{V_u(m+w)\} = \min_z V_u(z) \end{aligned}$$

gives the first equation.

QED.

B Appendix: Proofs

Proof of Remark 1

That M is homogenous of degree one on (c, b) follows from the homogeneity of degree zero of T, and hence the elasticities w.r.t c and b add up to one. That the elasticity w.r.t. b equals (minus) the elasticity w.r.t. R follows from the observation that M/c is a function of (b/Rc).

That M is increasing in b follows from differentiating the foc:

$$\frac{\partial M}{\partial b} = -\frac{\partial T}{\partial M}(M,c)\,/\,\frac{\partial^2 T}{\partial^2 M}(M,c)$$

and using that, by assumption, $\partial T/\partial M < 0$ and $\partial^2 T/\partial^2 M > 0$.

That the elasticity of M with respect to c is higher than $1/2$ follows from differentiating the foc and completing elasticities which gives:

$$\frac{c}{M}\frac{\partial M}{\partial c}=-\frac{\partial^2 T}{\partial M\partial c}\ \frac{c}{M}\ /\frac{\partial^2 T}{\partial^2 M}.$$

Using the homogeneity of degree zero of T we have:

$$\frac{\partial^2 T}{\partial M\partial M}M^2+2\frac{\partial^2 T}{\partial c\partial M}\ M\ c+\frac{\partial^2 T}{\partial c\partial c}c^2=0$$

and hence, using the assumption that $\partial^2 T/\partial c^2\geq 0$,

$$-\frac{\partial^2 T}{\partial c\partial M}\geq\frac{1}{2}\frac{\partial^2 T}{\partial M\partial M}\frac{M}{c}.$$

Using this inequality in the expression of the elasticity w.r.t. c we obtain the desired result. Given that the elasticity w.r.t. b is positive, and the the elasticities w.r.t. b and c add up to one, we know that the elasticity w.r.t. c is smaller than, or equal to, one.
QED.

Proof of Proposition 1. Fix an arbitrary sequence of contacts with the intermediary. For this sequence we show that:

$$\begin{aligned}&\sum_{j=0}^{\infty}e^{-\tau_j r}\left\{b\ \ I_{\tau_j}+\left(m\left(\tau_j^{+}\right)-m\left(\tau_j^{-}\right)\right)\right\}\\ =\ &\sum_{j=0}^{\infty}e^{-\tau_j r}\left\{b\ \ I_{\tau_j}+\int_0^{\tau_{j+1}-\tau_j}R\ m\left(\tau_j+t\right)\ e^{-rt}\ dt\right\}+\frac{c}{r}\end{aligned}$$

provided that $R=r+\pi$. For this it is sufficient to show that, provided that $R=r+\pi$,

$$\sum_{j=0}^{\infty}e^{-\tau_j r}\left(m\left(\tau_j^{+}\right)-m\left(\tau_j^{-}\right)\right)=\sum_{j=0}^{\infty}e^{-\tau_j r}\int_0^{\tau_{j+1}-\tau_j}R\ m\left(\tau_j+t\right)\ e^{-rt}\ dt+\frac{c}{r}.$$

To establish this equality write

$$\frac{c}{r}=\sum_{j=0}^{\infty}e^{-\tau_j r}\int_0^{\tau_{j+1}-\tau_j}c\ e^{-rt}\ dt$$

and using integration by parts we have

$$
\begin{aligned}
& R\int_0^{\tau_{j+1}-\tau_j} \ m\left(\tau_j+t\right) \ e^{-rt} \ dt \\
= & -\frac{R}{r}m\left(\tau_j+t\right) \ e^{-rt}\big|_0^{\tau_{j+1}-\tau_j} - \frac{R}{r}\int_0^{\tau_{j+1}-\tau_j} \left[m\left(t\right)\pi+c\right] \ e^{-rt}dt \\
= & -\frac{R}{r}m\left(\tau_{j+1}^-\right) \ e^{-r\left(\tau_{j+1}-\tau_j\right)} + \frac{R}{r}m\left(\tau_j^+\right) - \frac{R}{r}\int_0^{\tau_{j+1}-\tau_j} \left[m\left(t\right)\pi+c\right] \ e^{-rt}dt
\end{aligned}
$$

so that (using $R = r+\pi$)

$$
\begin{aligned}
& \sum_{j=0}^{\infty} e^{-\tau_j r}\int_0^{\tau_{j+1}-\tau_j} R \ m\left(\tau_j+t\right) \ e^{-rt} \ dt + \frac{c}{r} \\
= & \sum_{j=0}^{\infty} e^{-\tau_j r}\left\{\begin{array}{c} -\frac{R}{r}m\left(\tau_{j+1}^-\right) \ e^{-r\left(\tau_{j+1}-\tau_j\right)} + \frac{R}{r}m\left(\tau_j^+\right) - \frac{R}{r}\int_0^{\tau_{j+1}-\tau_j} c \ e^{-rt}dt \\ -\frac{R}{r}\int_0^{\tau_{j+1}-\tau_j} m\left(\tau_j+t\right)\pi \ e^{-rt}dt + \int_0^{\tau_{j+1}-\tau_j} c \ e^{-rt} \ dt \end{array}\right\} \\
= & \sum_{j=0}^{\infty} e^{-\tau_j r}\left\{\begin{array}{c} -\frac{R}{r}m\left(\tau_{j+1}^-\right) \ e^{-r\left(\tau_{j+1}-\tau_j\right)} + \frac{R}{r}m\left(\tau_j^+\right) - \left(\frac{r+\pi}{r}\right)\int_0^{\tau_{j+1}-\tau_j} c \ e^{-rt}dt \\ -\frac{R}{r}\int_0^{\tau_{j+1}-\tau_j} m\left(\tau_j+t\right)\pi \ e^{-rt}dt + \int_0^{\tau_{j+1}-\tau_j} c \ e^{-rt} \ dt \end{array}\right\} \\
= & \sum_{j=0}^{\infty} e^{-\tau_j r}\left\{\begin{array}{c} -\frac{R}{r}m\left(\tau_{j+1}^-\right) \ e^{-r\left(\tau_{j+1}-\tau_j\right)} + \frac{R}{r}m\left(\tau_j^+\right) \\ -\left(\frac{\pi}{r}\right)\int_0^{\tau_{j+1}-\tau_j}\left[c \ +m\left(\tau_j+t\right)\pi\right]e^{-rt}dt - \int_0^{\tau_{j+1}-\tau_j} m\left(\tau_j+t\right)\pi \ e^{-rt}dt \end{array}\right\}
\end{aligned}
$$

letting

$$
n\left(\tau_j+t\right) = e^{-rt}m\left(\tau_j+t\right)
$$

$$
\begin{aligned}
\frac{dn\left(\tau_j+t\right)}{dt} &= -re^{-rt}m\left(\tau_j+t\right) - e^{-rt}\left(c+\pi m\left(\tau_j+t\right)\right) \\
&= \left(-e^{-rt}\right)\left[rm\left(\tau_j+t\right)+c+\pi m\left(\tau_j+t\right)\right] \\
&= \left(-e^{-rt}\right)\left[Rm\left(\tau_j+t\right)+c\right]
\end{aligned}
$$

Compute the term

$$
\begin{aligned}
& -\left(\frac{\pi}{r}\right)\int_0^{\tau_{j+1}-\tau_j}\left[c \ +m\left(\tau_j+t\right)\pi\right]e^{-rt}dt - \int_0^{\tau_{j+1}-\tau_j} m\left(\tau_j+t\right)\pi \ e^{-rt}dt \\
= & -\left(\frac{\pi}{r}\right)\left[\int_0^{\tau_{j+1}-\tau_j}\left[c \ +m\left(\tau_j+t\right)\pi\right]e^{-rt}dt + \frac{r}{\pi}\int_0^{\tau_{j+1}-\tau_j} m\left(\tau_j+t\right)\pi \ e^{-rt}dt\right] \\
= & \left(\frac{\pi}{r}\right)\left[\int_0^{\tau_{j+1}-\tau_j}\left(-\right)\left[c \ +m\left(\tau_j+t\right)R\right]e^{-rt}dt\right] \\
= & \left(\frac{\pi}{r}\right)\left(n\left(\tau_{j+1}\right)-n\left(\tau_j\right)\right) = \left(\frac{\pi}{r}\right)\left(e^{-r\left(\tau_{j+1}-\tau_j\right)}m\left(\tau_{j+1}^-\right) - m\left(\tau_j^+\right)\right)
\end{aligned}
$$

Thus:

$$\begin{aligned}
& \sum_{j=0}^{\infty} e^{-\tau_j r} \int_0^{\tau_{j+1}-\tau_j} R\, m\left(\tau_j+t\right)\ e^{-rt}\ dt + \frac{c}{r} \\
= & \sum_{j=0}^{\infty} e^{-\tau_j r} \left\{ -\frac{R}{r} m\left(\tau_{j+1}^-\right)\ e^{-r(\tau_{j+1}-\tau_j)} + \frac{R}{r} m\left(\tau_j^+\right) + \left(\frac{\pi}{r}\right)\left(e^{-r(\tau_{j+1}-\tau_j)} m\left(\tau_{j+1}^-\right) - m\left(\tau_j^+\right)\right) \right\} \\
= & \sum_{j=0}^{\infty} e^{-\tau_j r} \left\{ -\frac{r+\pi}{r} m\left(\tau_{j+1}^-\right)\ e^{-r(\tau_{j+1}-\tau_j)} + \frac{r+\pi}{r} m\left(\tau_j^+\right) + \left(\frac{\pi}{r}\right)\left(e^{-r(\tau_{j+1}-\tau_j)} m\left(\tau_{j+1}^-\right) - m\left(\tau_j^+\right)\right. \right. \\
= & \sum_{j=0}^{\infty} e^{-\tau_j r} \left\{ -m\left(\tau_{j+1}^-\right)\ e^{-r(\tau_{j+1}-\tau_j)} + m\left(\tau_j^+\right) \right\} \\
= & \sum_{j=0}^{\infty} e^{-\tau_j r} m\left(\tau_j^+\right) - \sum_{j=0}^{\infty} e^{-\tau_{j+1} r} m\left(\tau_{j+1}^-\right)
\end{aligned}$$

and using that $m\left(\tau_0^-\right) = 0$, then

$$= \sum_{j=0}^{\infty} e^{-\tau_j r} m\left(\tau_j^+\right) - \sum_{j=0}^{\infty} e^{-\tau_j r} m\left(\tau_j^-\right)$$

Thus

$$\begin{aligned}
& \sum_{j=0}^{\infty} e^{-\tau_j r} \int_0^{\tau_{j+1}-\tau_j} R\, m\left(\tau_j+t\right)\ e^{-rt}\ dt + \frac{c}{r} \\
= & \sum_{j=0}^{\infty} e^{-\tau_j r} m\left(\tau_j^+\right) - \sum_{j=0}^{\infty} e^{-\tau_j r} m\left(\tau_j^-\right) \\
= & \sum_{j=0}^{\infty} e^{-\tau_j r} \left[m\left(\tau_j^+\right) - m\left(\tau_j^-\right) \right]
\end{aligned}$$

QED.

Proof of Proposition 2.

To solve for V^*, m^*, m^{**} and $V_u(\cdot)$ satisfying (14) and (15) we proceed as follows. Lemmas 1 solves for $V_u(A, V^*)$, Lemma 2 gives $A(V^*)$. Lemma 3 shows that $V_u(\cdot)$ is convex for any $V^* > 0$. Lemma 4 solves for m^*, using that since V_u is convex, m^* must satisfy $V_u'(m^*) = 0$. Finally, Lemma 5 gives V^* $(m^*.)$

Lemmas 2, 4 and 5 provide us with the following system of 3 equations in 3 unknown constants (A, m^*, V^*) :

$$\begin{aligned} A &= \frac{V^*(r+p)r + Rc/\left(1+\frac{\pi}{r+p}\right) + (r+p)^2 b}{c^2} & (58) \\ V^* &= \frac{R}{r} m^* & (59) \\ m^* &= \frac{c}{\pi}\left(\left[\frac{R}{Ac}/\left(1+\frac{\pi}{r+p}\right)\right]^{-\frac{\pi}{r+p+\pi}} - 1\right) & (60) \end{aligned}$$

As we show next, this system determines m^* as the solution of one non-linear equation.

Replacing equation (59) into (58) yields:

$$A = \frac{R(r+p)m^* + \frac{Rc(r+p)}{(r+p+\pi)} + (r+p)^2 b}{c^2}$$

We can rearrange equation (60) to get

$$A = \frac{R(r+p)}{(r+p+\pi)}\left(m^*\frac{\pi}{c}+1\right)^{1+\frac{r+p}{\pi}}$$

By equating the last two equations, collecting terms and rearranging, we get:

$$\left(m^*\frac{\pi}{c}+1\right)^{1+\frac{r+p}{\pi}} = 1 + (r+p+\pi)\frac{m^*}{c} + (r+p)(r+p+\pi)\frac{b}{Rc}$$

which is equation (16) in the main text.

We now show that this equation has a unique non negative solution. The equation can be written as

$$f(m^*) = g(m^*)$$

where

$$\begin{aligned} f(m^*) &\equiv (m^*\pi + c)^{1+\theta} \\ g(m^*) &\equiv c^\theta\left\{ m^*(\pi+r+p) + c + \frac{(r+p)\ b\ (r+p+\pi)}{R}\right\}. \end{aligned}$$

Notice that

$$f(0) = (c)^{1+\theta} < g(0) = c^{1+\theta} + c^\theta(r+p)\ b\ (r+p+\pi)/R,$$

that g is linear in m^* with slope $c^\theta (\pi + r + p)$. Since, by assumption $r+p+\pi > 0$

$$\begin{aligned} f'(m^*) &= (\pi + r + p)\ (m^*\pi + c)^\theta > 0 \\ f''(m^*) &= (\pi + r + p)(r + p)(m^*\pi + c)^{\theta - 1} > 0 \end{aligned}$$

so f is strictly increasing and strictly convex in m^*.

$$f'(0) = \pi\ (1 + \theta)(c)^\theta = (\pi + r + p)(c)^\theta = g'(0) = (c)^\theta (\pi + r + p)$$

there is a unique solution to $f(m^*) = g(m^*)$. If $\pi < 0$, the relevant range for m^* is $(0, -c/\pi)$, since $\lim_{m^* \to -c/\pi} f(m^*) = \infty$, and hence the unique solution of $f(m^*) = g(m^*)$ occurs in the interior of this range.

For $\pi = 0$, consider solving the limit as $\pi \to 0$ of $f(m^*)/c^\theta - g(m^*)/c^\theta = 0$, or

$$\begin{aligned} f(m^*)/c^\theta &\equiv (m^*\pi + c)^{1 + \frac{r+p}{\pi}} / c^{\frac{r+p}{\pi}} = (m^*\pi + c) \left\{ \left(1 + \frac{m^*}{c}\pi\right)^{\frac{1}{\pi}} \right\}^{r+p} \\ g(m^*)/c^\theta &\equiv m^*(\pi + r + p) + c + \frac{(r+p)\ b\ (r + p + \pi)}{R}. \end{aligned}$$

Taking the limit we have

$$\begin{aligned} \lim_{\pi \to 0} \left(f(m^*)/c^\theta \right) &= c \left\{ \lim_{\pi \to 0} \left(1 + \frac{m^*}{c}\pi\right)^{\frac{1}{\pi}} \right\}^{r+p} \\ &= c \left\{ \exp\left(\frac{m^*}{c}\right) \right\}^{r+p} = c \exp\left(m^* \left(\frac{r+p}{c} \right) \right) \end{aligned}$$

so the equation becomes:

$$\exp\left(m^* \left(\frac{r+p}{c} \right) \right) = 1 + m^* \left(\frac{r+p}{c} \right) + \frac{(r+p)^2 b}{cR}.$$

QED.

Lemma 1 *Let V^* be an arbitrary value. The differential equation*

$$rV_u(m) = Rm + p[V^* - V_u(m)] - V_u'(m)(c + \pi m) \tag{61}$$

has solution for $\pi \neq 0$:

$$V_u(m) = \left[\frac{pV^*(r+p) - Rc/\left(1+\frac{\pi}{r+p}\right)}{(r+p)^2} \right] + \left[\frac{R}{r+p} / \left(1 + \frac{\pi}{r+p}\right) \right] m \quad (62)$$
$$+ \left(\frac{c}{r+p} \right)^2 A \ \left[1 + \frac{\pi}{c} m \right]^{-\frac{r+p}{\pi}}$$

and for $\pi = 0$:

$$V_u(m) = \left[\frac{pV^*(r+p) - Rc}{(r+p)^2} \right] + \left[\frac{R}{r+p} \right] m + \left(\frac{c}{r+p} \right)^2 A \exp\left(-\frac{r+p}{c} m \right) \ . \quad (63)$$

for an arbitrary constant A.

Proof of Lemma 1. The ODF (61) has the form:

$$f(x) = a_0 + a_1 x + (a_2 + a_3 x) f'(x)$$

$$\begin{aligned} a_0 &= \frac{pV^*}{r+p} \\ a_1 &= \frac{R}{r+p} \\ a_2 &= -\frac{c}{r+p} \\ a_3 &= -\frac{\pi}{r+p} \end{aligned}$$

which has the solution

$$f(x) = A_0 + A_1 x + A \ \left[1 + \frac{A_2}{A_3} x \right]^{A_3}$$

To see that this is the solution, notice that

$$f'(x) = A_1 + A \ A_3 \ \left(\frac{A_2}{A_3} \right) \left[\ 1 + \frac{A_2}{A_3} x \right]^{(A_3 - 1)}$$

and thus which requires:

$$
\begin{aligned}
& A_0 + A_1 x + A \ \left[1 + \left(\frac{A_2}{A_3}\right) x\right]^{A_3} \\
= \ & a_0 + a_1 x + (a_2 + a_3 x) \left(A_1 + A \ A_3 \ \left(\frac{A_2}{A_3}\right) \left[\ 1 + \frac{A_2}{A_3} x\right]^{(A_3 - 1)}\right)
\end{aligned}
$$

which gives

$$
\begin{aligned}
A_0 &= a_0 + a_2 A_1 \\
A_1 x &= (a_1 + a_3 A_1)\, x
\end{aligned}
$$

or

$$
\begin{aligned}
A_1 &= a_1 / \left(1 - a_3\right) \\
A_0 &= a_0 + a_2 a_1 / \left(1 - a_3\right)
\end{aligned}
$$

$$
A \ \left[1 + \left(\frac{A_2}{A_3}\right) x\right]^{A_3} = (a_2 + a_3 x) \ \ A \ A_3 \left(\frac{A_2}{A_3}\right) \left[1 + \left(\frac{A_2}{A_3}\right) x\right]^{(A_3 - 1)}
$$

or

$$
\left[1 + \left(\frac{A_2}{A_3}\right) x\right]^{A_3} = (a_2 + a_3 x) \ \ A_3 \left(\frac{A_2}{A_3}\right) \left[1 + \left(\frac{A_2}{A_3}\right) x\right]^{(A_3 - 1)}
$$

or

$$
\left[1 + \left(\frac{A_2}{A_3}\right) x\right]^{A_3} = (a_2 + a_3 x) \ \ A_3 \left(\frac{A_2}{A_3}\right) \left[1 + \left(\frac{A_2}{A_3}\right) x\right]^{(A_3 - 1)} \frac{\left[1 + \left(\frac{A_2}{A_3}\right) x\right]}{\left[1 + \left(\frac{A_2}{A_3}\right) x\right]}
$$

$$
\left[1 + \left(\frac{A_2}{A_3}\right) x\right] = (a_2 + a_3 x) \ \ A_3 \left(\frac{A_2}{A_3}\right)
$$

or

$$
1 = a_2 \ A_2 \ \ \Longrightarrow \ A_2 \ = 1/a_2
$$

$$
\left(\frac{A_2}{A_3}\right) x = a_3 \ \ A_3 \left(\frac{A_2}{A_3}\right) \ x \Longrightarrow \ A_3 \ = 1/a_3
$$

$$
\begin{aligned}
A_2 &= 1/a_2 \\
A_3 &= 1/a_3
\end{aligned}
$$

Thus

$$
\begin{aligned}
V(m) &= A_0 + A_1 m + A \left[1 + \frac{A_2}{A_3} m\right]^{A_3} \\
&= a_0 + a_2 a_1 / (1 - a_3) \\
&\quad + a_1 / (1 - a_3)\ m \\
&\quad + A \left[1 + \frac{a_3}{a_2} m\right]^{1/a_3} \\
&= \frac{pV^*}{r+p} - \left[\frac{c}{r+p}\frac{R}{r+p} / \left(1 + \frac{\pi}{r+p}\right)\right] + \\
&\quad + \left[\frac{R}{r+p} / \left(1 + \frac{\pi}{r+p}\right)\right] m \\
&\quad + A \left[1 + \frac{\pi}{c} m\right]^{-\frac{r+p}{\pi}}
\end{aligned}
$$

As $\pi \to 0$

$$
\begin{aligned}
\lim_{\pi \to 0} \log \left[1 + \frac{\pi}{c} m\right]^{-\frac{r+p}{\pi}} &= -\lim_{\pi \to 0} \frac{r+p}{\pi} \log \left[1 + \frac{\pi}{c} m\right] \\
&= -\lim_{\pi \to 0} \frac{\log \left[1 + \frac{\pi}{c} m\right]}{\pi / (r+p)} = -\frac{\frac{m}{c}}{1/(r+p)}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\lim_{\pi \to 0} \left[1 + \frac{\pi}{c} m\right]^{-\frac{r+p}{\pi}} &= \lim_{\pi \to 0} \exp \left(\log \left[1 + \frac{\pi}{c} m\right]^{-\frac{r+p}{\pi}}\right) \\
&= \exp \left(\lim_{\pi \to 0} \log \left[1 + \frac{\pi}{c} m\right]^{-\frac{r+p}{\pi}}\right) \\
&= \exp \left(-\frac{r+p}{c} m\right)
\end{aligned}
$$

QED

Lemma 2 *Let V^* be an arbitrary non negative value. Let A be the constant that solves the ODE in Lemma 1. Imposing that this solution satisfies $V_u(0) = V^* + b$, the constant A is given by*

$$
A = \frac{V^* (r+p) r + Rc / \left(1 + \frac{\pi}{r+p}\right) + (r+p)^2 b}{c^2} > 0
$$

Proof of lemma 2. Using lemma 1:

$$V^* + b = V_u(0) = \frac{pV^*(r+p) - Rc/\left(1+\frac{\pi}{r+p}\right)}{(r+p)^2} + \left(\frac{c}{r+p}\right)^2 A$$

or

$$-\left(\frac{c}{r+p}\right)^2 A = \frac{pV^*(r+p) - Rc/\left(1+\frac{\pi}{r+p}\right)}{(r+p)^2} - (V^* + b)$$

or

$$\begin{aligned} A &= -\frac{pV^*(r+p) - Rc/\left(1+\frac{\pi}{r+p}\right) - (r+p)^2(V^*+b)}{c^2} \\ &= \frac{V^*(r+p)r + Rc/\left(1+\frac{\pi}{r+p}\right) + (r+p)^2 b}{c^2} > 0. \end{aligned}$$

QED.

Lemma 3 *Let V^* be an arbitrary value. The solution of V_u given in lemma 1, with the value of A given in Lemma 2 is a convex function of m.*

Proof of Lemma 3. Direct differentiation of V_u gives

$$\begin{aligned} &= \frac{pV^*}{r+p} - \left[\frac{c}{r+p}\frac{R}{r+p}/\left(1+\frac{\pi}{r+p}\right)\right] + \\ &\quad + \left[\frac{R}{r+p}/\left(1+\frac{\pi}{r+p}\right)\right] m \\ &\quad + A\left[1+\frac{\pi}{c}m\right]^{-\frac{r+p}{\pi}} \end{aligned}$$

$$\begin{aligned} V_u'(m) &= \left[\frac{R}{r+p}/\left(1+\frac{\pi}{r+p}\right)\right] \\ &\quad - \left(\frac{c}{r+p}\right) A \left[1+\frac{\pi}{c}m\right]^{-\frac{r+p}{\pi}-1} \\ V_u''(m) &= \left(\frac{\pi}{r+p}\right)\left(1+\frac{r+p}{\pi}\right) A \left[1+\frac{\pi}{c}m\right]^{-\frac{r+p}{\pi}-2} > 0 \end{aligned}$$

since, as shown in Lemma 2, $A > 0$. QED.

Lemma 4 *Let A be an arbitrary value for the constant that indexes the solution of the ODE for V_u in Lemma 1. The cash balances after a withdrawal, m^*, that solves*

$$V_u'(m^*) = 0$$

is given by:

$$m^* = \frac{c}{\pi}\left(\left[\frac{R}{Ac}/\left(1+\frac{\pi}{r+p}\right)\right]^{-\frac{\pi}{r+p+\pi}} - 1\right)$$

for $\pi \neq 0$ and

$$m^* = c\,\frac{\log\left(\frac{R}{Ac}\right)}{r+p}$$

for $\pi = 0$.

Proof of Lemma 4. We define m^* as

$$m^* = \arg\min_m V_u(m)$$

so the FOC for V_u when $\pi \neq 0$ (using equation 62) is_

$$V_u'(m^*) = \left[\frac{R}{r+p}/\left(1+\frac{\pi}{r+p}\right)\right] - \left(\frac{c}{r+p}\right) A\, \left[1+\frac{\pi}{c}m^*\right]^{-\frac{r+p}{\pi}-1} = 0$$

which gives

$$\left[\frac{R}{Ac}/\left(1+\frac{\pi}{r+p}\right)\right] = \left[1+\frac{\pi}{c}m^*\right]^{-\frac{r+p}{\pi}-1}$$

or

$$m^* = \frac{c}{\pi}\left(\left[\frac{R}{Ac}/\left(1+\frac{\pi}{r+p}\right)\right]^{-\frac{\pi}{r+p+\pi}} - 1\right)$$

Following the same steps for the case in which $\pi = 0$ (using equation 63) yields:

$$m^* = -\frac{c}{r+p}\ \log\left(\frac{R}{Ac}\right)$$

QED.

Lemma 5 *The value for the agents matched with a financial institution, V^*, is given by (for all m):*

$$V^* = \frac{R}{r}m^*$$

Proof of Proposition 5.

Recall the Bellman equation (10) :

$$rV_u(m) = Rm + p(V^* - V_u(m)) - V_u'(m)(c + m\pi)$$

At $m = m^*$ we have:

$$\begin{aligned} V_u'(m^*) &= 0 \\ V_u(m^*) &= V^* \end{aligned}$$

replacing these values in (10) and evaluating the Bellman equation at $m = m^*$

$$rV^* = Rm^* + p(V^* - V^*) - 0(c + m^*\pi)$$

or

$$rV^* = Rm^*$$

QED.

Proof of Proposition 3.

(i) The function $V_u(\cdot)$ is derived in Lemma 1. The expression for A comes from Lemma 2.

(ii) The solution for V^* comes from Lemma (5).

QED.

Proof of Proposition 4

Proof of 1). If m^* satisfies

$$0 = (m^*\pi + c)^{1+\frac{r+p}{\pi}} - c^{\frac{r+p}{\pi}}\left(m^*(r+\pi+p) + c\ + (r+p)\ (r+p+\pi)\ \frac{b}{R}\right)$$

then

$$\begin{aligned} & (m^*\lambda\pi + \lambda c)^{1+\frac{r+p}{\pi}} - (\lambda c)^{\frac{r+p}{\pi}}\left(\lambda m^*(r+\pi+p) + \lambda c\ + (r+p)\ (r+p+\pi)\ \frac{\lambda b}{R}\right) \\ = & \left\{\lambda^{1+\frac{r+p}{\pi}}(m^*\pi + c)^{1+\frac{r+p}{\pi}} - (\lambda c)^{\frac{r+p}{\pi}}\lambda\left(m^*(r+\pi+p) + c\ + (r+p)\ (r+p+\pi)\ \frac{b}{R}\right)\right\} \\ = & \lambda^{1+\frac{r+p}{\pi}}\left\{(m^*\pi + c)^{1+\frac{r+p}{\pi}} - (c)^{\frac{r+p}{\pi}}\left(m^*(r+\pi+p) + c\ + (r+p)\ (r+p+\pi)\frac{b}{R}\right)\right\} \\ = & 0. \end{aligned}$$

Proof of 2). We set $c = 1$ and write

$$f(m^*) = g(m^*)$$

for

$$
\begin{aligned}
f\left(m^*\right) &\equiv \left(m^*\pi+1\right)^{1+\frac{r+p}{\pi}} \\
g\left(m^*\right) &\equiv \left(m^*\left(r+\pi+p\right)+1+\left(r+p\right)\left(r+p+\pi\right)\frac{b}{R}\right)
\end{aligned}
$$

where $f\left(0\right)<g\left(0\right)$ for $b>0$, $g'\left(0\right)=f'\left(0\right)>0$, and $g''\left(m^*\right)=0$, and $f''\left(m^*\right)>0$ for all $m^*>0$. Thus, since g is increasing in b, we have that m^* is increasing in b. For $b=0$ we have

$$
\left(m^*\pi+1\right)^{1+\frac{r+p}{\pi}}=\left(m^*\left(r+\pi+p\right)+1\right)
$$

which implies that $m^*=0$.

For the next results we use the following:

$$
f\left(m\right)=g\left(m\right)
$$

is equivalent to

$$
\begin{aligned}
\frac{b}{R} &= \frac{1}{2}m^2+\frac{1}{3!}m^3\left(r+p-\pi\right)+\frac{1}{4!}m^4\left(r+p-\pi\right)\left(r+p-2\pi\right)+ \\
&\quad +\frac{1}{5!}m^5\left(r+p-\pi\right)\left(r+p-2\pi\right)\left(r+p-3\pi\right)+\cdots
\end{aligned}
$$

or

$$
\frac{b}{R}=m^2\left[\frac{1}{2}+\sum_{j=1}^{\infty}\frac{1}{\left(2+j\right)!}\left[\Pi_{s=1}^{j}\left(r+p-s\pi\right)\right]\ m^j\right] \tag{64}
$$

This result follows from expanding $\left(m\pi+1\right)^{1+\frac{r+p}{\pi}}$ around $m=0$.

That the elasticity of m^* with respect to b is smaller than $1/2$ and decreasing in p follows by differentiating (64) with respect to b.

Proof of 3). It follows from 1 and 2.

Proof of 4). It follows by inspection of (64), where b and R enter only as a ratio $\left(b/R\right)$.

Proof of 5). We notice that

$$
m^*=\sqrt{\frac{2b}{R}}+o\left(\sqrt{b}\right) .
$$

is equivalent to

$$
\left(m^*\right)^2=\frac{2b}{R}+\left[o\left(\sqrt{b}\right)\right]^2+2\sqrt{\frac{2b}{R}}\,o\left(\sqrt{b}\right).
$$

Inserting this expression into (64):

$$\begin{aligned}\frac{b}{R} &= \left\{\frac{1}{2}\left[\frac{2b}{R}+\left[o\left(\sqrt{b}\right)\right]^2+2\sqrt{\frac{2b}{R}}\,o\left(\sqrt{b}\right)\right]\right\} \\ &\quad \times\left\{1+2\sum_{j=1}^{\infty}\frac{1}{(2+j)\;!}\left[\Pi_{s=1}^{j}\left(r+p-s\pi\right)\right]\;m^j\right\}\end{aligned}$$

dividing by b/R

$$\begin{aligned}1 &= \left\{\frac{1}{2}\left[2+R\frac{\left[o\left(\sqrt{b}\right)\right]^2}{b}+2\sqrt{2R}\;\frac{o\left(\sqrt{b}\right)}{\sqrt{b}}\right]\right\} \\ &\quad \times\left\{1+2\sum_{j=1}^{\infty}\frac{1}{(2+j)\;!}\left[\Pi_{s=1}^{j}\left(r+p-s\pi\right)\right]\;m^j\right\}\end{aligned}$$

as $b\longrightarrow 0$ we have:

$$\begin{aligned}\frac{\left[o\left(\sqrt{b}\right)\right]^2}{b} &\longrightarrow 0 \\ \frac{o\left(\sqrt{b}\right)}{\sqrt{b}} &\longrightarrow 0 \\ m &\longrightarrow 0\end{aligned}$$

which verifies our approximation:

$$\begin{aligned}1 &= \lim_{b\to 0}\left\{\frac{1}{2}\left[2+R\frac{\left[o\left(\sqrt{b}\right)\right]^2}{b}+2\sqrt{2R}\;\frac{o\left(\sqrt{b}\right)}{\sqrt{b}}\right]\right\} \\ &\quad \times\lim_{b\to 0}\left\{1+2\sum_{j=1}^{\infty}\frac{1}{(2+j)\;!}\left[\Pi_{s=1}^{j}\left(r+p-s\pi\right)\right]\;m^*\left(b\right)^j\right\} \\ &= \frac{1}{2}\left[2+0+0\right]\times\left\{1+0\right\}\end{aligned}$$

Using the normalization that $c=1$ and the homogeneity of degree one of m^* with respect to (c,b) we obtain that

$$m^*/c=\sqrt{\frac{2\left(b/c\right)}{R}}+o\left(\sqrt{(b/c)}\right)$$

or

$$m^* = \sqrt{\frac{2\ b\ c}{R}} + c\ o\left(\sqrt{(b/c)}\right) \ .$$

Proof of 6). For $\pi = R - r = 0$ we have

$$\frac{b}{r} = \frac{1}{2}m^2 + m^2 \sum_{j=1}^{\infty} \frac{1}{(j+2)\ !} (r+p)^j \ \ m^j$$

To see that m^* is decreasing in p, notice that the RHS is increasing in p and m.
To see that $m^* (p \ + r)$ is increasing in p, write

$$\frac{b}{r} = m^2 \left[\frac{1}{2} + \sum_{j=1}^{\infty} \frac{1}{(j+2)\ !} \left[(r+p)\, m\right]^j\right]$$

and notice that, as shown above, as p increases, m^2 decreases, and thus

$$\frac{1}{2} + \sum_{j=1}^{\infty} \frac{1}{(j+2)\ !} \left[(r+p)\, m\right]^j$$

which is a function of $(r+p)\, m$, must be increasing. This implies that the elasticity of m with respect to p is smaller than $p/\,(p+r)$ since

$$0 < \frac{\partial}{\partial p} \ (m \ \ (p+r)) = m + (p+r)\, \frac{\partial m}{\partial p} \ = m \left[1 + \frac{(p+r)}{p} \frac{p}{m} \frac{\partial m}{\partial p}\right]$$

thus

$$\frac{(p+r)}{p} \frac{p}{m} \frac{\partial m}{\partial p} \geq -1$$

or

$$-\frac{(p+r)}{p} \frac{p}{m} \frac{\partial m}{\partial p} \leq 1$$

or

$$0 \leq -\frac{p}{m} \frac{\partial m}{\partial p} \leq \frac{p}{p+r} \ .$$

Proof of 7). Inserting $R = r + \pi$ into (64) we can write

$$\frac{b}{R} = m^2 \ \varphi\,(m, R) \tag{65}$$

for

$$\varphi(m,R) \equiv \frac{1}{2}+\sum_{j=1}^{\infty}\frac{1}{(2+j)\,!}H(R,j)\;m^{j}, \tag{66}$$

$$H(R,j) \equiv \left[\Pi_{s=1}^{j}\left(\,r+p-s\,(R-r)\,\right)\right]\;. \tag{67}$$

We show below that differentiating (65) yields

$$-\frac{R}{m}\frac{dm}{dR}=\frac{1+\frac{R}{\varphi}\varphi_R}{2+\frac{m}{\varphi}\varphi_m} \tag{68}$$

which shows that $\varepsilon < 1/2$ if $\varphi_R < 0$ and $\varphi_m > 0$.

We show below that

$$\frac{\partial H(R,j)}{\partial R}|_{\pi=0}=-\,(r+p)^{j-1}\;\left[\frac{(j+1)\;j}{2}\right]<0$$

$$\varphi_R = \sum_{j=1}^{\infty}\frac{1}{(2+j)\,!}\frac{\partial H(R,j)}{\partial R}|_{\pi=0}\;m^{j}<0 \tag{69}$$

$$\varphi_m = \sum_{j=1}^{\infty}\frac{1}{(2+j)\,!}H(R,j)\,|_{\pi=0}\;j\;m^{j-1}>0 \tag{70}$$

Thus for $\pi=0$ the elasticity $(-R/m)\,(dm/dR)<1/2$ since $\varphi_R<0$ and $\varphi_m>0$.

To see why (68) hold, rewrite (65) dividing both sides by b/R

$$1=m^2\frac{R}{b}\varphi(m,R)$$

Total differential w.r.t. to R

$$0=m\frac{R}{b}\varphi(m,R)\frac{dm}{dR}+\frac{m^2}{2}\frac{1}{b}\varphi(m,R)+\frac{m^2}{2}\frac{R}{b}\left[\varphi_R+\varphi_m\frac{dm}{dR}\right]$$

or (multiply both sides by b and collect terms)

$$-\frac{R}{m}\frac{dm}{dR}\left[m^2\varphi+\frac{m^3}{2}\varphi_m\right]=\frac{m^2}{2}\left[\varphi+\varphi_R R\right]$$

or

$$-\frac{R}{m}\frac{dm}{dR}=\frac{\frac{1}{2}\left[1+\frac{\varphi_R}{\varphi}R\right]}{\left[1+\frac{m}{2}\frac{\varphi_m}{\varphi}\right]}$$

To see why:

$$\frac{\partial H(R,j)}{\partial R}|_{\pi=0} = -(r+p)^{j-1} \left[\frac{(j+1)\ j}{2}\right] < 0$$

start with

$$\begin{aligned}
\frac{\partial H(R,j)}{\partial R} &= -\sum\nolimits_{s=1}^{j} s\ \Pi_{u=1,u\neq s}^{j} \left(\ r+p-u\left(R-r\right)\ \right) \\
&= -\sum_{s=1}^{j} \frac{s}{\left(\ r+p-s\left(R-r\right)\ \right)}\ \Pi_{u=1}^{j}\left(\ r+p-u\left(R-r\right)\ \right) \\
&= -H(R,j)\ \sum_{s=1}^{j} \frac{s}{\left(\ r+p-s\left(R-r\right)\ \right)}
\end{aligned}$$

which is smaller than zero at $\pi = 0$:

$$\begin{aligned}
\frac{\partial H(R,j)}{\partial R}|_{\pi=0} &= -H(R,j)\,|_{\pi=0} \left[\sum_{s=1}^{j} \frac{s}{\left(\ r+p-s\left(R-r\right)\ \right)}\right]|_{\pi=0} \\
&= -H(R,j)\,|_{\pi=0} \left[\sum_{s=1}^{j} \frac{s}{\left(\ r+p\ \right)}\right] \\
&= -(r+p)^{j-1} \left[\frac{(j+1)\ j}{2}\right] < 0.
\end{aligned}$$

QED

Proof of Proposition 5

Assume $p > 0$ and consider the case of the companion discrete time model (see Appendix A), where Δ denotes the length of the time period. Let Δn denote the fraction of agents who withdraw in period of length Δ. These include Δp agents who withdraw when they meet the intermediary and Δz agents who withdraw since they reach zero cash balances. Note that Δz is given by the agents with full balances (Δn) who do meet with an intermediary for the maximum number of periods that the optimal cash withdraw allows them to finance, denoted by τ^*/Δ, at which which point they hit zero balances and withdraw at a bank desk. Δz is thus given by:

$$\Delta z = \Delta n \cdot (1 - \Delta p)^{\tau^*/\Delta}$$

(for small Δ) which yields a recursion for Δn :

$$\Delta n = \Delta p + \Delta n \cdot (1 - \Delta p)^{\tau^*/\Delta} \tag{71}$$

We show below that τ^*/Δ is given by

$$\frac{\tau^*}{\Delta} = \frac{\log\left[(m^*/c)\frac{\pi}{1+\Delta\pi} + 1\right]}{\log(1+\Delta\pi)} \tag{72}$$

Thus the period before cash is replenished (in units of time), denoted by τ^*, is:

$$\tau^* = \lim_{\Delta\to 0}\frac{\tau^*}{\Delta}\Delta = \frac{\log[1+(m^*/c)\,\pi]}{\pi} \tag{73}$$

The real cash balances $m(i)$ for an agent not matched with an intermediary for i model periods satisfy:

$$\begin{aligned}
m(1) &= \frac{m^*}{1+\Delta\pi} - \Delta c, \\
m(2) &= \frac{m(1)}{1+\Delta\pi} - \Delta c, ... \\
m(i+1) &= \frac{m(i)}{1+\Delta\pi} - \Delta c, .. \\
m\left(\frac{\tau^*}{\Delta}\right) &= \frac{m\left(\frac{\tau^*}{\Delta} - 1\right)}{1+\Delta\pi} - \Delta c
\end{aligned}$$

or, assuming that $m\left(\frac{\tau^*}{\Delta}\right) = 0$ (actually for $\Delta > 0$, it just has to be smaller than Δc) :

$$\begin{aligned}
0 &= \frac{m^*}{(1+\Delta\pi)^{\frac{\tau^*}{\Delta}}} - \Delta c\left(1 + \frac{1}{1+\Delta\pi} + \left(\frac{1}{1+\Delta\pi}\right)^2 + \cdots + \left(\frac{1}{1+\Delta\pi}\right)^{\frac{\tau^*}{\Delta}-1}\right) \\
&= \frac{m^*}{(1+\Delta\pi)^{\frac{\tau^*}{\Delta}}} - \Delta c\left[\frac{1-\left(\frac{1}{1+\Delta\pi}\right)^{\frac{\tau^*}{\Delta}}}{1-\frac{1}{1+\Delta\pi}}\right]
\end{aligned}$$

This gives

$$\frac{m^*}{(1+\Delta\pi)^{\frac{\tau^*}{\Delta}}}\left(\frac{\Delta\pi}{1+\Delta\pi}\right) = \Delta c\left[1 - \left(\frac{1}{1+\Delta\pi}\right)^{\frac{\tau^*}{\Delta}}\right]$$

which, after collecting terms and rearranging, yields (72).

Then, from (71), the number of withdrawals per unit of time, n , is given by the following expression:

$$n = p + (1-\Delta p)^{\frac{\tau^*}{\Delta}}\, n$$

which yields

$$n = \frac{p}{1-(1-\Delta p)^{\frac{\tau^*}{\Delta}}}$$

Then

$$\begin{aligned}
\lim_{\Delta\to 0} n &= \frac{p}{\lim_{\Delta\to 0}\left(1-(1-\Delta p)^{\frac{\tau^*}{\Delta}}\right)} \\
&= \frac{p}{1-\lim_{\Delta\to 0}\left[(1-\Delta p)^{\frac{\tau^*}{\Delta}}\right]} = \frac{p}{1-\lim_{\Delta\to 0}\left[(1-\Delta p)^{\frac{1}{\Delta}}\right]^{\tau^*}} \\
&= \frac{p}{1-\left(\lim_{\Delta\to 0}\left[(1-\Delta p)^{\frac{1}{\Delta}}\right]\right)^{\frac{\log[1+(m^*/c)\pi]}{\pi}}} = \frac{p}{1-(\exp(-p))^{\frac{\log[1+(m^*/c)\pi]}{\pi}}}
\end{aligned}$$

Note that

$$\begin{aligned}
[\exp(-p)]^{\frac{\log[1+(m^*/c)\pi]}{\pi}} &= \exp\left(-\frac{p}{\pi}\ \log\left[1+(m^*/c)\,\pi\right]\right) \\
&= \exp\left(\log\left[(1+(m^*/c)\,\pi)^{-\frac{p}{\pi}}\right]\right) \\
&= (1+(m^*/c)\,\pi)^{-\frac{p}{\pi}}
\end{aligned}$$

Thus

$$\lim_{\Delta\to 0} n = \frac{p}{1-(\exp(-p))^{\frac{\log[1+(m^*/c)\pi]}{\pi}}} = \frac{p}{1-(1+(m^*/c)\,\pi)^{-\frac{p}{\pi}}} \tag{74}$$

For $\pi = 0$, notice that

$$\begin{aligned}
\lim_{\pi\to 0}(1+(m^*/c)\,\pi)^{-\frac{p}{\pi}} &= \left[\lim_{\pi\to 0}(1+(m^*/c)\,\pi)^{\frac{1}{\pi}}\right]^{-p} \\
&= \left[\exp\left(\frac{m^*}{c}\right)\right]^{-p} = \exp\left(-p\frac{m^*}{c}\right)
\end{aligned}$$

For $p = 0$, we can either take the the limit of the solution for n (equation 74), which gives

$$n = \frac{\pi}{\log\left(1+\frac{m^*\pi}{c}\right)}$$

or computed directly, since for $p = 0$ we have that $n = \tau^*$ solves

$$m^* = c\int_0^{\tau^*} \exp(\pi\ s)\ ds$$

which yields the same expression.

QED.

Proof of Proposition 6 .

(i) Let $H(m,t)$ be the cdf for m at time t. Then, define $\psi(m,t;\Delta)$ as

$$\psi(m,t;\Delta) = H(m,t) - H(m-\Delta(m\pi+c),\ t).$$

Thus, $\psi(m,t;\Delta)$ is the fraction of agents with money in the interval $[m,\ m - \Delta(m\pi + c)\ )$ at time t. Define h as

$$h(m,t;\Delta) = \frac{\psi(m,t;\Delta)}{\Delta(m\pi + c)}$$

Then, $\lim h(m,t;\Delta)$ as $\Delta \to 0$ is the density of H evaluated at m at time t.

For a small Δ given the law of motion of cash we have that

$$\psi(m,\ t + \Delta\ ;\Delta) = \psi(m + \Delta(m\pi + c),\ t\ ;\Delta)\ \ (1 - \Delta p)$$

Assuming that we are in the stationary distribution $h(m,t;\Delta)$ does not depend on t, so we write $h(m;\Delta)$. Using the definition of h :

$$h(m;\Delta) = h(m + \Delta(m\pi + c)\ \ ;\Delta)\ \ \frac{\Delta([m + \Delta(m\pi + c)]\,\pi + c)}{\Delta(m\pi + c)}(1 - \Delta p)$$

or

$$h(m;\Delta) = h(m + \Delta(m\pi + c)\ \ ;\Delta)\ \ (1 + \Delta\pi)(1 - \Delta p)$$

Assuming that h is differentiable in m we write

$$h(m + \Delta(m\pi + c)\ \ ;\Delta) = h(m\ ;\Delta) + \frac{\partial h}{\partial m}(m\ ;\Delta)\ \ [\Delta(m\pi + c)]$$

thus

$$h(m;\Delta) = \left[h(m\ ;\Delta) + \frac{\partial h}{\partial m}(m\ ;\Delta)\ \ [\Delta(m\pi + c)] + o(\Delta)\ \right]\ (1 + \Delta\pi)(1 - \Delta p)$$

or

$$h(m;\Delta) = h(m\ ;\Delta)(1 + \Delta\pi)(1 - \Delta p) + \left[\frac{\partial h}{\partial m}(m\ ;\Delta)\ \ [\Delta(m\pi + c)] + o(\Delta)\ \right]\ (1 + \Delta\pi)(1 - \Delta p)$$

or

$$h(m;\Delta) = h(m\ ;\Delta)\left(1 + \Delta(\pi - p) - \Delta^2 p\pi\right) + \left[\frac{\partial h}{\partial m}(m\ ;\Delta)\ \ [\Delta(m\pi + c)] + o(\Delta)\ \right]\ (1 + \Delta\pi)(1 - \Delta$$

or

$$h(m\ ;\Delta)\,\Delta(p - \pi + \Delta p\pi) = \left[\frac{\partial h}{\partial m}(m\ ;\Delta)\ \ [\Delta(m\pi + c)] + o(\Delta)\ \right]\ (1 + \Delta\pi)(1 - \Delta p)$$

dividing by Δ :

$$h(m\ ;\Delta)(p - \pi + \Delta p\pi) = \left[\frac{\partial h}{\partial m}(m\ ;\Delta)\ \ (m\pi + c) + \frac{o(\Delta)}{\Delta}\ \right]\ (1 + \Delta\pi)(1 - \Delta p)$$

and taking the limit as $\Delta \to 0$

$$h(m)(p-\pi) = \frac{\partial h}{\partial m}(m)\ (m\pi + c) \tag{75}$$

The solution of this ODE is

$$h(m) = 1/m^*$$

if $p = \pi$ and

$$h(m) = A\ \left[1 + \frac{\pi}{c} m\right]^{\frac{p-\pi}{\pi}}$$

for some constant A if $p \neq \pi$. The constant A solves:

$$1 = \int_0^{m^*} h(m)\, dm = A\ \int_0^{m^*} \left[1 + \frac{\pi}{c} m\right]^{\frac{p-\pi}{\pi}} dm$$

where

$$\begin{aligned} \int_0^{m^*} \left[1 + \frac{\pi}{c} m\right]^{\frac{p-\pi}{\pi}} dm &= \frac{(c/\pi)}{1 + \frac{p-\pi}{\pi}} \left[1 + \frac{\pi}{c} m\right]^{\frac{p-\pi}{\pi}+1} |_0^{m^*} \\ &= \frac{(c/\pi)}{1 + \frac{p-\pi}{\pi}} \left(\left[1 + \frac{\pi}{c} m^*\right]^{\frac{p}{\pi}} - 1 \right) \end{aligned}$$

Thus

$$A = 1\ /\ \left\{ \left(\frac{c}{p}\right) \left(\left[1 + \frac{\pi}{c} m^*\right]^{\frac{p}{\pi}} - 1 \right) \right\}$$

Thus

$$h(m) = \left(\frac{p}{c}\right) \left(\frac{\left[1 + \frac{\pi}{c} m\right]^{\frac{p}{\pi}-1}}{\left[1 + \frac{\pi}{c} m^*\right]^{\frac{p}{\pi}} - 1} \right)$$

for $\pi \neq 0$ and takes the form

$$h(m) = \frac{\frac{p}{c} \exp\left(\frac{mp}{c}\right)}{\exp\left(\frac{m^* p}{c}\right) - 1}$$

for $\pi = 0$.

(ii) We now show that the distribution of m that corresponds to a higher value of m^* is stochastically higher. Consider the CDFs $H(m; m^*)$ and let m_1^* and m_2^* be two values for the optimal return point with $m_2^* > m_1^*$. We argue that

$$H(m; m_1^*) > H(m; m_2^*)$$

for all $m \in [0, m_2^*)$. This follows because the corresponding densities satisfy:

$$\frac{h(m; m_2^*)}{h(m; m_1^*)} = \frac{\left[1 + \frac{\pi}{c} m_1^*\right]^{\frac{p}{\pi}} - 1}{\left[1 + \frac{\pi}{c} m_2^*\right]^{\frac{p}{\pi}} - 1} < 1$$

in their common domain, $m \in [0, m_1^*]$. In the interval $[m_1^*, m_2^*)$ we have

$$H(m; m_1^*) = 1 > H(m; m_2^*).$$

QED.

Proof of Proposition 7.
(i) Given the density

$$h(m) = \left(\frac{p}{c}\right)\left(\frac{\left[1 + \frac{\pi}{c} m\right]^{\frac{p}{\pi} - 1}}{\left[1 + \frac{\pi}{c} m^*\right]^{\frac{p}{\pi}} - 1}\right)$$

we have

$$M = \int_0^{m^*} m h(m)\, dm = \int_0^{m^*} m \left(\frac{p}{c}\right)\left(\frac{\left[1 + \frac{\pi}{c} m\right]^{\frac{p}{\pi} - 1}}{\left[1 + \frac{\pi}{c} m^*\right]^{\frac{p}{\pi}} - 1}\right) dm$$

Using integration by parts:

$$\begin{aligned}
&= \frac{\frac{p}{c}}{\left[1 + \frac{\pi}{c} m^*\right]^{\frac{p}{\pi}} - 1} \int_0^{m^*} \left[m\left(1 + \frac{\pi}{c} m\right)^{\frac{p}{\pi} - 1}\right] dm \\
&= \frac{\frac{p}{c}}{\left[1 + \frac{\pi}{c} m^*\right]^{\frac{p}{\pi}} - 1} \left[\frac{mc}{p}\left(1 + \frac{\pi}{c} m\right)^{\frac{p}{\pi}} |_0^{m^*} - \int_0^{m^*} \frac{c}{p}\left(1 + \frac{\pi}{c} m\right)^{\frac{p}{\pi}} dm\right] \\
&= \frac{\frac{p}{c}}{\left[1 + \frac{\pi}{c} m^*\right]^{\frac{p}{\pi}} - 1} \left[\frac{mc}{p}\left(1 + \frac{\pi}{c} m\right)^{\frac{p}{\pi}} - \frac{c^2}{p}\left(\frac{1}{p + \pi}\right)\left(1 + \frac{\pi}{c} m\right)^{\frac{p}{\pi} + 1}\right]_0^{m^*} \\
&= \frac{\frac{p}{c}}{\left[1 + \frac{\pi}{c} m^*\right]^{\frac{p}{\pi}} - 1} \left[\left(1 + \frac{\pi}{c} m\right)^{\frac{p}{\pi}} \frac{c}{p}\left(m - \frac{c}{(p + \pi)}\left(1 + \frac{\pi}{c} m\right)\right)\right]_0^{m^*} \\
&= \frac{1}{\left[1 + \frac{\pi}{c} m^*\right]^{\frac{p}{\pi}} - 1} \left\{\left(1 + \frac{\pi}{c} m\right)^{\frac{p}{\pi}} \left[m - \frac{c}{(p + \pi)}\left(1 + \frac{\pi}{c} m\right)\right]\right\}_0^{m^*} \\
&= \frac{1}{\left[1 + \frac{\pi}{c} m^*\right]^{\frac{p}{\pi}} - 1} \left\{\left(1 + \frac{\pi}{c} m^*\right)^{\frac{p}{\pi}} \left[m^* - \frac{c}{(p + \pi)}\left(1 + \frac{\pi}{c} m^*\right)\right] + \frac{c}{p + \pi}\right\}
\end{aligned}$$

Thus the average stock of money is:

$$M = \frac{1}{\left[1+\frac{\pi}{c}m^*\right]^{\frac{p}{\pi}}-1}\left\{\left(1+\frac{\pi}{c}m^*\right)^{\frac{p}{\pi}}\left[m^*-\frac{c}{(p+\pi)}\left(1+\frac{\pi}{c}m^*\right)\right]+\frac{c}{(p+\pi)}\right\}$$

For $\pi = 0$

$$\begin{aligned} M &= \frac{1}{e^{m^*\frac{p}{c}}-1}\left\{e^{m^*\frac{p}{c}}\left[m^*+\frac{c}{p}\right]-\frac{c}{p}\right\} = \\ &= \frac{1}{1-e^{-\frac{p}{c}m^*}}m^* - c/p \end{aligned}$$

(ii) Follows immediately from result (ii) of Proposition **6**
QED.

Proof of proposition 8. Differentiating

$$M/c = \frac{1}{p+\pi}\left[n\,\frac{m^*}{c}-1\right]$$

we have

$$R\frac{d}{dR}\left(\frac{M}{c}\right) = \frac{n\,(m^*/c)}{p+\pi}\left(\frac{R}{n}\;n_R\;+\frac{R}{\frac{m^*}{c}}\;\left(\frac{m^*}{c}\right)_R\right)$$

or

$$\frac{R}{(M/c)}\frac{d}{dR}\left(\frac{M}{c}\right) = \frac{n\,(m^*/c)}{\left[n\;\frac{m^*}{c}-1\right]}\left(\frac{R}{n}\;n_R\;+\frac{R}{\frac{m^*}{c}}\;\left(\frac{m^*}{c}\right)_R\right)$$

Using η_{xy} for the elasticity of x w.r.t. y :

$$-\eta_{M,R} = \omega\left(-n_{n\;R}\;-\eta_{m^*R}\right)$$

and defining

$$\omega = \frac{n\,(m^*/c)}{\left[n\;\frac{m^*}{c}-1\right]}$$

we have:

$$\left(-\eta_{M,R}\right)-\eta_{nR} = -\left(\omega+1\right)n_{n\;R}\;+\omega\left(-\eta_{m^*R}\right)$$

where

$$n_{n\;R} = \eta_{n\;m^*}\;\eta_{m^*R}$$

Then we can write the different in the absolute value of the elasticities as:

$$\left(-\eta_{M,R}\right)-\eta_{nR} = \left[\omega+(1+\omega)\,\eta_{n\;m^*}\;\right]\left(-\eta_{m^*R}\right)>0$$

or, since $(-\eta_{m^*R}) > 0$ as

$$\frac{\omega}{1+\omega} \geq -\eta_{n\ m^*}\ .$$

Developing the terms

$$\frac{\omega}{1+\omega} = \frac{\frac{n(m^*/c)}{n\ \frac{m^*}{c}-1}}{1+\frac{n(m^*/c)}{n\ \frac{m^*}{c}-1}} = \frac{n\,(m^*/c)}{n\,(m^*/c)-1+n\,(m^*/c)} = \frac{1}{2-\frac{1}{n(m^*/c)}}$$

For $\pi = 0$ we have

$$n = \frac{p}{1-\exp\left(-pm^*/c\right)}$$

hence

$$\begin{aligned} n\ \frac{m^*}{c} &= \frac{p\,(m^*/c)}{1-\exp\left(-pm^*c\right)} \\ -\eta_{n\ m^*} &= \exp\left(-pm^*/c\right)\frac{p\,(m^*/c)}{1-\exp\left(-p\,(m^*/c)\right)} \end{aligned}$$

The condition for $-\left(\eta_{MR}\right)-\eta_{nR} \geq 0$ is

$$\frac{\omega}{1+\omega} = \frac{1}{2-\frac{1}{\frac{p(m^*/c)}{1-\exp(-pm^*c)}}} \geq -\eta_{n\ m*} = \exp\left(-pm^*/c\right)\frac{p\,(m^*/c)}{1-\exp\left(-p\,(m^*/c)\right)}$$

or

$$\frac{1}{2-\frac{1}{y}} = \frac{y}{2y-1} \geq y\ \exp\left(-pm^*/c\right)$$

where

$$y \equiv \frac{p\,(m^*/c)}{1-\exp\left(-p\,(m^*/c)\right)}$$

which is equivalent to

$$1 \geq \exp\left(-pm^*/c\right)\ \left(2\frac{p\,(m^*/c)}{1-\exp\left(-p\,(m^*/c)\right)}-1\right)$$

or

$$1 \geq e^{-x}\ \left(\frac{2\ x}{1-e^{-x}}-1\right)$$

where $x = pm^*/c$. We can write this condition as

$$e^{x} \geq \ \left(\frac{2\ x}{1-e^{-x}}-1\right)$$

or

$$e^{x}\left(1-e^{-x}\right)\geq\left(2x-1+e^{-x}\right)$$

or

$$1\geq e^{-x}\ \left(\frac{2\ x}{1-e^{-x}}-1\right)$$

or

$$e^{x}-e^{-x}\geq 2x$$

which is satisfied since letting $\phi\left(x\right)=e^{x}-e^{-x}$ we have $\phi\left(0\right)=0$ and

$$\phi'\left(x\right)=e^{x}+e^{-x}=2\left(1+\frac{1}{2!}x^{2}+\frac{1}{4!}x^{4}+...\right)\geq 2.$$

QED.

Proof of proposition 11

The FOC of the steady state problem (1), using expressions (25) and (20), is:

$$-R=b\frac{\partial T\left(M\right)}{\partial M}=b\frac{\partial n\left(\mu^{-1}\left(M\right)\right)}{\partial m^{*}}\frac{\partial\mu^{-1}\left(M\right)}{\partial M}$$

or equivalently we could solve for $\tilde{m}^{*}$ satisfying

$$-R=b\frac{\partial n\left(m^{*}\right)/\partial m^{*}}{\partial\mu\left(m^{*}\right)/\partial m^{*}}$$

Direct computation gives after some cancellations:

$$-R=b\frac{-\frac{p}{c^{2}}}{\frac{\tilde{m}^{*}\pi}{c}\frac{p}{p+\pi}\left(1+\frac{\tilde{m}^{*}\pi}{c}\right)^{\frac{p}{\pi}}-\frac{\tilde{m}^{*}p}{c}-\frac{p}{p+\pi}\left(1-\left(1+\frac{\tilde{m}^{*}\pi}{c}\right)^{\frac{p}{\pi}}\right)}$$

Rearranging this expression one can show that $\tilde{m}^{*}$ solves for m^{*}, in (16), for the case of $r=0$.

QED.

Proof of Proposition 13

It is easily derived following the logic of lemma 1

Proof of Proposition 14

We will use repeated substitution on the system (37)-(41) to arrive to one non-linear equation in one unknown, namely $\underline{m}$.

From (37) and (38) we have:

$$V^{*}=\frac{R}{r}\ m^{*}. \tag{76}$$

So now we replace V^* by this expression and drop the equation (38), so we have a system of 4 eqns in 4 unknowns. We solve for $A_\varphi(m^*)$ using (37) :

$$\varphi_m(m^*, A_\varphi(m^*)) = 0$$

or

$$0 = \frac{R}{r+\pi} = \frac{c}{r}A_\varphi\left[1+\frac{\pi}{c}m^*\right]^{-\frac{r}{\pi}-1}$$

or

$$A_\varphi(m^*) = \frac{rR}{c(r+\pi)}\left[1+\frac{\pi}{c}m^*\right]^{1+\frac{r}{\pi}} \tag{77}$$

To solve for $A_\eta(m^*)$ we use (40):

$$\eta\left(0, (m^*R/r), A_\eta(m^*)\right) = (m^*R/r) + b$$

or

$$V^* + b = \frac{p(V^*+f) - Rc/(r+p+\pi)}{r+p} + \left(\frac{c}{r+p}\right)^2 A_\eta$$

or

$$A_\eta = \left(\frac{r+p}{c^2}\right)\left(rV^* + br + p(b-f) + \frac{Rc}{r+p+\pi}\right)$$

and use $rV^* = Rm^*$ to get:

$$A_\eta(m^*) = \frac{r+p}{c^2}\left(Rm^* + br + p(b-f) + \frac{Rc}{r+p+\pi}\right) \tag{78}$$

Now we replace A_η and A_φ into (39) and (41) so we get two non-linear equations:

$$\begin{aligned}\eta\left(\underline{m}, (m^*R/r), A_\eta(m^*)\right) &= (m^*R/r) + f \\ \varphi\left(\underline{m}, A_\varphi(m^*)\right) &= (m^*R/r) + f\end{aligned}$$

The first equation gives:

$$\begin{aligned}(m^*R/r) + f &= \left[\frac{p}{r+p}\left[\frac{m^*R}{r} + f\right] - \frac{Rc}{(r+p)(r+p+\pi)}\right] + \left[\frac{R}{r+p+\pi}\right]\underline{m} \\ &\quad + \left(\frac{c}{r+p}\right)^2 A_\eta(m^*)\ \left[1+\frac{\pi}{c}\underline{m}\right]^{-\frac{r+p}{\pi}}\end{aligned}$$

$$\begin{aligned}m^*\left(\frac{R}{r+p}\right) + f &= \left[\frac{p}{r+p}f - \frac{Rc}{(r+p)(r+p+\pi)}\right] + \left[\frac{R}{r+p+\pi}\right]\underline{m} \\ &\quad + \left(\frac{c}{r+p}\right)^2 A_\eta(m^*)\ \left[1+\frac{\pi}{c}\underline{m}\right]^{-\frac{r+p}{\pi}}\end{aligned}$$

and substituting:

$$A_\eta(m^*) = \frac{r+p}{c^2}\left(Rm^* + br + p(b-f) + \frac{Rc}{r+p+\pi}\right)$$

we get

$$\begin{aligned} m^*\left(\frac{R}{r+p}\right) + f &= \left[\frac{p}{r+p}f - \frac{Rc}{(r+p)(r+p+\pi)}\right] + \left[\frac{R}{r+p+\pi}\right]\underline{m} \\ &\quad + \left\{m^*\frac{R}{r+p} + b - \left(\frac{c}{r+p}\right)\left[\frac{pf}{c} - \frac{R}{r+\pi+p}\right]\right\}\ \left[1+\frac{\pi}{c}\underline{m}\right]^{-\frac{r+p}{\pi}} \end{aligned}$$

or

$$\begin{aligned} & m^*\left(\frac{R}{r+p}\right)\left[1 - \left[1+\frac{\pi}{c}\underline{m}\right]^{-\frac{r+p}{\pi}}\right] \\ = & \left[\frac{p}{r+p}f - \frac{Rc}{(r+p)(r+p+\pi)}\right] + \left[\frac{R}{r+p+\pi}\right]\underline{m} + \\ & \left\{b - \left(\frac{c}{r+p}\right)\left[\frac{pf}{c} - \frac{R}{r+\pi+p}\right]\right\}\ \left[1+\frac{\pi}{c}\underline{m}\right]^{-\frac{r+p}{\pi}} - f \end{aligned}$$

or

$$m^* = \frac{\left[\frac{p}{r+p}f - \frac{Rc}{(r+p)(r+p+\pi)}\right] + \left[\frac{R}{r+p+\pi}\right]\underline{m} + \left\{b - \left(\frac{c}{r+p}\right)\left[\frac{pf}{c} - \frac{R}{r+\pi+p}\right]\right\}\ \left[1+\frac{\pi}{c}\underline{m}\right]^{-\frac{r+p}{\pi}} - f}{\left(\frac{R}{r+p}\right)\left[1 - \left[1+\frac{\pi}{c}\underline{m}\right]^{-\frac{r+p}{\pi}}\right]}$$

$$m^* = \frac{\left(\frac{c}{r+p}\right)\left[\frac{p\ f}{c} - \frac{R}{(r+p+\pi)}\right]\left[1 - \left[1+\frac{\pi}{c}\underline{m}\right]^{-\frac{r+p}{\pi}}\right] + \left[\frac{R}{r+p+\pi}\right]\underline{m} + b\left[1+\frac{\pi}{c}\underline{m}\right]^{-\frac{r+p}{\pi}} - f}{\left(\frac{R}{r+p}\right)\left[1 - \left[1+\frac{\pi}{c}\underline{m}\right]^{-\frac{r+p}{\pi}}\right]}$$

Hence:

$$m_1^*(\underline{m}) \equiv \frac{\left(\frac{c}{r+p}\right)\left[\frac{p\ f}{c} - \frac{R}{(r+p+\pi)}\right]}{\left(\frac{R}{r+p}\right)} + \frac{\left[\frac{R}{r+p+\pi}\right]\underline{m} + b\left[1+\frac{\pi}{c}\underline{m}\right]^{-\frac{r+p}{\pi}} - f}{\left(\frac{R}{r+p}\right)\left[1 - \left[1+\frac{\pi}{c}\underline{m}\right]^{-\frac{r+p}{\pi}}\right]}$$

or

$$m_1^*(\underline{m}) \equiv \mu(\underline{m}) = \left(\frac{r+p}{R}\right)\left[\frac{c}{r+p}\left(\frac{p\ f}{c} - \frac{R}{(r+p+\pi)}\right) + \frac{\left(\frac{R}{r+p+\pi}\right)\underline{m} + b\left(1+\frac{\pi}{c}\underline{m}\right)^{-\frac{r+p}{\pi}} - f}{1 - \left(1+\frac{\pi}{c}\underline{m}\right)^{-\frac{r+p}{\pi}}}\right] \tag{79}$$

Notice that for $\pi > 0$, $m_1^*(\underline{m})$ is continuous in $(0,\infty)$ and that:

$$\begin{aligned}
\lim_{\underline{m}\to 0} m_1^*(\underline{m}) &= +\infty \\
\lim_{\underline{m}\to +\infty} m_1^*(\underline{m}) &= +\infty \\
\lim_{\underline{m}\to \infty} \frac{m_1^*(\underline{m})}{\underline{m}} &= \left(\frac{r+p}{r+p+\pi}\right) < 1 \ .
\end{aligned}$$

Now we solve the second equation:

$$(m^*R/r) + f = -\left[\frac{Rc}{(r+\pi)\,r}\right] + \left[\frac{R}{r+\pi}\right]\underline{m} + \left(\frac{c}{r}\right)^2 A_\varphi(m^*) \left[1+\frac{\pi}{c}\underline{m}\right]^{-\frac{r}{\pi}}$$

and using expression (77) for $A_\varphi$we get:

$$m^*R/r = -\left[\frac{Rc}{(r+\pi)\,r}\right] + \left[\frac{R}{r+\pi}\right]\underline{m} + \frac{Rc}{(r+\pi)\,r}\left[1+\frac{\pi}{c}m^*\right]^{1+\frac{r}{\pi}}\left[1+\frac{\pi}{c}\underline{m}\right]^{-\frac{r}{\pi}} - f$$

or

$$m^* = -\left[\frac{c}{(r+\pi)}\right] + \left[\frac{r}{r+\pi}\right]\underline{m} + \frac{c}{(r+\pi)}\frac{\left[1+\frac{\pi}{c}m^*\right]^{1+\frac{r}{\pi}}}{\left[1+\frac{\pi}{c}\underline{m}\right]^{\frac{r}{\pi}}} - f\frac{r}{R}$$

or

$$m^* = \left[\frac{r}{r+\pi}\right]\underline{m} + \frac{c}{(r+\pi)}\left(\frac{\left[1+\frac{\pi}{c}m^*\right]^{1+\frac{r}{\pi}}}{\left[1+\frac{\pi}{c}\underline{m}\right]^{\frac{r}{\pi}}} - 1\right) - f\frac{r}{R} \tag{80}$$

which implicitly defines $m_2^*(\underline{m})$.

To establish the properties of $m_2^*(\underline{m})$ define $\sigma(m^*,\underline{m})$ as

$$\sigma(m^*,\underline{m}) \equiv \left[\frac{r}{r+\pi}\right]\underline{m} + \frac{c}{(r+\pi)}\left(\frac{\left[1+\frac{\pi}{c}m^*\right]^{1+\frac{r}{\pi}}}{\left[1+\frac{\pi}{c}\underline{m}\right]^{\frac{r}{\pi}}} - 1\right) - f\frac{r}{R}$$

so that $m^* = m_2^*(\underline{m})$ is the solution of

$$m^* = \sigma(m^*,\underline{m})\,.$$

Notice that σ is increasing in m^* with

$$\begin{aligned}
\frac{\partial\sigma\left(m^*,\underline{m}\right)}{\partial m^*} &= \frac{\left[1+\frac{\pi}{c}m^*\right]^{\frac{r}{\pi}}}{\left[1+\frac{\pi}{c}\underline{m}\right]^{\frac{r}{\pi}}}, \\
\frac{\partial\sigma\left(\underline{m},\underline{m}\right)}{\partial m^*} &= 1, \\
\frac{\partial\sigma\left(m^*,\underline{m}\right)}{\partial m^*} &> 1 \text{ for } m^* > \underline{m}
\end{aligned}$$

and

$$\sigma\left(\underline{m},\underline{m}\right) = \underline{m} - f\frac{r}{R}$$

Thus there is always a solution $m^* = m_2^*\left(\underline{m}\right)$, of $m^* = \sigma\left(m^*,\underline{m}\right)$ with $m_2^*\left(\underline{m}\right) > \underline{m}$. Since

$$\frac{\partial\sigma\left(m^*,\underline{m}\right)}{\partial\underline{m}} = \left[\frac{r}{r+\pi}\right]\left(1-\frac{\left[1+\frac{\pi}{c}m^*\right]^{1+\frac{r}{\pi}}}{\left[1+\frac{\pi}{c}\underline{m}\right]^{1+\frac{r}{\pi}}}\right) < 0 \text{ if } m^* > \underline{m}$$

then

$$\frac{d}{d\underline{m}}m_2^*\left(\underline{m}\right) = \frac{\frac{\partial\sigma\left(m^*,\underline{m}\right)}{\partial\underline{m}}}{1-\frac{\partial\sigma\left(m^*,\underline{m}\right)}{\partial m^*}} > 0 \text{ and } \frac{m_2^*\left(\underline{m}\right)}{\underline{m}} > 1 \ .$$

Notice that $m_2^*\left(\underline{m}\right)$ is continuous in $(0,\infty)$ and that:

$$\begin{aligned}
\lim_{\underline{m}\to 0} m_2^*\left(\underline{m}\right) &< \infty \\
\lim_{\underline{m}\to\infty} \frac{m_2^*\left(\underline{m}\right)}{\underline{m}} &= 1
\end{aligned}$$

Hence, to summarize, we $m_1^*\left(\underline{m}\right)$ and $m_2^*\left(\underline{m}\right)$ continuous in $(0,\infty)$ and:

$$\begin{aligned}
m_1^*(0) &= \infty > m_2^*(0) \\
\lim_{\underline{m}\to\infty}\frac{m_1^*\left(\underline{m}\right)}{\underline{m}} &= \frac{r+p}{r+p+\pi} < 1 \le \lim_{\underline{m}\to\infty}\frac{m_2^*\left(\underline{m}\right)}{\underline{m}}
\end{aligned}$$

Thus by the intermediate value theorem, there is an $\hat{\underline{m}}$ such that

$$m_1^*\left(\hat{\underline{m}}\right) - m_2^*\left(\hat{\underline{m}}\right) = 0.$$

For $\pi < 0$ the range of the functions defined above is $[0, -\pi/c]$. By a straightforward adaptation of the arguments above one can show the existence of the solution of the two equations in this case.

That the solution is unique follows from the Principle of Optimality. If there would be two solutions of $m_1^*\left(\hat{\underline{m}}\right) - m_2^*\left(\hat{\underline{m}}\right) = 0$ there would be two bounded solutions to the Bellman equation, a contradiction with the principle of optimality.

Now we verify the guesses that the value function $V_u(m)$ is decreasing in a neighborhood of $m = 0$ and single peaked. The convexity of $V_u(m)$ is equivalent to showing that $A_\varphi > 0$ and $A_\eta > 0$ which can be readily established from (77) and (78) provided $b > f$. Moreover, since $A_\varphi > 0$ and $A_\eta > 0$, then $V_u(m)$ is strictly decreasing on $(0, m^*)$.

Finally we extend the value function to the range (m^*, ∞). Given the values already found for V^* and A_φ we find $\bar{m}$ as the solution to

$$\varphi(\bar{m}, A_\varphi) = V^* + f$$

i.e. $\bar{m}$ solves:

$$\left(\frac{R}{r+\pi}\right)\bar{m} + \left(\frac{c}{r}\right)^2 A_\varphi \left[1 + \frac{\pi}{c}\bar{m}\right]^{-\frac{r}{\pi}} = V^* + f + \frac{Rc/(r+\pi)}{r}.$$

Now given V^* and $\bar{m}$ we find the constant $\bar{A}_\eta$ by solving

$$\eta\left(\bar{m}, V^*, \bar{A}_\eta\right) = V^* + f$$

i.e. $\bar{A}_\eta$ is:

$$\bar{A}_\eta = \left(\frac{r+p}{c}\right)^2 \left(1 + \frac{\pi}{c}\bar{m}\right)^{\frac{r+p}{\pi}} \left(V^* + f - \frac{p(V^* + f) - Rc/(r+p+\pi)}{r+p} - \frac{R}{r+p+\pi}\bar{m}\right)$$

Lastly, given V^* and $\bar{A}_\eta$ we find m^{**} as the solution of η

$$\eta\left(m^{**}, V^*, \bar{A}_\eta\right) = V^* + b.$$

Now we establish that V_u is strictly increasing in (m^*, m^{**}). For this notice that since $\eta\left(\bar{m}, V^*, \bar{A}_\eta\right) = \varphi(\bar{m}, A_\varphi)$ then by inspecting the Bellman equation (32) it follows that they have the same derivative with respect to m at $\bar{m}$. Since $\varphi(\bar{m}, A_\varphi)$ is convex this derivative is strictly positive. There are two cases. If $\bar{A}_\eta$ is positive then $\eta\left(\bar{m}, V^*, \bar{A}_\eta\right)$ is convex in this range and hence V_u is increasing. If $\bar{A}_\eta$ is negative then $\eta\left(\bar{m}, V^*, \bar{A}_\eta\right)$ is concave but it is increasing since it cannot achieve a maximum since it is the sum of a linear increasing and a bounded concave function.
QED.

Proof of Proposition 15 In proposition 13 we show that $V_u(m)$ is analytical in the interval $[\underline{m},\ m^*]$, so we can write:

$$V_u(m) = V_u(m^*) + \sum_{i=1}^{\infty} \frac{1}{i!} V_u^i(m^*)(m - m^*)^i$$

We will use that $f = V_u(\underline{m}) - V_u(m^*)$ to write:

$$f = \sum_{i=1}^{\infty} \frac{1}{i!} V_u^i(m^*)(\underline{m} - m^*)^i$$

The next step is to find an expression for the $i-th$ order derivative: $V_u^i(m^*)$. Considering the Bellman equation in an interval of m^* :

$$rV_u(m) = Rm + V_u^1(m)[-c - \pi m]$$

and differentiating it w.r.t. m :

$$rV_u^1(m) = R + V_u^2(m)[-c - \pi m] - \pi V_u^1(m)$$

or

$$V_u^2(m) = \frac{R - (r + \pi) V_u^1(m)}{[c + \pi m]} > 0$$

evaluating at m^*, using that $V_u^1(m^*) = 0$ we obtain:

$$V_u^2(m) = \frac{R}{c + \pi m^*}$$

Differentiating again we have

$$[r + 2\pi] V_u^2(m) = -V_u^3(m)[c + \pi m]$$

and hence, by induction we can show that

$$[r + (1 + i)\ \pi] V_u^{i+1}(m) = -V_u^{i+2}(m)[c + \pi m]$$

for $i = 1, ...$ Then we can write:

$$\begin{aligned} V_u^{i+1}(m^*) &= (-1)^{i-1} V_u^2(m) \left[\frac{r + 2\pi}{c + m^*\pi}\right]\left[\frac{r + 3\pi}{c + m^*\pi}\right] \cdots \left[\frac{r + i\pi}{c + m^*\pi}\right] \\ &= (-1)^{i-1} \frac{R}{(c + m^*\pi)^i} \Pi_{j=2}^{i} [r + j\pi] \end{aligned}$$

for $i = 2, 3, \ldots$ Hence we have

$$\begin{aligned}
f &= \sum_{i=1}^{\infty} \frac{1}{i!} V_u^i (m^*) (\underline{m} - m^*)^i \\
&= \frac{1}{2} \left(\frac{R}{c + m^* \pi} \right) (\underline{m} - m^*)^2 + \\
&\quad + \sum_{i=2} \frac{1}{(i+1)!} \; (-1)^{i-1} \frac{R}{(c + m^* \pi)^i} \left(\Pi_{j=2}^{i} [r + j\pi] \right) (\underline{m} - m^*)^{i+1} \\
&= \frac{1}{2} R (c + m^* \pi) \left(\frac{\underline{m} - m^*}{c + m^* \pi} \right)^2 + \\
&\quad + \sum_{i=2} \frac{(-1)^{i-1}}{(i+1)!} \; R (c + m^* \pi) \left(\Pi_{j=2}^{i} [r + j\pi] \right) \left(\frac{\underline{m} - m^*}{c + m^* \pi} \right)^{i+1}
\end{aligned}$$

or

$$\frac{f}{R (c + m^* \pi)} = \frac{1}{2} \left(\frac{\underline{m} - m^*}{c + m^* \pi} \right)^2 + \sum_{i=2} \frac{(-1)^{i-1}}{(i+1)!} \; \left(\Pi_{j=2}^{i} [r + j\pi] \right) \left(\frac{\underline{m} - m^*}{c + m^* \pi} \right)^{i+1}$$

and hence

$$\begin{aligned}
\frac{f}{R (c + m^* \pi)} &= \left(\frac{\underline{m} - m^*}{c + m^* \pi} \right)^2 \left[\frac{1}{2} + \sum_{i=2} \frac{(-1)^{i-1}}{(i+1)!} \; \left(\frac{\underline{m} - m^*}{c + m^* \pi} \right)^{i-1} \left(\Pi_{j=2}^{i} [r + j\pi] \right) \right] \\
&= \left(\frac{m^* - \underline{m}}{c + m^* \pi} \right)^2 \left[\frac{1}{2} + \sum_{k=1} \frac{1}{(k+2)!} \; \left(\Pi_{j=2}^{k+1} [r + j\pi] \right) \left((-1) \frac{\underline{m} - m^*}{c + m^* \pi} \right)^k \right] \\
&= \left(\frac{m^* - \underline{m}}{c + m^* \pi} \right)^2 \left[\frac{1}{2} + \sum_{k=1} \frac{1}{(k+2)!} \; \left(\Pi_{j=2}^{k+1} [r + j\pi] \right) \left(\frac{m^* - \underline{m}}{c + m^* \pi} \right)^k \right]
\end{aligned}$$

To see how $z = (m^* - \underline{m}) / (c + m^* \pi) > 0$ depends on each of its 3 determinants we write

$$\theta = z^2 \; \psi (z, r, \pi)$$

where

$$\begin{aligned}
\psi (z, r, \pi) &= \left[\frac{1}{2} + \sum_{k=1} \frac{1}{(k+2)!} \; \left(\Pi_{j=2}^{k+1} [r + j\pi] \right) z^k \right] \\
\theta &= \frac{f}{R (c + m^* \pi)}
\end{aligned}$$

The function ψ is increasing in $z,\ r$ and π, hence:

$$\log\theta = 2\log z + \log\psi\left(z, r, \pi\right)$$

and

$$\frac{1}{\theta} = \frac{2}{z}\frac{\partial z}{\partial\theta} + \frac{\psi_z}{\psi}\frac{\partial z}{\partial\theta}$$

or

$$1 = 2\frac{\theta}{z}\frac{\partial z}{\partial\theta} + \frac{\psi_z}{\psi}z\frac{\theta}{z}\frac{\partial z}{\partial\theta}$$

or

$$\frac{\partial z}{\partial\theta}\frac{\theta}{z} = \frac{1/2}{\left(1 + z\ \psi_z/\left(2\psi\ \right)\right)}$$

Since $\frac{\psi_z}{\psi} > 0$ then $0 < \frac{\partial z}{\partial\theta}\frac{\theta}{z} < \frac{1}{2}$, i.e. the elasticity of z with respect to θ (hence to $\frac{f}{R}$) is smaller than one half.

Note moreover that (for given m^*)

$$\begin{aligned}
0 &= \frac{2}{z}\frac{\partial z}{\partial r} + \frac{1}{\psi}\left(\psi_z\frac{\partial z}{\partial r} + \psi_r\right) \\
0 &= 2\frac{r}{z}\frac{\partial z}{\partial r} + \frac{zr}{z\psi}\psi_z\frac{\partial z}{\partial r} + \frac{r}{\psi}\psi_r \\
0 &= \frac{r}{z}\frac{\partial z}{\partial r}\left(2 + \frac{z\psi_z}{\psi}\right) + \frac{r}{\psi}\psi_r \\
\frac{r}{z}\frac{\partial z}{\partial r} &= -\frac{\frac{r}{\psi}\psi_r}{\left(2 + \frac{z\psi_z}{\psi}\right)} < 0
\end{aligned}$$

which implies that z (hence the inaction range) is decreasing in r .

Note finally that (for given m^*):

$$\begin{aligned}
\frac{1}{\theta}\frac{\partial\theta}{\partial\pi} &= \frac{2}{z}\frac{\partial z}{\partial\pi} + \frac{1}{\psi}\left(\psi_z\frac{\partial z}{\partial\pi} + \psi_\pi\right) \\
\frac{\pi}{\theta}\frac{\partial\theta}{\partial\pi} &= 2\frac{\pi}{z}\frac{\partial z}{\partial\pi} + \frac{z\pi}{z\psi}\psi_z\frac{\partial z}{\partial\pi} + \frac{\pi}{\psi}\psi_\pi \\
\frac{\pi}{\theta}\frac{\partial\theta}{\partial\pi} - \frac{\pi}{\psi}\psi_\pi &= \frac{\pi}{z}\frac{\partial z}{\partial\pi}\left(2 + \frac{z\psi_z}{\psi}\right)
\end{aligned}$$

or

$$\frac{\pi}{z}\frac{\partial z}{\partial\pi} = \pi\frac{\frac{1}{\theta}\frac{\partial\theta}{\partial\pi} - \frac{1}{\psi}\psi_\pi}{2 + \frac{z\psi_z}{\psi}}$$

which implies that z (hence the inaction range) is decreasing in r if $\pi > 0$, increasing otherwise.

QED.

Proof of Proposition 16

Let Δn be the fraction of agents that withdraw during a period of time of length Δ :

$$\Delta n = \Delta p H\left(\underline{m}\right) + \Delta n \cdot \left(1 - \Delta p\right)^{\tau^*/\Delta} \tag{81}$$

where τ^*/Δ is the number of periods of length Δ that it takes for an agent with cash $\underline{m}$ consuming Δc to get cash $m = 0$ if it does not have any contact with the FI. This quantity solves

$$\underline{m} = c \int_0^{\tau^*} \exp\left(\pi \ s\right) \ ds$$

or

$$\tau^* = \frac{1}{\pi} \log\left(1 + \underline{m}\pi/c\right)$$

Let

$$\begin{aligned} H\left(\underline{m}\right) &= 1 - n\Delta \ \frac{\hat{\tau}}{\Delta} = 1 - n \ \hat{\tau} = \text{fraction with cash } \in \left(0, \underline{m}\right) \\ &= 1 - \text{ fraction with cash } \in \left(\underline{m}, m^*\right) \end{aligned}$$

where $\hat{\tau}/\Delta$ is number of periods of length Δ that an agent with m^* takes to get cash $m = \underline{m}$ consuming Δc . Notice that the fraction with cash $\in (\underline{m}, m^*)$ equals the fraction who withdraw $\times$ number of periods until cash reaches $\underline{m}$, or $n\Delta \ \times \ \hat{\tau}/\Delta = n \ \times \hat{\tau}$. We have

$$\left(m^* - \underline{m}\right) = c \int_0^{\hat{\tau}} \exp\left(\pi \ s\right) \ ds$$

or

$$\hat{\tau} = \frac{1}{\pi} \log\left(1 + \left(m^* - \underline{m}\right)\pi/c\right) \tag{82}$$

Then

$$\Delta n = \Delta p \left(1 - n\hat{\tau}\right) + \Delta n \cdot \left(1 - \Delta p\right)^{\tau^*/\Delta}$$

or

$$n \left[1 + p\hat{\tau} - \left(1 - \Delta p\right)^{\tau^*/\Delta}\right] = p$$

or

$$n\left(\Delta\right) = \frac{p}{\left[1 + p\hat{\tau} - \left(1 - \Delta p\right)^{\tau^*/\Delta}\right]}$$

or

$$\begin{aligned}\lim_{\Delta\to 0} n(\Delta) &= \lim_{\Delta\to 0}\frac{p}{1+p\hat{\tau}-\left[(1-\Delta p)^{1/\Delta}\right]^{\tau^*}}\\ &= \lim_{\Delta\to 0}\frac{p}{1+p\hat{\tau}-\exp(-p)^{\tau^*}}\end{aligned}$$

since

$$\log(1-\Delta p)^{1/\Delta}=\frac{1}{\Delta}\log(1-\Delta p)=\frac{-\Delta p+o(\Delta p)}{\Delta}=-p+\frac{o(\Delta p)}{\Delta}$$

or

$$\lim_{\Delta\to 0}\exp\left(\log(1-\Delta p)^{1/\Delta}\right)=\exp(-p)$$

and

$$\begin{aligned}\exp(-p)^{\tau^*} &= \exp(-p)^{\frac{1}{\pi}\log(1+\underline{m}\pi/c)}\\ &= \exp\left(\log(1+\underline{m}\pi/c)^{-p/\pi}\right)\\ &= (1+\underline{m}\pi/c)^{-\frac{p}{\pi}}\end{aligned}$$

so

$$n=\frac{p}{(p/\pi)\log(1+(m^*-\underline{m})\pi/c)+1-(1+\underline{m}\pi/c)^{-\frac{p}{\pi}}}$$

and

$$\begin{aligned}H(\underline{m}) &= 1-n\,\hat{\tau}=1-\frac{p\frac{1}{\pi}\log(1+(m^*-\underline{m})\pi/c)}{(p/\pi)\log(1+(m^*-\underline{m})\pi/c)+1-(1+\underline{m}\pi/c)^{-\frac{p}{\pi}}}\\ &= \frac{(p/\pi)\log(1+(m^*-\underline{m})\pi/c)+1-(1+\underline{m}\pi/c)^{-\frac{p}{\pi}}-(p/\pi)\log(1+(m^*-\underline{m})\pi/c)}{(p/\pi)\log(1+(m^*-\underline{m})\pi/c)+1-(1+\underline{m}\pi/c)^{-\frac{p}{\pi}}}\\ &= \frac{1-(1+\underline{m}\pi/c)^{-\frac{p}{\pi}}}{(p/\pi)\log(1+(m^*-\underline{m})\pi/c)+1-(1+\underline{m}\pi/c)^{-\frac{p}{\pi}}}\end{aligned}$$

QED.

Proof of Proposition 17

I. Case $m\in(0,\underline{m})$.

The density for real cash balances satisfies:

$$\frac{\partial h(m)}{\partial m}=\frac{(p-\pi)}{(\pi m+c)}h(m)$$

so that

$$h(m) = A_0 \left(1 + \frac{\pi}{c} m\right)^{\frac{p}{\pi} - 1}$$

for some positive constant A_0 with CDF

$$H(m) = \frac{c}{p} A_0 \left[1 + \frac{\pi}{c} m\right]^{\frac{p}{\pi}} - B_0$$

where B_0 is a constant to determine.

The CDF has to satisfy two boundary conditions:

$$\begin{aligned} H(0) &= 0, \\ H(\underline{m}) &= \frac{1 - (1 + \underline{m}\pi/c)^{-\frac{p}{\pi}}}{(p/\pi)\log(1 + (m^* - \underline{m})\pi/c) + 1 - (1 + \underline{m}\pi/c)^{-\frac{p}{\pi}}} \end{aligned}$$

These conditions imply:

$$\begin{aligned} H(m) &= \frac{c}{p} A_0 \left(1 + \frac{\pi}{c} m\right)^{\frac{p}{\pi}} - B_0 \\ A_0 &= \frac{p}{c} \frac{1}{\left[\left[1 + \frac{\pi}{c}\underline{m}\right]^{\frac{p}{\pi}} - 1\right]} H(\underline{m}) \\ B_0 &= \frac{c}{p} A_0 \end{aligned}$$

which yield equations (46) and (47) in the proposition.

Case II. For $m \in (\underline{m}, m^*)$.

The density satisfies

$$\frac{\partial h(m)}{\partial m} = \frac{-\pi}{(\pi m + c)} h(m)$$

so it is given by

$$h(m) = A_1 \left(1 + \frac{\pi}{c} m\right)^{-1}$$

for some constant A_1 with CDF

$$H(m) = \frac{c}{\pi} A_1 \log\left(1 + \frac{\pi}{c} m\right) - B_1$$

for some constant B_1.

The CDF has to satisfy two boundary conditions:

$$
\begin{aligned}
H\left(m^*\right) &= 1, \\
H\left(\underline{m}\right) &= \frac{1-\left(1+\underline{m}\pi/c\right)^{-\frac{p}{\pi}}}{\left(p/\pi\right)\log\left(1+\left(m^*-\underline{m}\right)\pi/c\right)+1-\left(1+\underline{m}\pi/c\right)^{-\frac{p}{\pi}}}
\end{aligned}
$$

These conditions imply:

$$
\begin{aligned}
H\left(m\right) &= \frac{c}{\pi}A_1\log\left(1+\frac{\pi}{c}m\right)-B_1 \\
A_1 &= \frac{\left(1-H\left(\underline{m}\right)\right)\left(\pi/c\right)}{\log\left(1+\frac{\pi}{c}m^*\right)-\log\left(1+\frac{\pi}{c}\underline{m}\right)} \\
B_1 &= \frac{c}{\pi}A_1\log\left(1+\frac{\pi}{c}m^*\right)-1
\end{aligned}
$$

which yield equations (49) and (91) in the proposition.
QED.

Proof of Proposition 18

$$
\int_0^{\underline{m}} mh\left(m\right)dm = \left[H\left(\underline{m}\right)\underline{m}-H\left(0\right)0-\int_0^{\underline{m}}H\left(m\right)dm\right]
$$

$$
\begin{aligned}
\int_0^{\underline{m}} H\left(m\right)dm &= \int_0^{\underline{m}}\frac{A_0}{p/c}\left(1+\frac{\pi}{c}m\right)^{\frac{p}{\pi}}dm-B_0\left(\underline{m}\right) \\
&= \frac{A_0}{\left(p/c\right)\left(p+\pi\right)/c}\left[\left(1+\frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}+1}-1\right]-B_0\underline{m} \\
&= \frac{A_0}{p/c}\left[\frac{\left(1+\frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}+1}-1}{\left(p+\pi\right)/c}-\underline{m}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{\underline{m}}^{m^*} mh\left(m\right)dm &= \left[H\left(m^*\right)m^*-H\left(\underline{m}\right)\underline{m}-\int_{\underline{m}}^{m^*}H\left(m\right)dm\right] \\
&= \left[m^*-H\left(\underline{m}\right)\underline{m}-\int_{\underline{m}}^{m^*}H\left(m\right)dm\right]
\end{aligned}
$$

$$
\int_{\underline{m}}^{m^*} H\left(m\right)dm = \int_{\underline{m}}^{m^*}\frac{c}{\pi}A_1\log\left(1+\frac{\pi}{c}m\right)dm-B_1\left(m^*-\underline{m}\right)
$$

where

$$\int_{\underline{m}}^{m^*} \log\left(1+\frac{\pi}{c}m\right)dm = \frac{c}{\pi}\left(1+\frac{\pi}{c}m\right)\left[\log\left(1+\frac{\pi}{c}m\right)-1\right]|_{\underline{m}}^{m^*}$$

Hence

$$\begin{aligned}
&\int_{\underline{m}}^{m^*} H(m)\,dm \\
= \;& A_1\left(\frac{c}{\pi}\right)^2\left\{\left(1+\frac{\pi}{c}m^*\right)\left[\log\left(1+\frac{\pi}{c}m^*\right)-1\right]-\left(1+\frac{\pi}{c}\underline{m}\right)\left[\log\left(1+\frac{\pi}{c}\underline{m}\right)-1\right]\right\}-B_1\left(m^*-\underline{m}\right)
\end{aligned}$$

Thus

$$\begin{aligned}
M \;&=\; m^* - \int_0^{\underline{m}} H(m)\,dm - \int_{\underline{m}}^{m^*} H(m)\,dm \\
&=\; \frac{A_0}{p/c}\left[\frac{\left[1+\frac{\pi}{c}\underline{m}\right]^{\frac{p}{\pi}+1}-1}{(p+\pi)/c}-\underline{m}\right]+B_1\left(m^*-\underline{m}\right)+ \\
&\quad -A_1\left(\frac{c}{\pi}\right)^2\left\{\left(1+\frac{\pi}{c}m^*\right)\left[\log\left(1+\frac{\pi}{c}m^*\right)-1\right]-\left(1+\frac{\pi}{c}\underline{m}\right)\left[\log\left(1+\frac{\pi}{c}\underline{m}\right)-1\right]\right\}
\end{aligned}$$

QED.

Proof of Proposition 19

The expression

$$\frac{\int_0^{\underline{m}}\left(m^*-m\right)h(m)\,dm}{H\left(\underline{m}\right)}$$

is the expected withdrawal conditional on being done by an agent with $m > 0$, or conditional on being a withdrawal that happens due to a chance meeting with the intermediary.

$$\int_0^{\underline{m}}\left(m^*-m\right)h(m)\,dm = m^*H\left(\underline{m}\right)-\int_0^{\underline{m}} mh(m)\,dm$$

$$\int_0^{\underline{m}} mh(m)\,dm = \underline{m}H\left(\underline{m}\right)-\int_0^{\underline{m}} H(m)\,dm$$

with

$$\int_0^{\underline{m}} H(m)\,dm = \frac{A_0}{p/c}\left[\frac{\left(1+\frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}+1}-1}{(p+\pi)/c}-\underline{m}\right]$$

Thus

$$\int_0^{\underline{m}} (m^* - m)\, h(m)\, dm = (m^* - \underline{m})\, H(\underline{m}) + \frac{A_0}{p/c}\left[\frac{\left(1+\frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}+1} - 1}{(p+\pi)/c} - \underline{m}\right]$$

$$\left(\frac{A_0}{p/c}\right)/H(\underline{m}) = \frac{1}{\left(1+\frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}} - 1}$$

so

$$\begin{aligned}\frac{\int_0^{\underline{m}} (m^* - m)\, h(m)\, dm}{H(\underline{m})} &= (m^* - \underline{m}) + \frac{A_0}{p/c}\left[\frac{\left(1+\frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}+1} - 1}{(p+\pi)/c} - \underline{m}\right] \\ &= (m^* - \underline{m}) + \frac{\frac{\left(1+\frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}+1}-1}{(p+\pi)/c} - \underline{m}}{\left(1+\frac{\pi}{c}\underline{m}\right)^{\frac{p}{\pi}} - 1}\end{aligned}$$

QED.

C Appendix: Expressions for zero inflation $(\pi = 0)$

This appendix collects the expression that are obtained in the case of $\pi = 0$. In most cases they have to be obtained by using L'Hopital rule in the corresponding formulas for the general case.

The expression for m^* in Proposition 2:which for $\pi = 0$ takes the form:

$$\exp\left(\frac{m^*}{c}(r+p)\right) = 1 + \frac{m^*}{c}(r+p) \; + (r+p)^2 \frac{b}{cR} \; . \tag{83}$$

The expression for the value function in Proposition 3 for $\pi = 0$ takes the form

$$V_u(m) = \left[\frac{pV^*(r+p) - Rc}{(r+p)^2}\right] + \left[\frac{R}{r+p}\right] m + \left(\frac{c}{r+p}\right)^2 A \exp\left(-\frac{r+p}{c} m\right) \; .$$

The expression for the expected number of trips per unit of time n in Proposition 5 for $\pi = 0$ takes the form

$$n(m^*; c, 0, p) = \frac{p}{1 - e^{-m^* \frac{p}{c}}} \tag{84}$$

The expression for the density of the distribution of real cash balances in Proposition 6 for $\pi = 0$ takes the form

$$h(m) = \frac{\frac{p}{c}\exp\left(\frac{mp}{c}\right)}{\exp\left(\frac{m^* p}{c}\right) - 1} \tag{85}$$

The expression for aggregate money balances in Proposition 7 for $\pi = 0$

$$M = \mu\left(m^*; c, \pi, p\right) \equiv c\left[\frac{1}{1-e^{-\frac{p}{c}m^*}}\frac{m^*}{c} - 1/p\right] . \qquad (86)$$

C.1 The model with costly random withdrawals for the $\pi = 0$ case

Proposition 21 *When $\pi = 0$, for a given V^* and $0 < \underline{m} < \bar{m}$ the solution of $V_u(m)$ for $m \in (\underline{m}, \bar{m})$ is given by:*

$$\begin{aligned} V_u(m) &= \varphi\left(m, A_\varphi\right) \equiv \\ &\equiv \left[\frac{-Rc}{r^2}\right] + \left[\frac{R}{r}\right] m + \left(\frac{c}{r}\right)^2 A_\varphi \exp\left(-\frac{r}{c}m\right) . \end{aligned}$$

and

$$\begin{aligned} V_u(m) &= \eta\left(m, V^*, A_\eta\right) \equiv \\ &\left[\frac{p\left(V^*+f\right)(r+p) - Rc}{(r+p)^2}\right] + \left[\frac{R}{r+p}\right] m + \left(\frac{c}{r+p}\right)^2 A_\eta \exp\left(-\frac{r+p}{c}m\right) . \end{aligned}$$

for $m \in (0,\ \underline{m})$ or $m \in (\bar{m},\ m^{**})$.

Proposition 22 *For $\pi = 0$ the range of inaction $(m^* - \underline{m})$ is given by:*

$$\frac{f\ c}{R} = \left[m^* - \underline{m}\right]^2 \left(\frac{1}{2} + \sum_{j=3}^{\infty} \frac{1}{j!}\left[\left(m^* - \underline{m}\right)\ \frac{r}{c}\right]^{j-2}\right) \qquad (87)$$

C.1.1 Range of inaction when $\pi = 0$

Calculations for $m^* - \underline{m}$ for the case of $\pi = 0$. To see how we obtain the result for $\pi = 0$, start with the expression for $z^* = m^* - \underline{m}$:

$$z^* = \frac{1}{r/c}\left(\exp\left[z^*\ \frac{r}{c}\right] - 1\right) - f\frac{r}{R}.$$

Write this expression as:

$$\begin{aligned} \exp\left[z^*\ \frac{r}{c}\right] &= 1 + \left[z^*\ \frac{r}{c}\right] + \frac{1}{2}\left[z^*\ \frac{r}{c}\right]^2 + \frac{1}{3!}\left[z^*\ \frac{r}{c}\right]^3 + ... \\ &= 1 + \left[z^*\ \frac{r}{c}\right] + \left[z^*\ \frac{r}{c}\right]^2 \left(\frac{1}{2} + \sum_{j=3}^{\infty}\frac{1}{j!}\left[z^*\ \frac{r}{c}\right]^{j-2}\right) \end{aligned}$$

hence

$$z^* = \frac{1}{r/c}\left(1 + \left[z^*\ \frac{r}{c}\right] + \left[z^*\ \frac{r}{c}\right]^2\left(\frac{1}{2} + \sum_{j=3}^{\infty}\frac{1}{j!}\left[z^*\ \frac{r}{c}\right]^{j-2}\right) - 1\right) - f\frac{r}{R}$$

or

$$f\frac{r}{R} = \left([z^*]^2\left(\frac{r}{c}\right)\left(\frac{1}{2} + \sum_{j=3}^{\infty}\frac{1}{j!}\left[z^*\ \frac{r}{c}\right]^{j-2}\right)\right)$$

or

$$\frac{f\ c}{R} = [m^* - \underline{m}]^2\left(\frac{1}{2} + \sum_{j=3}^{\infty}\frac{1}{j!}\left[z^*\ \frac{r}{c}\right]^{j-2}\right)$$

QED.

C.1.2 CDF for $\pi = 0$

For $m \in (0, \underline{m})$ we have:

$$h(m) = A_0 \exp(pm/c)$$

and

$$H(m) = \frac{A_0}{p/c}\exp(pm/c) - B_0$$

hence:

$$H(m) = \frac{A_0}{p/c}\exp(pm/c) - B_0 \tag{88}$$

$$H(\underline{m}) = \frac{1 - \exp(-p(\underline{m}/c))}{p(m^* - \underline{m})/c + 1 - \exp(-p(\underline{m}/c))}$$

$$A_0 = \frac{H(\underline{m})(p/c)}{[\exp(p\underline{m}/c) - 1]} \tag{89}$$

$$B_0 = \frac{A_0}{p/c} \tag{90}$$

For $m \in (\underline{m}, m^*)$ we have

$$h(m) = A_1 c$$

$$H(m) = A_1 m/c - B_1$$

Hence

$$\begin{aligned}
H(m) &= \frac{A_1}{\pi}\log\left(1+\frac{\pi}{c}m\right)-B_1 && (91)\\
[1-H(\underline{m})] &= \frac{p(m^*-\underline{m})/c}{p(m^*-\underline{m})/c+1-\exp(-p\underline{m}/c)}\\
A_1 &= \frac{1-H(\underline{m})}{(m^*-\underline{m})/c} && (92)\\
B_1 &= A_1 m^*/c-1 && (93)
\end{aligned}$$

C.1.3 Average money holdings for $\pi = 0$

Proposition 23 *The average (expected) real money holdings for $\pi = 0$ is*

$$\begin{aligned}
M &= m^* - \frac{A_0}{(p/c)}\left\{\frac{[\exp(p\underline{m}/c)-1]}{(p/c)}-\underline{m}\right\} && (94)\\
&\quad -\frac{A_1}{c}\left((m^*)^2-(\underline{m})^2\right)+[A_1m^*/c-1](m^*-\underline{m})
\end{aligned}$$

where A_0, A_1 and B_1 are given in (89),(92) and (93).

Proof. For $\pi = 0$ we have

$$\begin{aligned}
M &= m^* - \int_0^{\underline{m}} H(m)\,dm - \int_{\underline{m}}^{m^*} H(m)\,dm\\
&= m^* - \frac{A_0}{(p/c)}\left\{\frac{[\exp(p\underline{m}/c)-1]}{(p/c)}-\underline{m}\right\}\\
&\quad -\frac{A_1}{c}\left((m^*)^2-(\underline{m})^2\right)+[A_1m^*/c-1](m^*-\underline{m})
\end{aligned}$$

$$\begin{aligned}
\int_0^{\underline{m}} H(m)\,dm &= \int_0^{\underline{m}}\left[\frac{A_0}{p/c}\exp(pm/c)-B_0\right]dm\\
&= \frac{A_0}{(p/c)(p/c)}[\exp(p\underline{m}/c)-1]-B_0\underline{m}\\
&= \frac{A_0}{(p/c)}\left\{\frac{[\exp(p\underline{m}/c)-1]}{(p/c)}-\underline{m}\right\}
\end{aligned}$$

$$\int_{\underline{m}}^{m^*} H(m)\,dm = \int_{\underline{m}}^{m^*} [A_1 m/c - B_1]\,dm$$
$$= \frac{A_1}{c}\left((m^*)^2 - (\underline{m})^2\right) - [A_1 m^*/c - 1](m^* - \underline{m})$$

C.1.4 Average withdrawal for $\pi = 0$

Proposition 24 *If $\pi = 0$ the average withdrawal W is given by:*

$$W = m^* \left[1 - \frac{p}{n} H(\underline{m})\right] + \left[\frac{p}{n} H(\underline{m})\right] \frac{\int_0^{\underline{m}} (m^* - m)\, h(m)\, dm}{H(\underline{m})} \tag{95}$$

where

$$\frac{\int_0^{\underline{m}} (m^* - m)\, h(m)\, dm}{H(\underline{m})} = m^* - \underline{m} - \frac{\frac{[\exp(p\underline{m}/c) - 1]}{(p/c)} - \underline{m}}{\exp(p\underline{m}/c) - 1}$$

D Appendix: Calibration for b and f

How to find b and f given $(m^*,\ \underline{m},\ r, \pi, R)$

For convenience we rewrite equation (80) for $m_2^*(\cdot)$:

$$m^* = \left[\frac{r}{r+\pi}\right]\underline{m} + \frac{c}{(r+\pi)}\left(\frac{\left[1 + \frac{\pi}{c} m^*\right]^{1+\frac{r}{\pi}}}{\left[1 + \frac{\pi}{c}\underline{m}\right]^{\frac{r}{\pi}}} - 1\right) - f\frac{r}{R}$$

to find f. It is given by:

$$f = \frac{\left[\frac{r}{r+\pi}\right]\underline{m} + \frac{c}{(r+\pi)}\left(\frac{\left[1+\frac{\pi}{c} m^*\right]^{1+\frac{r}{\pi}}}{\left[1+\frac{\pi}{c}\underline{m}\right]^{\frac{r}{\pi}}} - 1\right) - m^*}{r/R}$$

Given f and $(m^*,\ \underline{m},\ r, \pi, R, p)$ use equation (79) for $m_1^*(\cdot)$:

$$m^* = \frac{\left(\frac{c}{r+p}\right)\left[\frac{p\ f}{c} - \frac{R}{(r+p+\pi)}\right]}{\left(\frac{R}{r+p}\right)} + \frac{\left[\frac{R}{r+p+\pi}\right]\underline{m} + b\left[1 + \frac{\pi}{c}\underline{m}\right]^{-\frac{r+p}{\pi}} - f}{\left(\frac{R}{r+p}\right)\left[1 - \left[1 + \frac{\pi}{c}\underline{m}\right]^{-\frac{r+p}{\pi}}\right]}$$

to find b. It is given by

$$b = \frac{\left(m^* - \frac{\left(\frac{c}{r+p}\right)\left[\frac{p}{c}\frac{f}{} - \frac{R}{(r+p+\pi)}\right]}{\left(\frac{R}{r+p}\right)}\right)\left(\frac{R}{r+p}\right)\left[1 - \left[1 + \frac{\pi}{c}\underline{m}\right]^{-\frac{r+p}{\pi}}\right] - \left[\frac{R}{r+p+\pi}\right]\underline{m} + f}{\left[1 + \frac{\pi}{c}\underline{m}\right]^{-\frac{r+p}{\pi}}}$$

E Appendix: Properties of function ξ

To see why $\frac{\partial \xi}{\partial n} < 0$ notice that

$$\begin{aligned}
\frac{\partial \xi}{\partial n} &= \left(\frac{1}{p}\right)^2 \left[\log\left(\frac{n}{n-p}\right) - \frac{p}{n-p}\right] \\
&= \left(\frac{1}{p}\right)^2 \left[\log\left(1 + \frac{p}{n-p}\right) - \frac{p}{n-p}\right] \\
&\leq \left(\frac{1}{p}\right)^2 \left[\frac{p}{n-p} - \frac{p}{n-p}\right] = 0
\end{aligned}$$

To see why $\frac{\partial \xi}{\partial p} > 0$ write

$$\begin{aligned}
\frac{\partial \xi}{\partial p} &= -\frac{1}{p^2}\left[-\frac{n}{p}\log\left(1 - \frac{p}{n}\right) - 1\right] + \frac{1}{p}\left[\frac{n}{p^2}\log\left(1 - \frac{p}{n}\right) + \frac{n}{p}\frac{1}{n-p}\right] \\
&= -\frac{1}{p^2}\left[-\frac{n}{p}\log\left(1 - \frac{p}{n}\right) - 1\right] + \frac{1}{p^2}\left[\frac{n}{p}\log\left(1 - \frac{p}{n}\right) + \frac{n}{n-p}\right] \\
&= \frac{1}{p^2}\left[2\frac{n}{p}\log\left(1 - \frac{p}{n}\right) + 1 + \frac{n}{n-p}\right] \\
&= \frac{1}{p^2}\left[2\frac{n}{p}\log\left(1 - \frac{p}{n}\right) + \frac{n}{p}\frac{p}{n} + \frac{n}{p}\frac{p}{n-p}\right] \\
&= \frac{1}{p^2}\frac{n}{p}\left[2\log\left(1 - \frac{p}{n}\right) + 1 + \frac{p/n}{1-p/n}\right]
\end{aligned}$$

or

$$\frac{\partial \xi}{\partial p} = \frac{1}{p^2}\frac{n}{p}\, h\left(p/n\right)$$

where we define h as

$$h\left(x\right) = 2\log\left(1-x\right) + 1 + \frac{x}{1-x}.$$

Notice that

$$h\left(0\right) = 1,$$

and

$$h'(x) = -\frac{2}{1-x} + \frac{1}{(1-x)^2} = \frac{1}{(1-x)^2}\left[-2(1-x)+1\right]$$

so

$$h'\left(\frac{1}{2}\right) = 0,\ h'(x) < 0 \text{ for } x < 1/2,\ h'(x) > 0 \text{ for } x > 1/2.$$

thus the minimum of h is achieved at $x = 1/2$. Hence

$$h\left(\frac{1}{2}\right) = 2\log\left(\frac{1}{2}\right) + 1 + 1 = 2\left(\log\left(\frac{1}{2}\right) + 1\right) > 0$$

and thus

$$\frac{\partial \xi}{\partial p} = \frac{1}{p^2}\frac{n}{p}\ h(p/n) \geq \frac{1}{p^2}\frac{n}{p}\ h\left(\frac{1}{2}\right) > 0.$$

Setting $f = \pi = 0$ into the expression for b :

$$b = \frac{\left(c\ m^*/c + \left(\frac{c}{r+p}\right)\right)\left(\frac{R}{r+p}\right)\left[1 - \exp\left(-(r+p)\,m^*/c\right)\right] - c\left[\frac{R}{r+p}\right] m^*/c}{\exp\left(-(r+p)\,m^*/c\right)}$$

or

$$\begin{aligned}
b/c &= \frac{\left(m^*/c + \frac{c}{r+p}\right)\left(\frac{R}{r+p}\right)\left[1 - \exp\left(-(r+p)\,m^*/c\right)\right] - \left[\frac{R}{r+p}\right] m^*/c}{\exp\left(-(r+p)\,m^*/c\right)} \\
&= \frac{\frac{1}{r+p}\left(\frac{R}{r+p}\right)\left[1 - \exp\left(-(r+p)\,m^*/c\right)\right] - \left(\frac{R}{r+p}\right)(m^*/c)\exp\left(-(r+p)\,m^*/c\right)}{\exp\left(-(r+p)\,m^*/c\right)} \\
&= \left(\frac{R}{r+p}\right)\frac{\frac{1}{r+p} - \left[\frac{1}{r+p} + (m^*/c)\right]\exp\left(-(r+p)\,m^*/c\right)}{\exp\left(-(r+p)\,m^*/c\right)} \\
&= \left(\frac{R}{(r+p)^2}\right)\frac{1 - \left[1 + (r+p)\,(m^*/c)\right]\exp\left(-(r+p)\,m^*/c\right)}{\exp\left(-(r+p)\,m^*/c\right)} \\
&= \left(\frac{R}{(r+p)^2}\right)\left(\ \exp\left((r+p)\,m^*/c\right) - \left[1 + (r+p)\,(m^*/c)\right]\ \right)
\end{aligned}$$

To obtain the expression for ζ use

$$\frac{W}{M} = \frac{m^*}{M} - \frac{p}{n}$$

and

$$\frac{M}{m^*} = \frac{1}{p}\left[n - \frac{1}{m^*/c}\right],$$

$$
\begin{aligned}
\frac{W}{M} &= p\left[n - \frac{1}{m^*/c}\right]^{-1} - \frac{p}{n} \\
&= \left[\frac{n}{p} - \frac{1}{pm^*/c}\right]^{-1} - \frac{p}{n}
\end{aligned}
$$

and using that

$$
n/p = \frac{1}{1 - e^{-m^* \frac{p}{c}}}
$$

or

$$
m^* \frac{p}{c} = -\log\left(1 - p/n\right)
$$

we have

$$
\begin{aligned}
\frac{W}{M} &= \left[\frac{n}{p} - \frac{1}{-\log\left(1 - p/n\right)}\right]^{-1} - \frac{p}{n} \\
&= \zeta\left(n, p\right) = \left[\frac{1}{p/n} + \frac{1}{\log\left(1 - p/n\right)}\right]^{-1} - \frac{p}{n}
\end{aligned}
$$

Define

$$
\begin{aligned}
\phi\left(x\right) &\equiv \left[\frac{1}{x} + \frac{1}{\log\left(1 - x\right)}\right]^{-1} - x \\
&= \left[\frac{x \log\left(1 - x\right)}{\log\left(1 - x\right) + x}\right] - x
\end{aligned}
$$

$$
\begin{aligned}
\phi\left(x\right) &= \left[\frac{x \log\left(1 - x\right)}{\log\left(1 - x\right) + x}\right] - x = \frac{x\ \left(-x - \frac{1}{2}x^2 + o\left(x^2\right)\right)}{-x - \frac{1}{2}x^2 + o\left(x^2\right) + x} - x = \frac{\left(-x - \frac{1}{2}x^2 + o\left(x^2\right)\right)}{-1 - \frac{1}{2}x + \frac{o\left(x^2\right)}{x} + 1} - x \\
&= \frac{\left(-x - \frac{1}{2}x^2 + o\left(x^2\right)\right)}{-\frac{1}{2}x + \frac{o\left(x^2\right)}{x}} - x = \frac{\left(-1 - \frac{1}{2}x + \frac{o\left(x^2\right)}{x}\right)}{-\frac{1}{2} + \frac{o\left(x^2\right)}{x^2}} - x \\
\lim_{x \to 0} \phi\left(x\right) &= \frac{-1}{-\frac{1}{2}} = 2
\end{aligned}
$$

$$
\phi\left(x\right) = \frac{x \log\left(1 - x\right)}{\log\left(1 - x\right) + x} - x = \frac{x}{1 + \frac{x}{\log\left(1 - x\right)}} - x
$$

so

$$
\phi\left(1\right) = \frac{1}{1 + \frac{1}{\log(0)}} - 1 = \frac{1}{1 + \frac{1}{-\infty}} - 1 = 0\ .
$$

To see why $\phi'\left(x\right) < 0$ notice that [TO BE COMPLETED]

F Appendix: Additional evidence

Table 11: Households' currency holdings

Variable	1989	1991	1993	1995	1998	2000	2002	2004
Average currency[a]								
Household w/o account	20.5	20.1	12.7	14.9	14.1	14.9	19.1	15.8
Household w. account								
w/o ATM	14.5	14.2	10.2	11.4	11.9	11.5	10.9	10.6
w. ATM	11.6	9.0	6.5	6.9	6.9	6.2	6.1	6.1
Average withdrawal[a]								
Household w/o ATM	na	11.2	15.7	12.7	14.7	13.8	13.3	13.2
Household w. ATM	na	6.1	6.6	5.7	5.9	5.5	5.5	5.6
Minimum currency[a,b]								
Household w/o ATM	3.1	3.1	3.4	2.5	4.6	4.2	4.0	na
Household w. ATM	2.9	2.4	2.3	1.7	2.2	2.3	2.1	na
Number of withdrawals[c]								
Household w/o ATM	na	18.4	12.3	13.1	19.8	16.5	17.5	17.9
Household w. ATM	na	49.6	48.0	49.5	58.6	61.7	56.7	63.1
N. of Observations	8,274	8,188	8,089	8,135	7,147	8,001	8,011	8,012

Notes: [a]Ratio to non-durable daily consumption. - [b]Reported level of currency holdings that triggers a withdrawal. - [c]Per year. Source: Bank of Italy - *Survey of Household Income and Wealth*; entries computed using sample weights.

Table 12: The demand for currency and financial diffusion

Dependent variable: $log(M)$	OLS estimates		empty col
log(consumption)	0.46	0.45	
	(0.01)	(0.01)	
log(interest rate)	-0.24	-0.44	
	(0.05)	(0.06)	
log(interest rate) · bank-service diffusion		0.41	
		(0.07)	
bank-service diffusion		-0.56	
		(0.09)	
Dummy ATM card	-0.24	-0.24	
	(0.01)	(0.01)	
Sample size	28,244	28,244	
R^2	0.20	0.20	

Note: OLS regressions based on 1993-2002 surveys; standard errors in parenthesis. The regressors also include a constant, year dummies and 103 province dummies. Bank-service diffusion is defined as bank branches per capita at the province level (see the text).

Table 13: Number of withdrawals (c/o Bank and ATMs) per unit of consumption

Dependent variable: $log(n/c)$	Household w/o ATM	Household w. ATM	
	Bank wdrl.	Atm wdrl.	Bank + Atm wdrl.
log(interest rate)	0.33	0.36	0.40
	(0.17)	(0.13)	(0.12)
log(interest rate) $\cdot$ bank-branch diffusion	0.13	-0.09	-0.09
	(0.07)	(0.06)	(0.05)
bank-branch diffusion	0.06	0.20	0.18
	(0.09)	(0.08)	(0.08)
Sample size	9,834	14,160	15,030

Notes: OLS regressions based on 1993-2002 surveys; the dependent variable is the number of bank (ATM) trips scaled by the household real consumption. Robust standard errors (in parenthesis) are computed by clustering observations at the province*year, the finest level of disaggregation at which the interest rate is available. The regressors also include a constant, year dummies and 103 province dummies. Bank-branch diffusion is defined as bank branches per capita at the city level; The net nominal interest rate is measured in percent (see Table 2).