INFORMATION ON INFLATION EXPECTATIONS CONTAINED IN THE PRICES OF FINANCIAL ASSETS*

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The techniques to extract information on market expectations about future developments in variables relevant to the monetary policy, obtained from financial asset prices, have been studied and used by central banks in the past few years. In this article we try to summarise the techniques available to extract inflation expectations and the uncertainty associated with them, from information contained in the prices of financial assets. In particular, we focus our attention on the estimation of probability density functions of break-even inflation expectations.

1. INTRODUCTION

Financial asset prices reflect market participants’ expectations as they are forward looking in nature. As is known, the current price of a financial asset equals the present value of the future expected asset payoff. Thus, when setting the price of a financial asset, investors create expectations about the future value of that asset and fix the discount rates — including the risk premia inherent in that investment — to be applied. In a liquid and efficient market, the price of an asset should reflect, in principle, both the market opinion and the discount rates determining that price. In turn, discount rates used for the setting of prices of financial assets are driven by two factors: the compensation for the decision to postpone consumption and for the uncertainty linked to the future stream payoffs.

As the prices of financial assets are permanently updated they reflect continuously revised expectations, including the most recent information available in the market. In addition to giving information on market expectations, prices of some financial assets, namely derivatives, may also provide relevant information on the degree of uncertainty attached by the market to future developments. For example, options on long-term government bonds may indicate the degree of uncertainty associated with future developments of the yield on those bonds. Assuming certain hypotheses, these assets may also provide information on the uncertainty associated with break-even inflation expectations.

Therefore, financial asset prices constitute a large source of information for central banks. The information given by them is important for monetary policy as it provides a means of confirming the assessment by the central bank itself of risks for price stability, thus contributing to determine the monetary policy response which is appropriate to offset those risks. Moreover, for the above mentioned reasons, obtaining inflation expectations in financial markets is a fundamental strategic central banking area, as the expectations about future developments of some macroeconomic variables affect economic agents’ response to monetary policy decisions in the present.

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This article is organised as follows: section 2 analyses the methods to extract indicators of inflation expectations and section 3 focuses on the methods to extract the existing uncertainty indicators.

2. INDICATORS OF INFLATION EXPECTATIONS

The term structure of interest rates (or yield curve) is the relationship between the yield to maturity of default-free zero-coupon bonds(1) and their maturity.

A nominal interest rate or spot interest rate implied in a government bond may be decomposed into three components: (i) real interest rate expected, required by investors for holding the bond up to its maturity date; (ii) compensation for the inflation rate expected over the life of the bond; and (iii) compensation for the several risk premia involved (namely the liquidity risk premium and the inflation risk premium). Algebraically, for a zero-coupon security, which matures at moment \( t+m \), this relationship, also called Fischer identity, may be written:

\[
E_t(\pi_{t,m}) = i_{t,m} - r_{t,m} + \lambda_{t,m},
\]

being \( i_{t,m} \), the spot interest rate at moment \( t \) with maturity \( m \), \( r_{t,m} \) the real interest rate expected at \( t \) with a maturity \( m \), \( E_t(\pi_{t,m}) \) the average inflation rate expected at \( t \) for the period from \( t \) to \( t+m \) and \( \lambda_{t,m} \) a measure of the premia inherent in the several risks involved in this investment for the same period.

Thus, the changes in the term structure have implicit changes in one or several of these components. In particular, the extraction of information on inflation expectations from the yield curve is a delicate exercise, given the several explanatory factors involved. Besides, the very-short-term end of the yield curve (up to two years) is very sensitive to changes or to expectations of a change in the central bank intervention rates and therefore this additional explanatory factor should be taken into consideration, together with other factors. The normal practice of extracting inflation expectations from yield curves has thus been to restrict the analysis to the longest end of the yield curve (over two years), assuming that, in these maturities, both real interest rates and risk premia involved are relatively stable.

2.1 Mishkin’s approach

Since the term structure of interest rates contains information about future inflation, there must be therefore a way of quantifying the relationship between interest rates and inflation expectations. Mishkin (1990a and b, 1991), proposed a simple approach of estimating from an econometric perspective the relationship between interest rates and inflation expectations. In this analysis we assume that economic agents have rational expectations. Mishkin starts from Fischer identity without risk premium:

\[
E_t(\pi_{t,m}) = i_{t,m} - r_{t,m},
\]

And also:

\[
\pi_{t,m} = E_t(\pi_{t,m}) + e_{t,m},
\]

where \( e_{t,m} \) are the prediction errors of the inflation rate. Given the fact that rational expectations are assumed, we have \( E_t(e_{t,m}) = 0 \).

From (2.2) and (2.3) we get, by substitution:

\[
\pi_{t,m} = i_{t,m} - r_{t,m} + e_{t,m}.
\]

Considering two different maturities, \( m \) and \( n \), we may analyse the information contained in the term structure of interest rates about the path of future inflation. From the equation (2.4) applied to each maturity we get, calculating the respective difference:

\[
\pi_{t,m} - \pi_{t,n} = i_{t,m} - i_{t,n} - (r_{t,m} - r_{t,n}) + e_{t,m} - e_{t,n},
\]

deckomposing the average real interest rate into sample average plus deviation from the average, for both maturities, \( m \) and \( n \), we get:

\[
r_{t,m} = \bar{r}_m + u_{t,m},
\]

\[
r_{t,n} = \bar{r}_n + u_{t,n},
\]

---

(1) A zero-coupon bond is an asset, which generates at maturity a previously fixed single financial flow.
Hence, we may rewrite equation (2.5) as an equation to predict the change in:

$$\pi_{t,m} - \pi_{t,0} = \alpha + \beta (i_{t,m} - i_{t,0}) + \eta_{t,m},$$

(2.8)

with $\alpha = \bar{r}_w - \bar{r}_e \varepsilon \eta_{t,m} = (e_{t,w} - e_{t,e}) - (u_{t,0} - u_{t,0})$. Equation (2.8) may be estimated consistently through the method of the ordinary least square (OLS), provided that the hypothesis of rational expectations and a constant term structure of real interest rates occur. In this context hypothesis $\beta = 1$ may be tested.

The acceptance of hypothesis $\beta = 1$ indicates that the change in the spread of the nominal interest rate is a non-biased estimator for the change in the future inflation rate.

Chart 2.1 illustrates the relationship between the 10-2 year inflation differential\(^{(2)}\) and the spread between 10-2 year interest rates in Germany for the period from January 1980 to December 1999 (monthly data). The relationship between the inflation spread and the interest rate spread is positive, which makes it possible to confirm that the term structure of interest rates contains information on future inflation expectations, albeit being skewed\(^{(3)}\).

2.2 Calculation of inflation expectations with index-linked bonds

A more robust method to extract inflation expectations from financial assets arose with the launching of the issuance of index-linked bonds in several countries. An index-linked bond is an asset whose coupon and/or principal payments are linked to a price index, providing its holder with a guaranteed real yield. This type of bonds differs from conventional bonds, which guarantee to their holders a previously defined nominal yield. However, similarly to conventional bonds and considering that in developed countries the risk of government bankruptcy is negligible, the value of this asset in practice does not depend on the credibility of the issuer.

Several countries issued inflation-indexed bonds\(^{(4)}\). In the euro area, only the French Treasury issues index-linked government bonds. The first issue of inflation-indexed bonds in France (Obligation Assimilable du Trésor Indexée (OATI)) took place in September 1998. Currently, index-linked bonds mature in 2009 and 2029, with an annual coupon of 3 and 3.4 per cent respectively. The payment of the coupons and of the redemption of the OATI is linked to the French Consumer Price Index, excluding tobacco. The indexation lag is of 3 months at the most, therefore being considered negligible.

Suppose that $PR_{t,m}$ is the price of an OATI with maturity $m$. We admit that this index-linked bond pays in real terms a fixed coupon $CR$ in each moment $t+s_j$, with $s_j = m\(^{(5)}\). At the maturity date, the holder of the bond will be paid by the amount, also in real terms, $MR$. The payment at moment $t+s_j$, in real terms, of the coupon $CR$ means that the holder of the bond will receive $(1 + \pi_{j,s_j})$ where $CR \pi_{j,s_j}$ is the average annual inflation rate at period $t$ to $t+s_j$.

\(^{(2)}\) The inflation differential $\pi_{10} - \pi_2$ was calculated by the difference between the average inflation rate recorded in the ten subsequent years and the average inflation in the two subsequent years. For example, for December 1989 the inflation differential corresponds to the average inflation recorded between December 1989 and December 1999 less the average inflation recorded between December 1989 and December 1991.

\(^{(3)}\) One-off estimate of 0.757 with standard deviation of 0.12.

\(^{(4)}\) For a more detailed analysis, see Deacon and Derry (1998).

\(^{(5)}\) Given the fact that we are considering the year as a time unit, we have $s_j = s_{j-1} + 1 (j = 1, 2, \ldots, J)$.
Similarly, a real redemption of \( MR \) at the maturity date of the bond means that the holder will receive by that time \( \left(1 + \tau_{j+m}^{\text{MR}}\right)^{m} \) MR. The Yield-To-Maturity Real (YTMR) rate is the rate which makes it possible to equalise the market price of the index-linked bond to the expected future cash-flows, in real terms, generated by it:

\[
PR_{j,m} = \sum_{j=1}^{m} \frac{CR}{(1 + YTM)^{j}} + \frac{MR}{(1 + YTM)^{m}} \tag{2.9}
\]

or, equivalently in terms of nominal cash-flows:

\[
PR_{j,m} = \sum_{j=1}^{m} \frac{\left(1 + \pi_{j+j+m}^{\text{CR}}\right)^{j} CR}{(1 + \tau_{j+j+m}^{\text{CR}})^{j}} + \frac{\left(1 + \pi_{j+j+m}^{\text{MR}}\right)^{m} MR}{(1 + \tau_{j+j+m}^{\text{MR}})^{m}} \tag{1.10}
\]

Thus, the appreciation of OATIs does not need any hypothesis on inflation expectations at the maturity date of the bond.

The non-existence of a similar type of instrument in the euro area led to the frequent use of inflation expectations derived from OATIs as a proxy for inflation expectations in the long run for the euro area as a whole\(^{(6)}\).

2.2.1 Methodologies for the calculation of the break-even inflation rate\(^{(7)}\) for the euro area

The simultaneous issue of inflation-indexed securities, guaranteeing to their holders a defined real yield, and of securities guaranteeing a defined nominal yield, for the same maturity, enabled the development of methodologies for extracting inflation expectations from the prices of these instruments.

In the French case, a simple method of extracting inflation expectations from index-linked bonds consists in the direct application of the Fischer identity without risk premium. The average inflation expectation in the period is thus calculated as the difference between the nominal Yield-To-Maturity (YTM) of a conventional bond and the real YTM implied in an index-linked bond, issued for the same maturity and by the same issuer, in this case the French government. Implicitly we assume that investors are risk neutral, i.e., for investors it is indifferent to hold in their portfolios index-linked bonds or conventional bonds. This implies that, in the absence of arbitrage opportunities, the nominal yield guaranteed by an index-linked bond equals the nominal yield of a conventional bond. This equality enables the substitution of the nominal YTM of the index-linked bond by the YTM of a conventional bond. It should be stressed that the inflation expectation thus obtained is an average inflation expectation in the period from the present to the maturity date of the securities taken into consideration.

For example, subtracting the real YTM of an index-linked bond with the same residual maturity from the nominal YTM of a conventional bond with a five-year residual maturity, we get an average inflation expectation in the 5-year period up to the maturity date of both bonds.

2.2.2 Limitations to the use of index-linked bonds

The difference between the real and the nominal YTM may reveal the change in inflation expectations over time. However, this is a skewed indicator of the average level of inflation expected in a given period. This results basically from two sets of factors: first, the two types of bonds (index-linked and conventional) do not have the same type of characteristics, viz. time to maturity, structure of cash flows and degree of liquidity. Second, an inflation risk premium is inherent in nominal bonds; in principle, this type of risk premium does not exist in index-linked bonds.

With regard to the first set of factors, the problem arises from the fact that the average inflation expectation in a given period — being derived from only two securities — is very sensitive to the securities taken into consideration in this calculation. On the one hand, it is difficult to find two bonds, one of them index-linked to inflation and the other one nominal, with the same residual maturity. On the other hand, even if the two bonds have the same residual maturity, they can give rise

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\(^{(6)}\) It should be noted that the break-even inflation rate implied in these bonds is derived from the French CPI and not from the Harmonised Index of Consumer Prices (HICP), which is used by the Eurosystem in its definition of price stability. However, the difference is not relevant in practical terms.

\(^{(7)}\) Break-even inflation is a measure of inflation expectations extracted from index-linked bonds.
to a different cash flow structure\(^{(8)}\). An alternative way would be to calculate the average inflation expectation in the period from two securities with the same “duration”\(^{(9)}\). However, according to Deacon et al. (1994), this alternative would create difficulties, due to the fact that average inflation expectations thus obtained mask the residual maturities of the securities used for their calculation.

In addition to these factors, it should also be noted that index-linked bonds show a lower degree of liquidity than conventional bonds. This leads investors to demand a higher liquidity premium in order to hold index-linked bonds in their portfolios. As a result, real YTM is probably overestimated while the break-even inflation expectation is underestimated.

With regard to the second set of factors, the problem results from the fact that the conventional bond is still subject to inflation risk. Hence, the Fischer identity should include a measure of the inflation risk premium. We conclude that when the liquidity premium is not taken into consideration, break-even inflation rates may be underestimated, while they may be overestimated when the inflation premium is not taken into consideration.

The fact that the inflation premium is not considered in the calculation of expectations regarding break-even inflation is probably the major limitation of the methodology described above. For example, Remolona et al. (1998) — combining information on the real and the nominal yield curve — estimated for the United Kingdom from July 1982 to July 1997 inflation expectations, inflation risk premium, real interest rate and risk premium related to real factors. In this work, the real discount factor (price kernel) is a linear function of a real factor, while the nominal discount factor is a linear function of two factors, one of them identified with inflation and the other one identified with the real factor. For that period, the average inflation risk premium for the 2-year maturity was around 100 basis points, having declined to 70 basis points after the pound sterling left the European Monetary System. In sum, this work showed that, for the English case, the break-even inflation rate obtained by the direct application of the Fischer identity without risk premium overestimates average inflation expectations for the period. This type of research cannot be applied to the French case, due to the fact that only 10 and 30-year maturity bonds were issued.

2.2.3 An indicator of inflation expectations for the euro area

Given the difficulty in estimating adequately liquidity and inflation risk premia, average inflation expectations for the euro area implied in the prices of index-linked and conventional government bonds will be estimated below, assuming that the balance of these premia is relatively low in the euro area. This exercise is obviously interesting for the monetary policy, as it makes it possible for monetary authorities to calculate, at a very high frequency (daily), an indicator of economic agents’ expectations about the future inflation rate\(^{(10)}\).

Chart 2.2 illustrates the daily evolution of the 10-year nominal YTM\(^{(11)}\) in France, as well as the evolution of its two components: the real YTM and the break-even inflation rate, for the period from January 1999 to August 2000. As mentioned above, since index-linked government bonds exist only in France, the average inflation expectation obtained from OATIs is frequently used as a proxy for inflation in the long run for the euro area as a whole.

Considering the period as a whole, the upward trend of the nominal YTM in the euro area becomes apparent. However, the contribution of each component to this upward trend is not symmetrical. The growth of the nominal YTM in the first half of 1999 was mainly due to the increase in

\(^{(8)}\) For example, a nominal bond with a 10-year maturity, which has a semi-annual coupon payment of 6.5 per cent and a principal payment of 100 m.u., will give rise to a semi-annual coupon payment of 3.25 m.u. In the case of an index-linked bond with a real interest rate of 4.25 per cent and assuming an inflation rate of 3 per cent, the coupon received varies between 2.13 and 2.82 m.u. and the equivalent payment will be of 134 m.u. In real terms, the value of the coupons of an index-linked bond will be constant (equal to 2.13 m.u.), while a conventional bond will give rise to settlements, which decline due to the erosion effect caused by inflation.

\(^{(9)}\) “Duration” of a bond is defined as the average time the investor must wait to obtain the redemption of the principal invested, in which the average is calculated weighting the periods by the amounts to be received in those periods.

\(^{(10)}\) It should be recalled that surveys on consumer prices are an alternative method of extracting inflation expectations.

\(^{(11)}\) Nominal and real interest rates used as from this point correspond to YTMs implied in the prices of the respective bonds.
the break-even inflation rate, the real YTM having declined slightly. In the second half of 1999 the trend was reversed. The real YTM started an upward trend up to the first quarter of 2000, while the break-even inflation rate remained relatively stable, or even with a slightly downward trend. In the past few months (approximately since early 2000), the nominal YTM has not evinced a clear trend, due to the fact that neither the real interest rate expectation nor the break-even inflation rate have recorded significant changes.

Despite the short sample period available, the break-even inflation rate implied in index-linked bonds, which represents an average expectation for a ten-year period, seems to have the properties of a “leading indicator” for the current year-on-year inflation rate. Chart 2.3 compares the evolution in the past two years of the year-on-year inflation rate in France with inflation expectations implied in French index-linked bonds. Indeed, it can be seen that the latter anticipated relatively in advance the recent rise in the inflation rate.

3. INDICATORS OF UNCERTAINTY

As referred to in section 2, the prices of financial assets provide information on the future path expected by investors for certain macroeconomic variables, namely the levels of economic activity and inflation. In particular, the analysis over time of nominal interest rates evinces how economic agents revise their expectations about future levels for these variables. However, prices of financial assets only make it possible to infer the average level expected for the level of macroeconomic variables and do not provide information on the degree of uncertainty attached to these expectations by the market. In mathematical terms, the degree of uncertainty associated with a random variable is measured by the moments of order above one, namely the second, the third and the fourth moments. Since economic conditions change over time, the uncertainty assessed by the market regarding future inflation and economic activity is also likely to change. Besides, investors are expected, under certain circumstances, to attach to different scenarios different probabilities, which may give rise to multimodal or asymmetric distribution of probabilities implied for the different prices in the future. Under these circumstances, the calculation of uncertainty indicators is particularly indicative of the overall market sentiment. Due to their prospective characteristics in relation to the price of the underlying asset, derivatives prices intrinsically contain information related to different

(12) The second centred moment of a random variable defines the degree of dispersion of the observations around the average, the third moment defines the skewness and the fourth moment the kurtosis.
aspects of uncertainty. For example, options on long-term government bonds may be useful to gauge the degree of uncertainty associated with future developments in the yields of these bonds.

If the yield of a long-term bond may be decomposed in ex ante real interest rate, average inflation expectations up to the maturity date and a component related to several risk premia, any measure of uncertainty extracted from the option prices on bonds must also reflect the uncertainty attached to these components. However, there is no consensus on the way in which uncertainty related to inflation expectations can be separated from uncertainty associated with the path of the real interest rate or the risk premia. The fact that expectations about these components affect the pricing of derivatives of long-term bonds therefore requires the assumption of certain simplifying hypotheses in relation to these factors, with a view to extracting and interpreting prospective information.

3.1 Main concepts

Financial derivatives prices (forward, futures and options contracts) reflect economic agents’ expectations about future price developments in underlying assets. However, while forward and futures contracts only provide information on the expected value of the prices of underlying assets, options premia allow for the estimation of the probability attached by the market to the various possible prices of underlying assets.

Thus, options prices contain relevant information for financial institutions and the private sector in general. This information is also potentially useful for monetary authorities, namely for constructing uncertainty indicators, assessing the impact of monetary policy measures and to help to identify anomalies in the operation of financial markets. All these issues have been increasingly focused in the literature.

In order to derive an uncertainty indicator associated with a financial asset, account is taken of the valuation in period $t$ of a European call option with exercise date $T$. In a world of risk-neutral agents, the price of an option (premium) is obtained by calculating its expected return, discounted at the risk-free interest rate, in relation to the risk-neutral probability measure. If $S_T$ is the price of the underlying asset at time $T$, and $X$ the strike price of the European option, the price of a call option at $t$ is given by:

$$C(X) = e^{-r(T-t)} \int_0^\infty \max[S_T - X, 0] q(S_T) dS_T, \quad (3.1)$$

where $r = T - t$ and $q(S_T)$ is the risk-neutral probability density function (PDF) of the price of asset $S_T$, conditional on the current asset price. Differentiating (3.1) with respect to the strike price gives:

$$\frac{\partial C}{\partial X} = -e^{-r(T-t)} \left( 1 - \int_x^\infty q(S_T) dS_T \right), \quad (3.2)$$

that is:

$$1 + \frac{\partial C}{\partial X} e^{r(T-t)} = P_0[S_T \leq X]. \quad (3.3)$$

Thus, the distribution function of the future price of the underlying asset is obtained. The risk-neutral probability density function is derived through the differentiation of (3.2) with respect to the strike price:

$$q(X) = e^{r(T-t)} \frac{\partial^2 C(X)}{\partial X^2}. \quad (3.4)$$

This PDF is the uncertainty measure associated with a given future average expectation. The various estimation methods for this function are explained below.

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(13) An option gives the holder the right, but not the obligation, to conduct a specific financial transaction at a certain future date, at a predetermined price, against the payment of a premium to the seller of an option (a call option gives the right to buy, while a put option gives the right to sell). The predetermined price for the future transaction is known as the exercise price or strike price and the date at which or up to which an option can be exercised is known as the maturity date or exercise date. European options can only be exercised at expiration, while American options can be exercised at any time up to and including the maturity date. An option has a number of designations according to its results. Thus, it is said to be in-the-money when after being immediately exercised a profit is implied for its holder. Conversely, when after being immediately exercised a loss is implied for its holder, this option is said to be out-of-the-money. If the exercise of an option contract does not imply a gain or a loss to the holder, this option is said to be at-the-money.
3.2 Techniques to extract PDFs through prices of options on interest rates

The methods to extract PDFs from the premia of options on financial assets can be grouped into four approaches. In the first approach, the PDF is nonparametrically estimated, that is with no functional restrictions on the stochastic process that governs the underlying financial asset, the call pricing function, the implied volatility or the PDF.

In the second approach, a functional form of the PDF or of the stochastic process followed by the underlying asset is specified. Should the latter be chosen, it is necessary to derive the PDF implied in this stochastic process. The parameters of the PDF, whether it has been derived or directly specified, are estimated by minimising the distance between the observed premia and the theoretical premia that are generated by the functional form assumed.

In the third approach, a specific density considered liable of being a PDF is estimated in a first phase, usually starting by a simple density function. The PDF is subsequently re-estimated as the density estimated in the first phase and (ii) it minimises the distance from the observed premia.

In the fourth approach the PDF is derived directly from some parametric specification of the call pricing function (or the put pricing function) or of the volatility implied in options.

Following the proposals of academic literature, the empirical studies developed have frequently chosen the second approach, specifying the stochastic process followed by the underlying asset and inducing the respective PDF implied in this stochastic process. The PDF parameters are estimated by minimising the distance between the observed options premia and the theoretical premia (directly or indirectly) generated by the functional form specified for the PDF.

The most commonly used functional form for the PDF is the mixture of several lognormal distributions(14). In the case of a European option and of a mixture of two distributions, we have the following optimisation problem:

\[
\begin{align*}
\text{Min} & \sum_{j=1}^{N} \sum_{i=1}^{2} \left[ \left( \hat{C}(X_{j}, \tau) - C_{i}^{0} \right)^{2} + \left( \hat{P}(X_{j}, \tau) - P_{i}^{0} \right)^{2} \right] \\
+ & \lambda \left[ \theta e^{-\frac{\mu_{1}}{2} \tau} + (1-\theta) e^{-\frac{\mu_{2}}{2} \tau} \right] \left( s_{T} - X_{j} \right) \left( s_{T} - X_{j} \right) \right] \end{align*}
\]

where:

- \( \hat{C}(X_{j}, \tau) \) and \( \hat{P}(X_{j}, \tau) \) are the theoretical premia of call and put options for the different strike prices and a given maturity \( \tau \),
- \( C_{j}^{0} \) and \( P_{j}^{0} \) are the premia observed for these strike prices of the call and put options,
- \( q\left( \mu_{j}, \sigma_{j}, S_{T} \right) \)

that is, \( q\left( \mu_{j}, \sigma_{j}, S_{T} \right) \) is the lognormal density function for call and put options premia, where the parameters \( \mu_{1} \) and \( \mu_{2} \) are the means of the normal corresponding distributions and \( \sigma_{1} \) and \( \sigma_{2} \) the respective variances and finally,

- \( \theta \) and \( (1-\theta) \) are the weight attached to each distribution.

The first two segments of the objective function are the sum of the squared deviations between the observed premia and the estimated premia of call options and put options. The last segment of the objective function reflects the squared difference between the estimated mean of the distribution and the future price. From the theoretical point of view, the distribution mean should equal the future price, whereby \( \lambda > 0 \) reflects the penalty coefficient for the distance between the two measures. Where \( \lambda = 0 \) the penalty is nil, while where \( \lambda \) is

very large \((\lambda \to \infty)\) this will be equivalent to considering a conditioned optimisation problem with an equality restriction between the distribution mean and the future price.

When fixing a value for \(\lambda\), whenever there is a sufficiently large number of call and put prices with the same time-to-maturity, simultaneously observed but with different strike prices, the parameters of equation (3.8), as well as \(\theta\) (where \(\lambda > 0\)), can be determined by the minimisation of expression (3.5). In line with the current practice in other central banks, namely the Bank of England and Banca d’Italia, the specification considered by Banco de Portugal is a mixture of two lognormal distributions\(^{(15)}\) and the penalty parameter was fixed at 1.

An American-style option gives the holder the right to exercise it at any time up to and including the maturity date. Thus, the theoretical premia of the option at the maturity date. Thus, the theoretical premia of the option is only exercised at the maturity date. The American-style option, i.e. in case the American-style option is only exercised at the maturity date. The upper bound is equal to the value of the European option, i.e. in case the American-style option is only exercised at the maturity date. The lower bound is equal to the value of the American-style option if the maturity corresponds to the present moment, i.e. the sum that its holder receives for having exercised his option immediately. Equations (3.6) and (3.7) are substituted by:

\[
\hat{C}(X, \tau) = w_1 C^i(X, \tau) + (1 - w_1) C^o(X, \tau) \\
C^i(X, \tau) = \max \left( e^{\lambda S_1} - X_j, \int_{S_j}^{\infty} \theta q(\mu_j, \sigma_j; S_j) + + (1 - \theta) q(\mu_j, \sigma_j; S_j) (S_j - X_j) dS_j \right) \\
C^o(X, \tau) = \max \left( e^{\lambda S_1} - X_j, e^{\lambda - \tau} \int_{S_j}^{\infty} \theta q(\mu_j, \sigma_j; S_j) + + (1 - \theta) q(\mu_j, \sigma_j; S_j) (S_j - X_j) dS_j \right) \\
\hat{R}(X, \tau) = w_1 P^i(X, \tau) + (1 - w_1) P^o(X, \tau)
\]

(15) In Adão, Cassola and Luís (1998) it was concluded that the estimation of the density functions through the combination of two lognormal distributions has several advantages compared to the alternative methods, given that it allows for PDFs that are simultaneously flexible — asymmetric density functions, exhibiting skewness and/or multimodal functions — and consistent with the empirical data observed.

\[
P^i(X, \tau) = \max \left\{ X_j - e^{\lambda - \tau} S_j, \int_{S_j}^{\infty} \theta q(\mu_j, \sigma_j; S_j) + + (1 - \theta) q(\mu_j, \sigma_j; S_j) (X_j - S_j) dS_j \right\} \\
P^o(X, \tau) = \max \left\{ X_j - e^{\lambda - \tau} S_j, e^{\lambda - \tau} \int_{S_j}^{\infty} \theta q(\mu_j, \sigma_j; S_j) + + (1 - \theta) q(\mu_j, \sigma_j; S_j) (X_j - S_j) dS_j \right\}
\]

(3.10)

\[h = \begin{cases} 
1 & \text{if the option is in-the-money} \\
2 & \text{if the option is out-of-the-money}
\end{cases}
\]

In the next section PDFs are derived for the euro area long-term bond yields. Since options on euro area forward interest rates are of the American style, the expressions used were (3.9) and (3.10).

3.3 Derivation of PDFs for the euro area long-term bond yields

Chart 3.1 shows two PDFs of the euro area long-term bond yields. The proxy for this rate is the 10-year German government bond yields. Given that the options and futures contracts with the highest degree of liquidity in the euro area are the German Bund contracts\(^{(16)}\), these should, in principle, reflect investors’ expectations about the level of the euro area long-term bond yields.\(^{(17)}\) The sample dates are 19 June and 22 June 2000.

\(^{(16)}\) Derivatives contracts on German (Bund) government bonds are traded on the EUREX derivatives exchange. The underlying asset of the options is a futures contract traded on this market. In turn, the underlying asset of the futures contract is a long-term public debt instrument issued by the German Federal Government, with a residual maturity of between 8.5 and 10.5 years and an annual coupon of 6 per cent. Options on Bund futures are of the American style, and can be exercised at any time up to and including the maturity date. Maturities coincide with the months when the underlying futures contract matures, which include the March, June, September and December cycle. The last trading day of the contract is the 10th business day of the month in which the contract matures.

\(^{(17)}\) PDFs were originally estimated for the German long-term bond prices and not for interest rates. The bond prices thus obtained were converted into interest rates by using the bond with the cheapest delivery on the estimation day (i.e. when the options prices are observed). The PDF associated with the bond prices was transformed into the PDF associated with interest rates by using the first derivative of the price with respect to the interest rate, according to the normal formula of the change of the variable.
The aim is to analyse the impact of the decision of the Organisation of Petroleum Exporting Countries (OPEC) of 21 June to increase crude oil production, in order to contain the rise of the oil price recorded in the previous months.

The probability density functions estimated for 19 June and 22 June show that the investors’ reaction to the shock was significant, albeit contrary to expectations. In fact, the probability mass shifted strongly to the right, mirroring an upward (not downward) revision by investors of their expectations regarding the level of 10-year interest rate which will prevail in the euro area in December. The most probable value for the 10-year interest rate at the end of the year (measured by the distribution mode) increased by around 14 b.p., moving from 4.97 per cent on 19 June to 5.11 per cent on 22 June (Table 3.1). The upward revision of expectations was accompanied by an increased uncertainty, confirmed by a longer PDF on 29 June. PDFs are positively skewed on both 19 and 22 of June, i.e. investors attached a higher probability to the forward interest rate in December 2000, being higher than the distribution mode and not the opposite.

3.4 Extracting PDFs for inflation expectations

The measure of the degree of uncertainty associated with the 10-year interest rate will be, however, more elucidative, if it can be broken down into its various components, viz. the uncertainty about the real interest rate, the inflation rate expectations and the risk premium. This paper builds on the possibility of extracting uncertainty indicators associated with inflation expectations through a methodology similar to that suggested in Söderlind and Svensson (1997).

If there were \( n \) prices of options on real and nominal interest rates, with maturity \( T \), these prices could be used, under certain assumptions, to extract the implied risk-neutral probability density functions for the future inflation expectations. If there are no derivatives on future inflation expectations or on real interest rates, assumptions can nevertheless be made on the manner in which investors would define their expectations, should they exist.

Consider the Fisher identity, with reference not to the present time \( t \), but to the maturity time of the options, \( T \):

\[
i_{T,m} = r_{T,m} + E_T \left( \pi_{T,m} \right) + \lambda_{T,m},
\]

where \( i_{T,m} \) is the nominal interest rate at \( T \) for the maturity \( m \) (the maturity of the bond implied in the option, which will be \( T+m \)), \( r_{T,m} \) the real interest rate at \( T \) expected in the period between \( T \) and \( T+m \), \( E_T \left( \pi_{T,m} \right) \) the inflation rate expected at \( T \) for the same period and \( \lambda_{T,m} \) a measure of the premia associated with the various risks involved in this investment.

The extent of the risk premium \( \lambda_{T,m} \) (which represents the consolidated effect of the liquidity and inflation premium) is assumed to be minor. Even if it is not, it is not admitted to have significant

### Chart 3.1

**PDF FOR DECEMBER 2000 OF THE EURO AREA 10-YEAR INTEREST RATE ON 19 AND 22 JUNE 2000**

![Chart depicting probability density functions for 19 June and 22 June 2000 for the euro area 10-year interest rate.]

### Table 3.1

**STATISTICS ON EXPECTATIONS ABOUT THE 10-YEAR INTEREST RATE**

<table>
<thead>
<tr>
<th>Per cent</th>
<th>19 June 2000</th>
<th>22 June 2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>5.01</td>
<td>5.11</td>
</tr>
<tr>
<td>Mode</td>
<td>4.97</td>
<td>5.11</td>
</tr>
<tr>
<td>Median</td>
<td>5.00</td>
<td>5.11</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>0.37</td>
<td>0.42</td>
</tr>
</tbody>
</table>
variability over time \((\Delta l_{T,n} = 0)\), whereby the compared analysis for different moments in time remains valid. Under this hypothesis, the nominal interest rate is decomposed in two components: the real interest rate and the future inflation expectation, i.e.: \(i_{T,n} = r_{T,n} + E_T(\pi_{T,n})\).

If the real interest rate and the inflation expectations were two independent variables, then inflation expectations could be extracted from information on nominal and real interest rates. The hypothesis of a nil conditional covariance between the two variables is equivalent to considering that the conditional variance of \(r\) and the conditional covariance between \(i\) and \(r\) are statistically equal. Both variables had a similar behaviour in the sample considered (the last 100 observations), and the difference between them does not seem to be significant.

Admitting that:

\[
r_{T,n} \sim N\left(r_{T,n}, \sigma_{1,T}\right),
\]

(3.12)

where \(r_{T,n}\) is the real rate recorded at the present moment \(t\) and \(\sigma_{1,T}\) the variance of \(r_{T,n}\), conditional on the information available in \(t\). Considering also that the PDF of the nominal interest rate is the result of the combination of two normal distributions:

\[
i \sim \theta. N\left(\mu_i, \sigma_i\right) + (1 - \theta). N\left(\mu_{1-i}, \sigma_{1-i}\right).
\]

(3.13)

Then, under the hypothesis that the real interest rate and inflation expectations are independent from each other, the PDF of inflation expectations is also the result of the linear combination of two normal distributions:

\[
E_T(\pi_{T,n}) \sim \theta. N\left(\mu_{\pi}, r_{T,n}, \sigma_{\pi} - \sigma_{1,T}\right) + (1 - \theta). N\left(\mu_{2-\pi}, r_{T,n}, \sigma_{2-\pi} - \sigma_{1,T}\right).
\]

(3.14)

Note that (3.14) sets up a model for the first moment of the probability distribution of the future inflation \(E_T(\pi_{T,n})\) implied in the behaviour of financial markets. It is not therefore a model for the distribution of the inflation rate \(\pi_{T,n}\) itself. The equation (3.14) is relevant since it is reasonable to admit that there is a close relationship between the PDF of the inflation expectation and the PDF of inflation in the full sense.

In practice, the construction of the PDF of euro area inflation expectations assumes that the Bund bond yield is a proxy for the euro area nominal interest rate and that the French index-linked bond yield is a proxy for the euro area real interest rate. Thus, the probability distribution of the nominal interest rate was obtained through Bund options, according to the explanations of the previous section. In turn, the parameters of the distribution of the real interest rate were obtained from historical data on the real interest rate implied in French index-linked bonds. In order to avoid comparability problems arising from changes in the spread between French and German long rates, the observed values of real rates were previously adjusted by adding to them the spread between French and German nominal rates.

Chart 3.2 shows the forecasts made before (19 June) and after (22 June) the OPEC meeting with regard to the PDF of the expectations in December 2000 about the average euro area inflation rate for the period from 2001 to 2010. Currently, the aim is to analyse to what extent the change in the uncertainty linked with the expectations for the 10-year nominal interest rate was due to a change in the uncertainty linked with inflation expectations in that period.

The change in the estimates of the PDF of inflation expectations shows that the investors’ response to the shock was significant, albeit contrary
to expectations. The probability mass of the PDFs of inflation expectations shifted to the right, i.e. investors revised their inflation expectations upwards. The most probable value for the average inflation rate expected for the ten-year period starting in January 2001 (measured by the distribution mode) increased by around 10 b.p., moving from 1.50 per cent on 19 June to 1.60 per cent on 22 June (Table 3.2).

When comparing the data from Table 3.1 with those from Table 3.2, the 14 b.p. revision of the most probable value for the euro area 10-year nominal interest rate in December 2000 can be explained by the 10 b.p. revision of inflation expectations. In sum, the analysis of the PDF of inflation expectations confirms the analysis made in the previous section, i.e. the OPEC countries’ decision on 21 June was not considered to be sufficient to reverse the upward trend of the oil price, and therefore it did not have the desired impact on financial markets.

4. CONCLUSION

This paper summarises the most recent theoretical and applied research on the manner in which government bond prices and their derivatives can be used to extract information on the market expectations regarding future inflation. The contribution from index-linked bonds to the construction of inflation expectation indicators was noticeable. In the second part of this paper probability density functions of “implied inflation expectations” were derived. The preliminary results obtained, despite the fragility of their simplifying hypotheses, are encouraging.

REFERENCES


